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Characterization of expansion-related properties of modular graphs

Yilun Shang

Department of Computer and Information Sciences, Northumbria University, Newcastle NE1 8ST, UK

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ABSTRACT

A fundamental organizing principle of real-world complex networked systems is modularity, where networks have interactions at different levels. In this paper we consider a modular graph G having modules with arbitrary intraconnections and random interconnections between activated vertices in different modules. The vertices in different modules are activated with probability r and linked by an interconnecting edge with probability p independently. We present results regarding the Cheeger constant, robustness, algebraic connectivity as well as the smallest eigenvalue for the Dirichlet Laplacian matrix of G with high probability. Our results suggest that $r = \sqrt{(\ln n)/n}$ is a potential scaling for the recently observed external field-like phenomena of modular networks in statistical mechanics.

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1. Introduction

The past two decades have witnessed a significant advance in understanding large graph and network structures with the growing prosperity of network science [1]. Classical graph theory, be it deterministic or random, has been boosted and diversified by a variety of complex network models, which have close relevance to real-life applications in nature, technology and human society. Modular graphs or networks with community structures have received a considerable attention [2–6] as it addresses a key limitation in classical graph theory, where individual graphs are treated as isolated systems. Many real-world networks, however, have shown a hierarchical modular topology, in which small groups of modules of vertices are connected more closely to each other than to the graph at large. This fundamental organizing principle in networks has been well embodied in complex systems such as the Internet [7], neural networks [8], protein complexes [9], and socioeconomic ties in geographical regions [10]. Modular graphs have been extensively investigated in mathematics, physics and computer science under different names like multi-layer network [2], community structure [11,12], stochastic block model [13–15], heterogeneous or multitype random graph [16,17], and planted partition model [18].

Recently, a new model of modular graph has been introduced in [19] to study network robustness. The graph consists of multiple interacting modules with random connections linking different modules. In each module, only a fraction r of vertices (called interconnected nodes) is allowed to connect with other modules. It is found that, irrespective of intraconnections within each module, the interconnections between modules behave in a manner analogous to a ghost field in percolation or an external field in Ising spin systems near the magnetic–paramagnetic criticality [20]. First-order phase transition of this model has been studied through generalized core percolation [21] and cascading failure [22], where Widom's scaling identity from statistical mechanics is shown to be satisfied in both cases. The model has also

E-mail address: yilun.shang@northumbria.ac.uk.

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been applied in the study of structural resilience of spatial networks [23] and epidemic spreading processes [24]. In [25], it is shown that an optimal level of interconnected nodes can be determined for the modular graph model in terms of giant component size.

In this paper, we will continue this line of research and investigate the topological properties of modular graphs [19] from the perspective of random graph theory [26]. Note that all the above mentioned works are based on mean-field approximation techniques borrowed from statistical physics. To fix the notation, we define the modular graph $G = (V, E)$ as follows. For $1 \le i \le m$, let $G_i = (V_i, E_i)$ be a graph (or module) with the vertex set V_i having $|V_i| = n$ vertices and the edge set denoted by E_i. Each vertex in V_i is activated with probability $r = r(n)$ independently to become an interconnected node. Only interconnected nodes are allowed to establish links outside their own modules. On top of the m modules, the modular graph $G = (V, E)$ has the vertex set $V = \cup_{i=1}^m V_i$ and the edge set $E = (\cup_{i=1}^m E_i) \cup E_I$, where E_I is formed by interconnections between a pair $\{v_i, v_j\}$ of interconnected nodes in V, namely, $v_i \in V_{i_1}$ and $v_j \in V_{i_2}$ with $i_1 \ne i_2$. The intraconnections in $\cup_{i=1}^m E_i$ are not specified in our model, whereas the interconnections are established with probability $p = p(n)$ independently between interconnected nodes. A vertex being an interconnected node is also assumed to be independent of establishing an interconnection. This model is more general than some well-known random graphs such as Erdős–Rényi random graph, random multipartite graph, and some stochastic block models [26,27].

We are interested in phase transitions of some expansion-related properties of the modular graph model $G(V, E)$ in the thermodynamic limit, namely, as n tends to infinity. The rest of the paper is organized as follows. Section 2 contains our main results starting from the Cheeger constant and graph robustness to algebraic connectivity and the smallest Dirichlet graph Laplacian eigenvalue. The conclusion is drawn in Section 3.

2. Main results

Let $[m]$ denote the set $\{1, 2, \dots, m\}$. For simplicity, the set $\{v_i\}_{i\in[n]}$ can represent the set of vertices in any module of G. The maximum and minimum degrees of a graph H are denoted by $d_{\max}(H)$ and $d_{\min}(H)$, respectively. We use the standard Landau asymptotic notations O, o and Θ to capture the rates of growth of functions with respect to the order n throughout the paper; c.f. [26]. We write $G_I = (V, E_I)$ for the special case of modular graph G with $E_i = \emptyset$ for $i \in [m]$.

2.1. Cheeger constant of modular graphs

The Cheeger constant is a key edge expansion index for graphs. Given a vertex set $S \subseteq V$, the edge boundary of S is defined as $\partial S = \{\{v_i, v_j\} \in E : v_i \in S, v_j \in V \backslash S\}$. The Cheeger constant of G is defined by minimizing the ratio of the order of edge boundary and the order of the set: $h(G) = \min\{|\partial S|/|S| : S \subseteq V, 0 < |S| \le |V|/2\}$. It is easy to see that $h(G) > 0$ if and only if G is connected. Moreover, $h(G) \le d_{\min}(G)$. The well-known Cheeger inequality relates the expansion property to the spectral gap [28]:

$$\frac{h(G)^2}{2d_{\max}(G)} \le \lambda(G) \le 2h(G), \tag{1}$$

where $\lambda(G)$ is the second smallest Laplacian eigenvalue of G. Recall that the Laplacian for a simple graph is positive semidefinite and the smallest Laplacian eigenvalue is zero. The first result concerns the Cheeger constant of G_I.

Theorem 1. *Let $m \ge 2$ and $k \ge 0$ be two constants. Suppose $\omega(n) = o(\ln \ln n)$.*

(i) If $p = \frac{\ln n + k \ln \ln n + \omega(n)}{(m-1)nr^2}$, then

$$\mathbb{P}(h(G_I) > k) = 1 - O(e^{-\omega(n)}). \tag{2}$$

(ii) If $p = \frac{\ln n + k \ln \ln n - \omega(n)}{(m-1)nr^2}$, then

$$\mathbb{P}(h(G_I) > k) = O\left(e^{-e^{\frac{\omega(n)}{2}}}\right). \tag{3}$$

Proof. (i) Consider a vertex set $S \subseteq V$ with $|S \cap V_i| = s_i$ for $i \in [m]$ and $\sum_{i=1}^m s_i = s = |S|$. Suppose $1 \le s \le nm/2$. Define $\mathcal{A}_S = \{|\partial S| \le sk\}$ to be the event of no more than sk edges leaving S. To bound the probability of this event, we note that $|\partial S|$ follows the distribution of an independent sum of binomial distributions: $|\partial S| \sim \sum_{i=1}^m \sum_{j=1, j\ne i}^m \mathrm{Bin}(s_i(n - s_i), r^2 p)$. Since $\sum_{i=1}^m \sum_{j=1,j\ne i}^m s_i(n - s_i) = n(m-1)s - s^2 + \sum_{i=1}^m s_i^2$, we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{A}_S) &= \sum_{j=0}^{sk} \binom{n(m-1)s - s^2 + \sum_{i=1}^m s_i^2}{j} r^{2j} p^j (1 - r^2 p)^{n(m-1)s - s^2 + \sum_{i=1}^m s_i^2 - j} \\ &\le \sum_{j=0}^{sk} \binom{n(m-1)s}{j} r^{2j} p^j (1 - r^2 p)^{n(m-1)s - (1 - \frac{1}{m})s^2 - j},\end{aligned} \tag{4}$$

where we have applied $0 \le s^2 - \sum_{i=1}^m s_i^2 \le (1 - \frac{1}{m})s^2$ by the inequality of arithmetic and quadratic means. Since $k \le sk \le nmk/2$ and $0 \le j \le sk$, we estimate the ratio of two adjacent terms in the expression (4) as

$$\frac{\binom{n(m-1)s}{j+1} r^{2j+2} p^{j+1} (1-r^2p)^{n(m-1)s-(1-1/m)s^2-j-1}}{\binom{n(m-1)s}{j} r^{2j} p^j (1-r^2p)^{n(m-1)s-(1-1/m)s^2-j}} \ge \frac{(n(m-1)-k)r^2p}{2k(1-r^2p)} \ge \frac{\ln n}{3k}$$

for large n by the definition of p. Therefore, there is some constant $c > 0$ such that

$$\mathbb{P}(\mathcal{A}_S) \le c\binom{n(m-1)s}{sk} r^{2sk} p^{sk} (1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-sk}.$$

Next, taking the sum over all possible sets S, we obtain

$$\begin{aligned}
\sum_{\substack{S\subseteq V\\|S|=s}} \mathbb{P}(\mathcal{A}_S) \le& c\binom{nm}{s}\binom{n(m-1)s}{sk} r^{2sk} p^{sk} (1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-sk} \\
\le& c\left(\frac{nme}{s}\right)^s \left(\frac{n(m-1)epr^2}{k}\right)^{sk} (1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-sk} \\
\le& c\left(\frac{2me^{k+1}}{k^k}\right)^s \left(\frac{ne^{-r^2pn(m-1)}(n(m-1)r^2p)^k}{s(1-r^2p)^{(1-\frac{1}{m})s}}\right)^s \\
\le& 2c\left(\frac{2me^{k+1}}{k^k}\right)^s \left(\frac{e^{-\omega(n)}e^{-(1-\frac{1}{m})s\ln(1-r^2p)}}{s}\right)^s \\
=& 2c\left(\frac{2me^{k+1}}{k^k}\right)^s \left(\frac{e^{-\omega(n)}e^{(1-\frac{1}{m})sr^2p}e^{r^4p^2(1-\frac{1}{m})s\sum_{i=2}^{\infty}\frac{(r^2p)^{i-2}}{i}}}{s}\right)^s,
\end{aligned} \tag{5}$$

where we have applied the estimates $\binom{a}{b} \le (ae/b)^b$, $1 - b \le e^{-b}$ for any $b \le a$, and the definition of p. Since $\sum_{i=2}^{\infty}(r^2p)^{i-2}/i < \sum_{i=2}^{\infty}(r^2p)^{i-2} = (1-r^2p)^{-1}$, we have $e^{r^4p^2(1-\frac{1}{m})s\sum_{i=2}^{\infty}\frac{(r^2p)^{i-2}}{i}} \le 2$ for any large n by the definition of p. Hence, using (5) we have

$$\sum_{\substack{S\subseteq V\\|S|=s}} \mathbb{P}(\mathcal{A}_S) \le 2c\left(\frac{4me^{k+1}}{k^k}\right)^s e^{-\omega(n)s}\varphi(s)^s, \tag{6}$$

where $\varphi(s) = s^{-1}e^{(1-1/m)sr^2p}$. A direct calculation gives rise to $\varphi(s) \le \max\{\varphi(1), \varphi(nm/2)\}$ for $1 \le s \le nm/2$. By the choice of p, we know that $1 < \varphi(1) < e$ and $\varphi(nm/2) = o(1)$ as $n \to \infty$. It follows from (6) that

$$\sum_{\substack{S\subseteq V\\|S|=s}} \mathbb{P}(\mathcal{A}_S) \le 2c\left(\frac{4me^{k+2-\omega(n)}}{k^k}\right)^s. \tag{7}$$

By using (7), the probability of existing a set S with $1 \le s \le nm/2$ such that $\mathcal{A}_S$ occurs is bound from the above by

$$\sum_{s=1}^{\frac{nm}{2}} \sum_{\substack{S\subseteq V\\|S|=s}} \mathbb{P}(\mathcal{A}_S) \le 2c\sum_{s=1}^{\frac{nm}{2}}\left(\frac{4me^{k+2-\omega(n)}}{k^k}\right)^s \le 2c\cdot\frac{\frac{4me^{k+2-\omega(n)}}{k^k}}{1-\frac{4me^{k+2-\omega(n)}}{k^k}} = O(e^{-\omega(n)}).$$

This indicates $\mathbb{P}(h(G_I) > k) \ge 1 - O(e^{-\omega(n)})$ and (2) is proved.

(ii) Let $V_1 = \{v_i\}_{i\in[n]}$ and d_i be the degree of vertex v_i in G_I. Define a random variable $\xi = \sum_{i=1}^n \xi_i$, where $\xi_i = 0$ if $d_i \ge k+1$ and $\xi_i = 1$ if $d_i \le k$. It suffices to prove the following

$$\mathbb{P}(\xi = 0) = O\left(e^{-e^{\frac{\omega(n)}{2}}}\right). \tag{8}$$

In fact, we have $\mathbb{P}(h(G_I) > k) \le \mathbb{P}(d_{\min}(G_I) > k) = \mathbb{P}(d_{\min}(G_I) \ge k+1) \le \mathbb{P}(\xi = 0)$ and (3) will be true if (8) holds.

To estimate the probability of $\xi = 0$, we note that by independence,

$$\mathbb{P}(\xi = 0) = (1 - \mathbb{P}(\xi_1 = 1))^n \le e^{-n\mathbb{P}(\xi_1=1)}. \tag{9}$$

Moreover, we have

$$\begin{aligned}
\mathbb{P}(\xi_1 = 1) =& \sum_{i=0}^{k}\binom{n(m-1)}{i} r^{2i}p^i(1-r^2p)^{n(m-1)-i} \\
\ge& \binom{n(m-1)}{k} r^{2k}p^k(1-r^2p)^{n(m-1)}.
\end{aligned} \tag{10}$$

By the choice of p, we obtain

$$\frac{(1-r^2p)^{n(m-1)}}{e^{-n(m-1)r^2p}} = e^{-n(m-1)\sum_{j=2}^{\infty}\frac{(r^2p)^j}{j}} \geq \frac{\sqrt{2}}{2}$$

for large n, and $\binom{n(m-1)}{k} \geq (\sqrt{2})^{-1}n^k(m-1)^k/k!$ by [26, Lem. 21.1]. Combining these with (10) yields

$$\begin{aligned}\mathbb{P}(\xi_1=1) \geq & \frac{n^k(m-1)^k}{2k!}\cdot\frac{(\ln n+k\ln\ln n-\omega(n))^k}{(m-1)^kn^k}\cdot e^{-\ln n-k\ln\ln n+\omega(n)} \\ = & \frac{(\ln n+k\ln\ln n-\omega(n))^k}{2nk!(\ln n)^k}e^{\omega(n)} \geq \frac{e^{\omega(n)}}{2nk!}.\end{aligned} \tag{11}$$

In the light of (9) and (11), we derive that

$$\mathbb{P}(\xi=0) \leq e^{-\frac{1}{2k!}e^{\omega(n)}} = O\left(e^{-e^{\frac{\omega(n)}{2}}}\right)$$

since $k \geq 0$. This completes the proof of part (ii). □

As the property of $h(G_I) > k$ is monotonic, Theorem 1 indicates that $p = (\ln n + k\ln\ln n)/(m-1)nr^2$ is a sharp threshold for edge expansion. Moreover, by $h(G_I) \leq d_{\min}(G_I)$ and the proof of (ii), we have essentially shown that $p = (\ln n + k\ln\ln n)/(m-1)nr^2$ is also a sharp threshold for the weaker property $d_{\min}(G_I) \geq k+1$. Namely, if $p = \frac{\ln n+k\ln\ln n+\omega(n)}{(m-1)nr^2}$, then $d_{\min}(G_I) \geq k+1$ w.h.p.; If $p = \frac{\ln n+k\ln\ln n-\omega(n)}{(m-1)nr^2}$, then $d_{\min}(G_I) \leq k$ w.h.p.. Here, w.h.p. refers to 'with high probability' [26], meaning that $\lim_{n\to\infty}\mathbb{P}(\mathcal{A}) = 1$ for some property or event $\mathcal{A}$.

The next result concerns the maximum degree and the Cheeger constant of G_I off the criticality. Define

$$\theta = \limsup_{n\to\infty}\frac{\ln n}{(m-1)npr^2}. \tag{12}$$

When $\theta \in [0, 1)$, we define the following constant

$$\zeta = \max\{a \in (0, 2/25] : a(3+\ln 2-\ln a) = 1-\theta\}. \tag{13}$$

It is easy to check that ζ in (13) exists. This is just a technical definition, which helps us to formulate a probabilistic upper bound for the edge boundary (see (18) below).

Theorem 2. *Let $m \geq 2$.*

(i) If $\theta \in (0, \frac{27}{25})$, for any $\varepsilon \in (0, 1/2]$,

$$\mathbb{P}\left(d_{\max}(G_I) \leq n(m-1)pr^2\left(1+\frac{9}{5}\left(\frac{\ln n}{(m-1)npr^2}\right)^{\frac{1}{2}-\varepsilon}\right)\right) = 1-O\left(n^{1-\frac{27}{25\theta^{2\varepsilon}}}\right). \tag{14}$$

If $\theta = 0$, the above probability is $1-o(1)$.

(ii) If $\theta \in (0, 1)$, then

$$\mathbb{P}(\zeta npr^2 \leq h(G_I)) = 1-O\left(n^{-\frac{\zeta}{(m-1)\theta}}\right). \tag{15}$$

If $\theta = 0$, the above probability is $1-o(1)$.

Proof. (i) We first consider the case of $\theta \in (0, \frac{27}{25})$. Let d_i be the degree of a vertex $v_i \in V$ and $d_i \sim \mathrm{Bin}((m-1)n, r^2p)$. Hence, $\mathbb{E}d_i = n(m-1)r^2p$. By Chernoff's bound [26, Cor. 21.7], it is easy to check that

$$\mathbb{P}(d_i \geq (1+\delta)\mathbb{E}d_i) \leq e^{-\frac{\delta^2\mathbb{E}d_i}{3}} \tag{16}$$

holds for $0 < \delta \leq 9/5$. Taking $\delta = \frac{9}{5}((\ln n)/(m-1)npr^2)^{1/2-\varepsilon}$, it follows from (16) that

$$\begin{aligned}\mathbb{P}(d_{\max}(G_I) \geq (1+\delta)\mathbb{E}d_i) \leq & mn\mathbb{P}(d_i \geq (1+\delta)\mathbb{E}d_i) \\ \leq & me^{\ln n-\frac{27}{25}(\ln n)\left(\frac{\ln n}{(m-1)npr^2}\right)^{-2\varepsilon}} \\ = & O\left(n^{1-\frac{27}{25\theta^{2\varepsilon}}}\right).\end{aligned}$$

This readily yields (14). The case of $\theta = 0$ can be shown by taking the limit $\theta \to 0$ in the above proof.

(ii) Suppose $\theta \in (0, 1)$. Similarly as in the proof of Theorem 1(i), we consider a vertex set $S \subseteq V$ with $|S \cap V_i| = s_i$ for $i \in [m]$ and $\sum_{i=1}^m s_i = s = |S|$. Suppose $1 \leq s \leq nm/2$. Let $\mathcal{B}_S = \{|\partial S| < \zeta snpr^2\}$ be the event of no less than $\zeta snpr^2$ edges leaving S, where ζ is defined in (13). The random variable $|\partial S|$ follows the distribution of an independent sum of binomial distributions as $|\partial S| \sim \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \mathrm{Bin}(s_i(n-s_i), r^2p)$. Arguing similarly as in (4), we derive

$$\mathbb{P}(\mathcal{B}_S) \le \sum_{j=0}^{\zeta snpr^2} \binom{n(m-1)s}{j} r^{2j}p^j(1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-j}. \tag{17}$$

Since $m \ge 2$ and $0 \le j \le \zeta snpr^2$, we estimate the ratio of two adjacent terms in the expression (17) as

$$\begin{aligned}\frac{\binom{n(m-1)s}{j+1} r^{2j+2}p^{j+1}(1-r^2p)^{n(m-1)s-(1-1/m)s^2-j-1}}{\binom{n(m-1)s}{j} r^{2j}p^{j}(1-r^2p)^{n(m-1)s-(1-1/m)s^2-j}} &\ge \frac{(n(m-1)-\zeta npr^2)r^2p}{2\zeta npr^2(1-r^2p)}\\ &\ge \frac{(1-\zeta r^2p)}{2\zeta(1-r^2p)}\\ &\ge \frac{1}{2\zeta} > 1\end{aligned}$$

for large n, where we have used the definitions of ζ and p. Therefore, there is some constant $c > 0$ such that

$$\mathbb{P}(\mathcal{B}_S) \le c\binom{n(m-1)s}{\zeta snpr^2} r^{2\zeta snpr^2}p^{\zeta snpr^2}(1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-\zeta snpr^2}.$$

Next, taking the sum over all possible sets S, we obtain

$$\begin{aligned}\sum_{\substack{S\subseteq V\\ |S|=s}} \mathbb{P}(\mathcal{B}_S) &\le c\binom{nm}{s}\binom{n(m-1)s}{\zeta snpr^2} r^{2\zeta snpr^2}p^{\zeta snpr^2}(1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-\zeta snpr^2}\\ &\le c\left(\frac{nme}{s}\right)^s\left(\frac{(m-1)e}{\zeta}\right)^{\zeta snpr^2}(1-r^2p)^{n(m-1)s-(1-\frac{1}{m})s^2-\zeta snpr^2}\\ &\le c\left(\frac{(m-1)e}{\zeta}\right)^{\zeta snpr^2} e^{s\ln(\frac{nme}{s})}e^{-r^2p\left(n(m-1)s-(1-\frac{1}{m})s^2-\zeta snpr^2\right)}\\ &:= ce^{s\psi(s)},\end{aligned} \tag{18}$$

where we have applied the estimates $\binom{a}{b} \le (ae/b)^b$, $1-b \le e^{-b}$ for any $b \le a$, and the definition of p. Here, the function ψ in (18) is defined as follows

$$\psi(s) = 1+\ln m - \ln s + r^2p\left(1-\frac{1}{m}\right)s + npr^2\chi(s), \tag{19}$$

where

$$\chi(\zeta) = \zeta + \zeta\ln(m-1) - \zeta\ln\zeta - (m-1) + \zeta pr^2 + \frac{\ln n}{npr^2}. \tag{20}$$

By checking the derivative of ψ, it is easy to see that $\psi(s) \le \max\{\psi(1), \psi(nm/2)\}$ for $1 \le s \le nm/2$. By (19) and (20), we have $\psi(1) = 1+\ln m + r^2p(1-1/m) + npr^2\chi(\zeta)$ and $\psi(nm/2) = 1+\ln 2 + npr^2\big(\chi(\zeta)+(m-1)/2-(\ln n)/npr^2\big)$. Since $\frac{\partial\chi(\zeta)}{\partial\zeta} = \ln(m-1) - \ln\zeta + pr^2 > 0$, we know that $\chi(\zeta)$ is increasing. By a direct calculation, we have $\chi(0) = (\ln n)/npr^2 - (m-1) \le (m-1)(\theta-1) < 0$. Let $\chi(\zeta) = \chi(0)+\sigma$, where

$$\sigma = \zeta(1+\ln(m-1)+pr^2-\ln\zeta) \tag{21}$$

by (20). We will have a closer look at $\psi(1)$ and $\psi(nm/2)$.

Firstly, regarding $\psi(nm/2)$, by (19), (20) and (21) we have

$$\begin{aligned}\psi\left(\frac{nm}{2}\right) &= 1+\ln 2 + npr^2\left(\chi(0)+\sigma+\frac{m-1}{2}-\frac{\ln n}{npr^2}\right)\\ &= 1+\ln 2 + npr^2\left(\sigma - \frac{m-1}{2}\right).\end{aligned} \tag{22}$$

In view of (13), we have $\zeta(3+\ln 2-\ln\zeta) \le 1/2$. Therefore, $\zeta(3+\ln m-\ln\zeta) \le (m-1)/2$ for any $m \ge 2$. Furthermore,

$$\zeta+\sigma = \zeta(2+\ln(m-1)+pr^2-\ln\zeta) < \zeta(3+\ln m-\ln\zeta) \le \frac{m-1}{2}. \tag{23}$$

Combining (22) and (23), we have

$$\psi\left(\frac{nm}{2}\right) \le -\zeta npr^2. \tag{24}$$

for large n.

Secondly, regarding $\psi(1)$, by (19), (20) and (21) we have

$$\begin{aligned}\psi(1) =&1+\ln m+r^2p\Big(1-\frac{1}{m}\Big)+npr^2(\chi(0)+\sigma)\\ =&1+\ln m+r^2p\Big(1-\frac{1}{m}\Big)+npr^2\Big(\frac{\ln n}{npr^2}-(m-1)+\sigma\Big).\end{aligned} \tag{25}$$

By (13), we have $\zeta(3+\ln m-\ln\zeta)\le(m-1)(1-\theta)$ for $m\ge 2$. Therefore,

$$\zeta+\sigma=\zeta(2+\ln(m-1)+pr^2-\ln\zeta)<\zeta(3+\ln m-\ln\zeta)\le(m-1)(1-\theta). \tag{26}$$

In view of the definition of p and (26), we obtain

$$\frac{\ln n}{npr^2}-(m-1)+\sigma\le(m-1)(1-\theta)+\sigma<-\zeta. \tag{27}$$

Combining (25) and (27) yields

$$\psi(1)\le-\zeta npr^2. \tag{28}$$

for large n.

It follows from (24), (28) and the above comments that $\psi(s)\le-\zeta npr^2$ for large n. By (18), we obtain

$$\sum_{\substack{S\subseteq V\\|S|=s}}\mathbb{P}(\mathcal{B}_S)\le ce^{-s\zeta npr^2}.$$

Accordingly,

$$\begin{aligned}\mathbb{P}(h(G_I)<\zeta npr^2)\le&\sum_{s=1}^{\frac{nm}{2}}\sum_{\substack{S\subseteq V\\|S|=s}}\mathbb{P}(\mathcal{B}_S)\le c\sum_{s=1}^{\frac{nm}{2}}e^{-s\zeta npr^2}\\ &\le c\sum_{s=1}^{\infty}e^{-s\zeta npr^2}=\frac{ce^{-\zeta npr^2}}{1-e^{-\zeta npr^2}}=O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big).\end{aligned}$$

This proves (15). The case of $\theta=0$ can be shown by taking the limit $\theta\to0$ in the above proof. □

The Cheeger constant result for the modular graph G is stated in the following theorem.

Theorem 3. *Let $m\ge2$. If $\theta\in(0,1)$,*

$$\mathbb{P}\Big(\zeta npr^2\le h(G)\le(m-1)(n+\sqrt{n})pr^2\Big)=1-O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big), \tag{29}$$

where θ and ζ are defined in (12) and (13). *If $\theta=0$, the above probability is $1-o(1)$.*

Proof. Suppose $\theta\in(0,1)$. Regarding the vertex set V_1, we have $|\partial V_1|\sim\mathrm{Bin}(n^2(m-1),r^2p)$, namely it is binomial distributed. Hence, the expectation is $\mathbb{E}|\partial V_1|=n^2(m-1)r^2p$. By Chernoff's bound with $\delta=1/\sqrt{n}$, we obtain

$$\mathbb{P}(|\partial V_1|\ge(1+\delta)\mathbb{E}|\partial V_1|)\le e^{-\frac{n(m-1)r^2p}{3}}\le e^{-\frac{\ln n}{3\theta}}=n^{-\frac{1}{3\theta}} \tag{30}$$

for large n, where we have used the definition of θ. Since $h(G)\le|\partial V_1|/|V_1|$ holds, we have

$$\begin{aligned}\mathbb{P}(h(G)\le(m-1)(n+\sqrt{n})pr^2)\ge&\mathbb{P}\Big(\frac{|\partial V_1|}{|V_1|}\le(m-1)(n+\sqrt{n})pr^2\Big)\\ =&\mathbb{P}(|\partial V_1|\le(1+\delta)\mathbb{E}|\partial V_1|)\\ \ge&1-n^{-\frac{1}{3\theta}}\end{aligned} \tag{31}$$

in view of (30).

On the other hand, using the monotonicity of the Cheeger constant and (15), we obtain

$$\mathbb{P}(h(G)\ge\zeta npr^2)\ge\mathbb{P}(h(G_I)\ge\zeta npr^2)=1-O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big). \tag{32}$$

Note that

$$\frac{n^{-\frac{1}{3\theta}}}{n^{-\frac{\zeta}{(m-1)\theta}}}=n^{-\frac{1}{\theta}(\frac{1}{3}-\frac{\zeta}{m-1})}\to0$$

as $n\to\infty$. Combining this with (31) and (32) readily yields the result. The case of $\theta=0$ can be shown by taking the limit $\theta\to0$ in the above proof. □

Theorem 3 indicates that the Cheeger constant $h(G) = \Theta(npr^2)$ w.h.p., namely, the modular graph G is a $\Theta(npr^2)$-expander when $\theta < 1$.

Remark 1. From the above results we observe that the critical activation probability r of interconnected nodes has to satisfy the condition

$$\sqrt{\frac{\ln n}{n}} \leq r \leq 1$$

since the edge probability p is within the interval [0, 1]. This is consistent with the experiments performed in [19–24]. In these works, the network size is taken typically as $n = 10^7$ giving $\sqrt{(\ln n)/n} = 0.013$ and the correct phase transition exponents in Widom's relationship for the zero external field are observed when $r \leq 10^{-2}$. Since the activation probability r is only taken as a constant in these numerical experiments, the scaling of r with respect to network size is not revealed. Our expansion results suggest a potential scaling at zero external field as $r = \sqrt{(\ln n)/n}$.

Different from the Cheeger constant, which concerns the expansion of a set of vertices, the analogous concept of robustness index [29] focuses on the expansion of a single vertex. Given a graph H, a vertex set $S \subset H$ is k-reachable if there is a vertex $v_i \in S$ satisfying $|N_i \backslash S| \geq k$, where N_i represents the neighborhood of vertex v_i in H. The graph H is called k-robust if for any pair of disjoint non-empty sets $S_1, S_2 \in H$, at least one of them is k-reachable. It is known that 1-robustness is equivalent to 1-connectivity and in general k-robustness is stronger than k-connectivity [29–31].

We have the following threshold results of robustness in modular graphs. The proofs are straightforward and left to the readers.

Corollary 1. *Let $m \geq 2$ and $k \geq 0$ be two constants. Suppose $\omega(n) = o(\ln \ln n)$.*

(i) If $p = \frac{\ln n + k \ln \ln n + \omega(n)}{(m-1)nr^2}$, then

$$\mathbb{P}(G_I \text{ is } (k+1)\text{-robust}) = 1 - O(e^{-\omega(n)})$$

and

$$\mathbb{P}(G \text{ is } (k+1)\text{-robust}) = 1 - O(e^{-\omega(n)}). \tag{33}$$

(ii) If $p = \frac{\ln n + k \ln \ln n - \omega(n)}{(m-1)nr^2}$, then

$$\mathbb{P}(G_I \text{ is } (k+1)\text{-robust}) = O\left(e^{-e^{\frac{\omega(n)}{2}}}\right).$$

Since the robustness property of a graph is monotonic, Corollary 1 indicates that $p = (\ln n + k \ln \ln n)/(m-1)nr^2$ is a sharp threshold for robustness of G_I. The next result shows the robustness of G in the dense regime, extending (33) near the critical point.

Corollary 2. *Let $m \geq 2$. If $\theta \in (0, 1)$,*

$$\mathbb{P}\left(G \text{ is } \lceil \zeta npr^2 \rceil\text{-robust}\right) = 1 - O\left(n^{-\frac{\zeta}{(m-1)\theta}}\right),$$

where θ and ζ are defined in (12) and *(13). If $\theta = 0$, the above probability is $1 - o(1)$.*

Remark 2. When $\theta \in (0, 1)$, using (14) we obtain

$$\mathbb{P}(d_{\min}(G_I) \leq 2n(m-1)pr^2) = 1 - O\left(n^{1-\frac{27}{25\theta}}\right).$$

Therefore,

$$\mathbb{P}(G_I \text{ is } \lceil 2n(m-1)pr^2 \rceil\text{-robust}) \leq \mathbb{P}(d_{\min}(G_I) \geq 2n(m-1)pr^2) = O\left(n^{1-\frac{27}{25\theta}}\right).$$

It is easy to see that

$$\mathbb{P}\left(G_I \text{ is } k\text{-robust with } k \in [\lceil \zeta npr^2 \rceil, \lceil 2n(m-1)pr^2 \rceil]\right) = 1 - O\left(n^{-\frac{\zeta}{(m-1)\theta}}\right).$$

When $\theta = 0$, the above probability is $1 - o(1)$. In other words, the modular graph G_I is $\Theta(npr^2)$-robust w.h.p. when $\theta \in [0, 1)$.

2.2. Laplacian eigenvalues of modular graphs

In this section, we examine the spectral gap of the Laplacian matrix of modular graph G. The second smallest eigenvalue of the Laplacian, denoted by $\lambda(G)$, is often referred to as the algebraic connectivity [28]. The Cheeger inequality (1) establishes the relationship between algebraic connectivity and Cheeger's constant.

Theorem 4. *Let $m \geq 2$. If $\theta \in (0, 1)$,*

$$\mathbb{P}\Big(\frac{\zeta^2 npr^2}{4(m-1)} \leq \lambda(G) \leq 2(m-1)(n+\sqrt{n})pr^2\Big) = 1 - O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big), \tag{34}$$

where θ and ζ are defined in (12) *and* (13). *If $\theta = 0$, the above probability is $1 - o(1)$.*

Proof. Assume $\theta \in (0, 1)$. By the right-hand side of (1) and (29) we have

$$\begin{aligned}\mathbb{P}(\lambda(G) \leq 2(m-1)(n+\sqrt{n})pr^2) \geq & \mathbb{P}\Big(h(G) \leq (m-1)(n+\sqrt{n})pr^2\Big) \\ =& 1 - O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big).\end{aligned} \tag{35}$$

It follows from (14) and (15) that

$$\mathbb{P}(d_{\max}(G_I) \leq 2n(m-1)pr^2 \text{ and } h(G_I) \geq \zeta npr^2) = 1 - O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big). \tag{36}$$

Applying (36) and the left-hand side of (1), we derive

$$\mathbb{P}\Big(\lambda(G_I) \geq \frac{\zeta^2 npr^2}{4(m-1)}\Big) = 1 - O\Big(n^{-\frac{\zeta}{(m-1)\theta}}\Big). \tag{37}$$

Combining (35) and (37) yields the desired result (34). The case of $\theta = 0$ can be shown as above by using the corresponding results in Theorem 2. □

Theorem 4 implies that $\lambda(G) = \Theta(npr^2)$ w.h.p. when $\theta < 1$. On the other hand, when $\theta > 1$, taking $k = 0$ in Theorem 1(ii), we have $\mathbb{P}(d_{\min}(G_I) = 0) = 1 - o(1)$. Accordingly, $\mathbb{P}(\lambda(G_I) = 0) \geq \mathbb{P}(d_{\min}(G_I) = 0) = 1 - o(1)$. The positivity of $\lambda(G)$ in this case depends on the intraconnections of G_i for $i \in [m]$.

Next, we consider the Dirichlet graph Laplacian (also known as grounded Laplacian) matrix of the modular graphs. A Dirichlet Laplacian matrix of a graph H is defined as a principal submatrix of the Laplacian of H [32]. From the perspective of graphs, a Dirichlet graph Laplacian corresponds to the resulting Laplacian from removing a certain number of vertices. If H is connected, it is known that its Dirichlet graph Laplacian matrices are positive definite [32,33]. Regarding our modular graph G, we here consider a special case of removing all vertices in one module, say G_1. Let $\mu(G)$ be the smallest eigenvalue for this Dirichlet Laplacian matrix of G.

Remark 3. In networked multiagent systems, the communication network between agents is often modeled by a modular graph, where each module represents agents performing a different task. In the leader-following consensus control, a module G_1 of vertices (i.e., agents) can be viewed as leaders and all vertices in other modules are followers [34,35]. In this scenario, the smallest eigenvalue $\mu(G)$ quantifies the performance (especially, the convergence rate) of the agent consensus dynamics [36,37].

Theorem 5. *Let $m \geq 2$. If $\theta \in (0, 1)$,*

$$\mathbb{P}\big(\zeta npr^2 \leq \mu(G) \leq 2npr^2\big) = 1 - O\big(n^{-\frac{\zeta}{\theta}}\big), \tag{38}$$

where θ and ζ are defined in (12) *and* (13). *If $\theta = 0$, the above probability is $1 - o(1)$.*

Proof. Assume $\theta \in (0, 1)$ and fix any $j_0 \in [m]\backslash\{1\}$. Write $V_{j_0} = \{v_i\}_{i\in[n]}$ and let $H = G_I[V_1 \cup V_{j_0}]$ be the induced subgraph of G_I over $V_1 \cup V_{j_0}$. Note that H is a random bipartite graph. Denote the degree of v_i in H by d_i for $i \in [n]$.

Taking $m = 2$ in (15) and noting that $h(H) \leq d_{\min}(H) \leq \min_{i\in[n]} d_i$, we obtain

$$\mathbb{P}\Big(\zeta npr^2 \leq \min_{i\in[n]} d_i\Big) \geq \mathbb{P}\big(\zeta npr^2 \leq h(H)\big) = 1 - O\big(n^{-\frac{\zeta}{\theta}}\big). \tag{39}$$

By choosing $\varepsilon = 1/2$ and $m = 2$ in (14), we obtain

$$\mathbb{P}\Big(\max_{i\in[n]} d_i \leq 2npr^2\Big) \geq \mathbb{P}\big(d_{\max}(H) \leq 2npr^2\big) = 1 - O\big(n^{1-\frac{27}{25\theta}}\big). \tag{40}$$

Since (39) and (40) hold for any $j_0 \in [m]\backslash\{1\}$, we slightly abuse the notation by letting d_i represent the number of neighbors within V_1 in the graph G for any vertex $i \in V\backslash V_1$, and we have

$$\mathbb{P}\Big(\zeta npr^2 \leq \min_{i\in V\backslash V_1} d_i \leq \max_{i\in V\backslash V_1} d_i \leq 2npr^2\Big) = 1 - O\big(n^{-\frac{\zeta}{\theta}}\big), \tag{41}$$

where we have noted $n^{1-\frac{27}{25\theta}} = o\big(n^{-\frac{\zeta}{\theta}}\big)$. In the rest of the proof, we will stick to this generalized connotation of d_i.

Let $0_{n(m-1)} \in \mathbb{R}^{n(m-1)}$ be the vector with all entries being zero. Let $1_S \in \mathbb{R}^{n(m-1)}$ be the indicator vector for a set $S \subseteq V\backslash V_1$, where the kth entry is 1 if $k \in S$ and 0 if $k \notin S$. Denote by $L(G) \in \mathbb{R}^{n(m-1)\times n(m-1)}$ the Dirichlet graph Laplacian matrix under consideration. Therefore, by using the Rayleigh–Ritz theorem, we have

$$\mu(G) = \min_{0_{n(m-1)} \neq x \in \mathbb{R}^{n(m-1)}} \frac{x^{\mathsf{T}} L(G) x}{x^{\mathsf{T}} x}, \tag{42}$$

where T means the matrix transpose. For any non-empty set $S \subseteq V\backslash V_1$, setting $x = |S|^{-1/2} 1_S$ we have

$$\frac{x^{\mathsf{T}} L(G) x}{x^{\mathsf{T}} x} = \frac{|\partial S|}{|S|} = \frac{\sum_{i\in S} d_i}{|S|} \le \max_{i\in S} d_i \le \max_{i\in V\backslash V_1} d_i. \tag{43}$$

Combining (42) and (43), we have

$$\mu(G) \le \max_{i\in V\backslash V_1} d_i. \tag{44}$$

On the other hand, let $x = (x_1, x_2, \ldots, x_{n(m-1)})^{\mathsf{T}} \in \mathbb{R}^{n(m-1)}$ be a non-negative eigenvector corresponding to $\mu(G)$, namely, $\mu(G)x = L(G)x$. The existence of x is guaranteed by the Perron–Frobenius theorem. Hence, $\mu(G) 1^{\mathsf{T}}_{V\backslash V_1} x = 1^{\mathsf{T}}_{V\backslash V_1} L(G) x$. This means

$$\mu(G) 1^{\mathsf{T}}_{V\backslash V_1} x = \sum_{i=1}^{n(m-1)} d_i x_i \ge 1^{\mathsf{T}}_{V\backslash V_1} x \cdot \min_{i\in V\backslash V_1} d_i,$$

which yields $\mu(G) \ge \min_{i\in V\backslash V_1} d_i$. Combining this with (41) and (44), we obtain (38). The case of $\theta = 0$ can be shown similarly by using the corresponding results in Theorem 2. □

Theorem 5 implies that $\mu(G) = \Theta(npr^2)$ w.h.p. when $\theta < 1$.

3. Conclusions

An attractive feature of the modular graph model considered here is that our results do not depend on the intra-connections of G in each module. This allows flexibility as the Cheeger constant, robustness, as well as the (Dirichlet) Laplacian spectra have diverse applications in complex networks and physical systems. That said, the bounds are tend to be conservative as one would expect.

For the smallest eigenvalue of the Dirichlet Laplacian, we have the following conjecture.

Conjecture 1. *Let $m \ge 2$ and S be the removed vertex set. If $\theta < 1$ and $V\backslash S = \Omega(n)$, then the smallest eigenvalue of the Dirichlet Laplacian of G is $\Theta(npr^2)$ w.h.p.*

Data availability

No data was used for the research described in the article.

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