Magnetospheric Multiscale measurements of turbulent electric fields in earth's magnetosheath: How do plasma conditions influence the balance of terms in generalized Ohm's law?

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Harry C. Lewis; Julia E. Stawarz; Luca Franci; Lorenzo Matteini; Kristopher Klein; Chadi S. Salem; James L. Burch; Robert E. Ergun; Barbara L. Giles; Christopher T. Russell; Per-Arne Lindqvist

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Magnetospheric Multiscale measurements of turbulent electric fields in earth’s magnetosheath: How do plasma conditions influence the balance of terms in generalized Ohm’s law?

ABSTRACT

Turbulence is ubiquitous within space plasmas, where it is associated with numerous nonlinear interactions. Magnetospheric Multiscale (MMS) provides the unique opportunity to decompose the electric field (E) dynamics into contributions from different linear and nonlinear processes via direct measurements of the terms in generalized Ohm’s law. Using high-resolution multipoint measurements, we compute the magnetohydrodynamic (\(E_{\text{MHD}}\)), Hall (\(E_{\text{Hall}}\)), electron pressure (\(E_{p}\)), and electron inertia (\(E_{\text{inertia}}\)) terms for 60 turbulent magnetosheath intervals, to uncover the varying contributions to the dynamics as a function of scale for different plasma conditions. We identify key spectral characteristics of the Ohm’s law terms: the Hall scale, \(k_{\text{Hall}}\), where \(E_{\text{Hall}}\) becomes dominant over \(E_{\text{MHD}}\); the relative amplitude of \(E_{p}\) to \(E_{\text{Hall}}\), which is constant in the sub-ion range; and the relative scaling of the nonlinear and linear components of \(E_{\text{MHD}}\) and of \(E_{\text{Hall}}\), which are independent of scale. We find expressions for the characteristics as a function of plasma conditions. The underlying relationship between turbulent fluctuation amplitudes and ambient plasma conditions is discussed. Depending on the interval, we observe that \(E_{\text{MHD}}\) and \(E_{\text{Hall}}\) can be dominated by either nonlinear or linear dynamics. We find that \(E_{p}\) is dominated by its linear contributions, with a tendency for electron temperature fluctuations to dominate at small scales. The findings are not consistent with existing linear kinetic Alfvén/C19/C19 wave theory for isothermal fluctuations. Our work shows how contributions to turbulent dynamics change in different plasma conditions, which may provide insight into other turbulent plasma environments.

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I. INTRODUCTION

Turbulence is a complex fluid flow phenomenon, which is found in nonconducting fluids and plasma systems throughout the universe. Turbulence arises due to nonlinear interactions, which lead to a cross-scale energy cascade, facilitating the transfer of energy from large scales to small scales. The applications of turbulence to space plasma systems span many different scales throughout the universe, including the geospace environment [e.g., Refs. 1–3], planetary magnetospheres [e.g., Refs. 4 and 5], the solar wind [e.g., Refs. 6–10], interstellar medium [e.g., Refs. 11 and 12], and accretion disks [e.g., Ref. 13]. Turbulence in a magnetized plasma is an especially complicated process due to couplings between fluid flow and electromagnetic forces, from which arise several different characteristic spatial and temporal scales, numerous plasma wave modes, and anisotropic fluctuations driven by a nonzero mean magnetic field. Understanding the full turbulent energy cascade—from macrophysical injection to microphysical dissipation—is of importance to understanding how such turbulent systems evolve.

Earth’s magnetosheath, which consists of a turbulent region of shocked solar wind plasma draped over the dayside magnetosphere, is an important natural laboratory for turbulence studies [e.g., Refs. 1–3 and 14–16]. Turbulent fluctuations in the magnetosheath are partly associated with solar wind turbulence that has been processed by the shock and partly associated with new fluctuations driven by shock dynamics.17–20 Conditions within the magnetosheath are beneficial for in situ measurements, which can build up velocity distributions from many particles as a result of the high density compared to the solar wind. Rapid measurements of the plasma moments mean that small-scale structures can be investigated. At small scales, the single-fluid magnetohydrodynamic (MHD) approximation breaks down, leading to fundamental changes in the behavior of the plasma. The latest generation of spacecraft missions can directly probe turbulence at such scales to small scales. The applications of turbulence to space plasma systems span many different scales throughout the universe, including the geospace environment [e.g., Refs. 11 and 12], and accretion disks [e.g., Ref. 13].

Turbulence generates small-scale structures that are thought to be linked to dissipation. In the absence of collisional viscosity, the turbulent energy cascade in the collisionless plasma reaches kinetic scales, where there are numerous candidates for dissipation mechanisms [e.g., Refs. 20–28]. Such mechanisms include: resonant interactions, such as Landau damping;29 trapping of particles and acceleration by nonlinear electric fields;30,31 stochastic heating of particles in the turbulent electromagnetic field;32 and processes occurring at current sheets, such as magnetic reconnection.33 Turbulence is the primary mechanism by which energy from the largest fluctuation scales can be transferred to kinetic scales, at which it is dissipated into fluid flow and particle energies via various energy conversion channels [e.g., Refs. 25–28]. However, the nature of the exact mechanisms responsible for energy dissipation in collisionless plasmas is still an open question [see Ref. 33].

Electric fields (E) play an important role in shaping the dynamics of the magnetic fields (B) within a plasma. Magnetic fields are unable to do work directly on the particles, so many pathways to dissipation, as well as processes for exchanging energy between the bulk flow and the magnetic field, involve a coupling of the electric field to the particles through the current density (j) via a non-zero j ⋅ E.27,29,34 In a plasma, a number of different physical effects contribute to generating E, which are encapsulated in terms of generalized Ohm’s law [e.g., Ref. 35]
transitions to $E_{\text{Hall}}$ around the ion scales. The relative contribution of $E_p$ compared to $E_{\text{Hall}}$ was approximately constant across scales, and the spatial part of $E_{\text{magn}}$ was small across the range of scales currently accessible by spacecraft measurements. Stawarz et al. also investigated the ratio of nonlinear to linear components of $E_{\text{MHD}}$ and $E_{\text{Hall}}$, discovering that the former becomes more nonlinear at small scales due to increasing alignment between $B$ and $u$ fluctuations as a result of the Hall effect modifying the MHD magnetic field and that the latter remains constant across all scales.

In this work, we aim to examine how the characteristics of the electric field spectrum change when the ambient plasma conditions are varied, by analyzing a survey of turbulent magnetosheath intervals observed by the Magnetospheric Multiscale mission (MMS). This paper is organized as follows: Sec. II outlines how we use spacecraft data to compute the terms in Ohm’s law as a function of plasma conditions. Section II A examines the variability of the spatial scale at which the dominant $E_{\text{MHD}}$ fluctuations give way to $E_{\text{Hall}}$. Section II B examines how plasma conditions impact the relative amplitude of $E_p$ compared to $E_{\text{Hall}}$. Section II C investigates the ratio of nonlinear and linear contributions for $E_{\text{MHD}}$ and $E_{\text{Hall}}$. Section II D examines the ratio of nonlinear and linear components of $E_p$. Section II E examines the underlying dependences on ambient plasma conditions.

II. DATA

With the launch of MMS in 2015, it became possible to directly measure generalized Ohm’s law down to sub-ion scales. The earliest works to do this were primarily focused on structures associated with magnetic reconnection, with studies taking place on laboratory plasmas and data from the Cluster mission. MMS enables the evaluation of Ohm’s law via high instrumental resolution, combined with tetrahedral multipoint measurements, which allow the calculation of spatial gradients to first order using the finite-difference “barycentric estimator” technique. Due to downlink limitations, the highest-resolution “burst” mode measurements from MMS are only available for certain time intervals. Field measurements are provided by the FIELDS suite, with magnetic field data from the fluxgate magnetometer (FGM) providing $B$ measurements at a rate of 128 vectors/s, and electric field data from the electric field double probes (EDP), which measures the observed electric field $E_{\text{obs}}$ at 8192 vectors/s. Velocity distributions and corresponding particle moments are obtained from fast plasma investigation (FPI) suite. We use the electron moments $T_e$, $P_e$, $v_e$, and $n_e$ which have a 30 ms cadence, and ion moments $T_i$, $v_i$, which have a 150 ms cadence. We use spin tone data products to de-spin $v_e$ and $v_i$ as well as counting statistics to generate particle moment uncertainty estimations, which we use to flag intervals that are reaching the particle noise floor.

We examine the database of magnetosheath turbulence intervals compiled by Stawarz et al. The database contains 60 intervals of continuous burst mode observations in the magnetosheath that are at least several minutes in length and have no large-scale inhomogeneities, and in which Taylor’s hypothesis is valid. (The bulk flow speed $u_0$ is sufficiently fast to neglect dynamic changes in the plasma.) We utilize these intervals to study the relative importance of the terms in generalized Ohm’s law for a range of steady plasma conditions using power spectra in wavenumber space. The intervals fill a wide range of parameter space, covering around 2.5 orders of magnitude in $\beta_\rho$ and 2.7 in $\beta_\omega$. The intervals feature strong variability in plasma variables, such as $B$, $u$, and $E$, which is characteristic of the turbulent magnetosheath. Contributions to the dynamics come from an ensemble of structures at different scale-sizes, as can be seen for an example interval in Figs. 1(a)–1(c). To examine how the fluctuations behave as a function of scale, we take the power spectrum, which is an important measure for turbulence that breaks down the intensity of fluctuations corresponding to different wavenumbers. In turbulence theory, the slope of power spectra in different scale ranges is linked to the energy cascade rate, which itself is linked to the physical processes driving the turbulent decay of fluctuations. We obtain power spectra using Welch’s average periodogram. To estimate a wavenumber spectrum, we invoke Taylor’s hypothesis to express the frequency spectra obtained from time series data in terms of wavenumber using the relation $k = 2\pi u_0/\omega$, where $u_0$ is the average bulk speed of the plasma, obtained for each interval by taking the magnitude of the mean ion bulk velocity within that interval. Hereafter, we denote the wave number power spectrum of an arbitrary time-evolving quantity $q$ obtained by this method as $PSD_{q}\left(k\right) = \frac{\langle \left| q - \langle q \rangle \right|^2 \rangle}{\langle q^2 \rangle}$.

Throughout this study, the characteristic scales associated with species $s$ are the gyroradius, $r_s = \sqrt{2m_s k_B T_e} / eB_0$, and the inertial length, $l_s = \sqrt{m_s / \rho_s}$, where $k_B$ is the Boltzmann constant, $T_e$ is the scalar temperature for species $s$, $B_0 = |B_0|$ is the guide field strength, $\rho_s$ is the vacuum permeability, and $q_s = (q_e - q_i)$ denotes the average value of a generic quantity $q$. At this stage, it is helpful to define two additional characteristic scales: the magnetic correlation length (the largest scale of turbulent fluctuations of the magnetic field in a system) $\lambda_s$, for which we use the values computed by Stawarz et al. and the average spacecraft separation $R_{sc} = \langle |R_i - R_j| \rangle$ for position $R$ and spacecraft index $i \neq j \in \{1, 2, 3, 4\}$.

For each measurement interval, we compute $E_{\text{MHD}}$, $E_{\text{Hall}}$, $E_p$, and $E_{\text{inertia}}$ following the method outlined within Stawarz et al. To remove the $E$ component coming from the advection of magnetic structure in the background flow, we subtract $u_0 \times B$ from $E_{\text{obs}}$ (defining the result as the electric field in the plasma frame, $E_p = E_{\text{obs}} + u_0 \times B$) and $E_{\text{MHD}}$. Figure 1(d) shows power spectra typical of $B$, $u$, $E$, and Fig. 1(e) shows power spectra of the generalized Ohm’s law terms for the example interval in Figs. 1(a)–1(c). $E_{\text{MHD}}$ and $E_{\text{Hall}}$ can be measured independently by all four MMS spacecraft, whereas $E_p$ and $E_{\text{inertia}}$ contain gradient terms computed using the barycentric estimator method, which is only evaluable at the barycenter of the MMS tetrahedron. To obtain a single set of variables that represent measurements from all four spacecraft, averaging is required. Variables that are computed individually and then averaged after taking the power spectrum are referred to as “single spacecraft” (1SC) variables, whereas those which are averaged over all four spacecraft before computing derived quantities are known as “barycenter” variables, denoted by $\text{bary}$. To compute $E_{\text{bary}} = -u \times B$, the single-fluid velocity is calculated from the data as $u = (m v_e + m v_i) / (m + m_i)$. At observable scales, $u$ is dominated by $v_e$, so we downsample $v_i$ and $B$ onto the ion sampling timescale. Barycenter averaging takes place after computing the power spectrum such that $PSD_{E_{\text{bary}}} = \langle PSD_{E_{\text{MHD}}(1SC)} \rangle_{\text{bary}}$. For $E_{\text{Hall}} = -j \times B$, the current density $j$ is computed as $j = ne(v_e - v_i)\big|_{bary}$, where $v_i$ is interpolated onto the electron timescale, since $j$ is dominated by $v_e$ at small scales. In addition, we verify the current using the curlometer method, $\mu_0 j = \nabla \times B$. When computing $E_{\text{inertia}}$, we cannot compute the $\partial / \partial t$ term, since
any temporal variations have been equated to spatial ones as per Taylor’s hypothesis.

In this study, we investigate the relative behavior of \( \mathbf{E}_{\text{MHD}} \), \( \mathbf{E}_{\text{Hall}} \), and \( \mathbf{E}_p \) by comparing their spectral behavior. When comparing \( \mathbf{E}_{\text{Hall}} \) to \( \mathbf{E}_{\text{MHD}} \), the former term is averaged onto the measurement cadence of the latter (the ion timescale), and then the average of the ISC power spectra is taken such that \( \langle \mathbf{E}_{\text{Hall}}/\mathbf{E}_{\text{MHD}} \rangle^2 = \langle \text{PSD} \rangle_{\text{Hall,ISC}}/\langle \text{PSD} \rangle_{\text{MHD,ISC}} \rangle_{\text{bary}} \). Conversely, when comparing \( \mathbf{E}_p \) to \( \mathbf{E}_{\text{Hall}} \), \( \mathbf{E}_{\text{Hall}} \) stays on its original sampling cadence and the power spectrum of the barycenter average is used, such that \( \langle \mathbf{E}_p/\mathbf{E}_{\text{Hall}} \rangle^2 = \text{PSD} \rangle_{\text{Hall,ISC}}/\langle \text{PSD} \rangle_{\mathbf{E}_p} \). Consequently, \( \mathbf{E}_p/\mathbf{E}_{\text{Hall}} \) cannot be evaluated at length scales smaller than \( R_{\text{sc}} \).

III. DIMENSIONAL ANALYSIS

Equation (1) couples the evolution of \( \mathbf{E} \) to \( \mathbf{B} \) and the particle moments. Each term describes the contribution to the electric field from a different physical effect, so the interplay between terms determines the overall behavior of the total electric field. When considering a turbulent environment, it is useful to split quantities into a mean and fluctuating component, such that \( q = q_0 + \delta q \), where \( q_0 = \langle q \rangle \) is the average value and \( \delta q = q - \langle q \rangle \) is the deviation from the mean.

Considering \( \mathbf{E}_{\text{MHD}} \), \( \mathbf{E}_{\text{Hall}} \), and \( \mathbf{E}_p \), which are typically the three main components of generalized Ohm’s law at observable scales, it is possible to identify quantifiable features of the spectra, which encode the relationship between different terms. The MHD and Hall dynamics can be characterized by the length scale at which the Hall term becomes dominant over the MHD term, hereafter referred to as the Hall scale \( (k_{\text{Hall}}) \). By taking the ratio \( \mathbf{E}_{\text{Hall}}/\mathbf{E}_{\text{MHD}} \) and neglecting angles between vectors, it is possible to obtain an expression for the scale-dependent relationship between these two terms, given by

\[
\frac{\mathbf{E}_{\text{Hall}}}{\mathbf{E}_{\text{MHD}}} = \left| \frac{1}{n_e} \mathbf{j} \times \mathbf{B} \right| \approx k_{\text{Hall}} \frac{\delta b_A}{\delta u} \tag{3}
\]

where \( k \) is a wavenumber coming from \( \mathbf{j} = \nabla \times \mathbf{B}/\mu_0 \) and the magnetic fluctuation in Alfvén units \( \delta b_A = \delta B/\sqrt{\mu_0 \rho n} \) (where \( n = n_e + \delta n \) is the total density). Defining \( k_{\text{Hall}} \) as the value of \( k \) at which \( \frac{\mathbf{E}_{\text{MHD}}}{\mathbf{E}_{\text{Hall}}} = 1 \) gives

\[
k_{\text{Hall}} = \frac{\delta u}{d_i \delta b_A}. \tag{4}
\]

Similarly, the electron pressure and Hall dynamics, which typically follow similar power-laws, except with higher spectral power in \( \mathbf{E}_{\text{Hall}} \), can be quantified by the ratio of fluctuation power between the two terms

\[
\frac{\text{PSD} \mathbf{E}_{\text{Hall}}}{\text{PSD} \mathbf{E}_{\text{Hall}}} \approx k_{\text{Hall}} \frac{\delta b_A}{\delta u} \tag{5}
\]
In the case of "combined" linear term, and the nonlinear contribution \( \delta \) given, respectively, by

\[
\frac{E_{P}}{E_{Hall}} \approx \frac{1}{ne} \nabla \cdot \mathbf{P}_{e} \sim \frac{\beta_{e} \delta P_{e}}{P_{e,0}} B_{0},
\]

where the electron plasma beta \( \beta_{e} \approx 2 \mu_{B} n_{e} q_{e} B_{0}^{2} \), and \( \langle |B| \rangle \) is the average magnitude of the magnetic field, which becomes \( B_{0} \) in the case of small fluctuation amplitudes.

Each of these terms also contains a combination of linear and nonlinear contributions. Since nonlinear variables are associated with scale-transfer terms related to turbulence, whereas linear components in isolation produce wave-like behavior, measuring how nonlinear a term is at a given scale is a way of understanding which processes are driving the dynamics of that term. Turbulence theories often involve assumptions about the nature of the nonlinear fluctuations, typically related to the rate of nonlinear interactions, \( \tau_{nl} \) as compared to linear wave periods, \( \tau_{d} \). The Iroshnikov–Kraichnan model of turbulence is a dimensional analysis description of counterpropagating Alfvén waves for weak turbulence, where the nonlinear timescales are sufficiently slow that it takes many interactions to significantly alter the dynamics \( (\tau_{nl} \ll \tau_{d}) \). Critical balance takes into account scale-dependent anisotropy, requiring that \( \tau_{nl} \sim \tau_{d} \) at each scale, in a statistical sense. Galtier et al. derived an exact theory of incompressible MHD turbulence where the nonlinear interactions are negligible. The theory of dynamic alignment accounts for the reduction of the nonlinear interaction at small scales due to the increasing alignment between velocity and magnetic fluctuations. More recent theories, for example, refined critical balance, also have a dependence on the relative importance of nonlinear and linear interactions.

We split each Ohm’s law term into contributions consisting of background (non-fluctuating), linear (one fluctuating variable) and nonlinear (multiple fluctuating variables) elements. Estimating the ratio of nonlinear components to linear components for the MHD and Hall terms gives

\[
\begin{align*}
&\left| -\frac{\delta \mathbf{u} \times \delta \mathbf{B}}{\delta \mathbf{B}/ne} \right| \sim B_{0}, \\
&\frac{\delta j \times \delta \mathbf{B}/ne}{\delta j \times \mathbf{B}_{0}/ne} \sim B_{0}.
\end{align*}
\]

In the case of \( \mathbf{E}_{P} \), two linear terms are present: one associated with density fluctuations \( \delta n \) and the other associated with temperature fluctuations \( \delta T_{e} \). The electric fields associated with \( \delta n \), \( \delta T_{e} \), the "combined" linear term, and the nonlinear contribution \( \delta n \delta T_{e} \) are given, respectively, by

\[
\begin{align*}
\mathbf{E}_{\delta n} &= -\mathbf{T}_{e,0} \nabla \cdot \delta n/ne, \\
\mathbf{E}_{\delta T_{e}} &= -n_{0} \nabla \cdot \mathbf{T}_{e}/ne, \\
\mathbf{E}_{\delta T_{e}+\delta n} &= -\nabla \cdot (n_{0} \mathbf{T}_{e} + \delta n \mathbf{T}_{e,0})/ne, \\
\mathbf{E}_{\delta n \delta T_{e}} &= -\nabla \cdot (\delta n \mathbf{T}_{e})/ne.
\end{align*}
\]

Estimating the ratio of nonlinear terms to individual linear components, we obtain the expressions

\[
\frac{\mathbf{E}_{\delta n \delta T_{e}}}{\mathbf{E}_{\delta n}} \sim \frac{\delta n}{n_{0}},
\]

where we have assumed that the divergence of the nonlinear component is the same as that of the linear components. From this point onward, we denote the ratio of nonlinear and linear components of an electric field term as \( R_{nl/lin} = \frac{E_{nonlinear}}{E_{linear}} \).

There are a number of approximations which go into the dimensional analysis expressions used in this section. Scale dependence of the alignment between different vectors is ignored, in addition to geometric differences in the scale dependence arising from divergences and curls. We also assume that fluctuating quantities can be characterized by a single number quantity. One could envisage various ways to characterize a scale-dependent fluctuation: the root mean square \( \delta q_{rms} = \sqrt{\langle \delta q^{2} \rangle} \) is equivalent to the integral of the power spectrum over all wavenumbers (for a turbulent environment, this quantity is typically weighted toward the large scales for certain quantities); or the integral across some range \( \Delta k = [k_{1} - \delta k, k + \delta k] \), which gives the power associated with the wavenumber band \( \Delta k \). We encounter the quotient of two fluctuating quantities, \( \delta q_{1}(k)/\delta q_{2}(k) \) especially if the ratio is not decaying with \( k \)—this quantity can equivalently be integrated or averaged over some \( \Delta k \). For brevity, we denote \( \delta q_{1}(k)/\delta q_{2}(k) \) as \( \langle \delta q_{1}/\delta q_{2} \rangle \) from this point onward.

IV. RESULTS

The dimensional analysis expressions appeal to characteristic length scales and fluctuation amplitudes, whereas we are considering how these expressions manifest themselves over the ensemble of structures that make up a turbulent plasma. We cannot a priori use the dimensional expressions contained within Eqs. (3)–(14) because there are multiple ways that one might reasonably use to quantify the characteristic values for any given interval. In this section, we directly compute the terms in generalized Ohm’s law for the 60 MMS intervals of magnetosheath data discussed in Sec. II. Using the dimensional analysis discussed in Sec. III as a tool to contextualize the observations, we examine how the relative contributions of the different terms vary across the intervals and what that implies about the nature of the turbulent dynamics.

A. The Hall scale

The wavenumber \( k_{H} \) corresponds to the largest wavenumber at which the turbulent electric field is dominated by fluctuations in the MHD term. Figure 2(a) shows the ratio of \( E_{H} \) to \( E_{H_{MID}} \) as a function of scale for an example interval, with \( k_{H} \) defined as the wavenumber representing the intersection of this curve with \( E_{H} / E_{H_{MID}} = 1 \). The dimensional analysis expression in Eq. (4) expresses this scale as a function of 1 and \( \delta u / \delta B_{x} \). At least for this example interval, \( k_{H} \) falls between \( k_{d1} = 1 \) and \( k_{p1} = 1 \), suggesting that \( \delta u / \delta B_{x} \) is playing a nontrivial role in setting the location of \( k_{H} \) in this interval.
FIG. 2. (a) The ratio of the spectra of $E_{\text{Hall},0.13c}$ to $E_{\text{MHD},0.13c}$ for interval “110” (2016-12-08/10:16:52–10:21:02), including the best-fit line derived from a smoothed spectrum (not shown). (b) The ratio of 1 SC spectra of $\delta u / \delta b_A$ for the same interval. Measured $k_{\text{Hall}}$ is denoted in panels (a) and (b) by the green dash-dotted vertical line. (c) The trend between measured $k_{\text{Hall}}$ and the dimensional expression from Eq. (4), using $\langle \delta u / \delta b_A \rangle_{\text{rms}}$, for all intervals. The diagonal dashed line corresponds to agreement, and the example interval “110” is surrounded by a hallow black circle to identify it.

however, even in the context of incompressible MHD turbulence and in the solar wind, an excess of magnetic energy is an observed feature\(^1\). At smaller scales, starting around $k_{\text{i0}} = 1$ and $k_{\text{Hall}}$, the magnetic energy becomes increasingly dominant, consistent with the behavior of kinetic scale dynamics and the reduced importance of turbulence.

In Fig. 2(c), we examine the location of $k_{\text{Hall}}$ for all 60 of the intervals examined in this study. Comparing the measured values of $k_{\text{Hall}}$ to the dimensional analysis expression from Eq. (4), we find that the Alfvénic assumption ($\langle \delta u / \delta b_A \rangle = 1$) and using $\delta u_{\text{rms}} / \delta b_{\text{rms}}$, which is weighted toward the large-scale fluctuations, systematically overestimate the value of $k_{\text{Hall}}$. We further examine $\delta u / \delta b_A$ evaluated at a variety of relevant scales based on the square root of the ratio of power spectral densities, including at $k_{\text{d0}} = 1$ and $k_{\text{rms}} = 1$, as well as evaluating $k_{\text{d1}} \delta u / \delta b_A = 1$ as a scale-dependent expression, defining $k_{\text{d1}} = 1 / d_i \times \langle \delta u / \delta b_A \rangle_{\text{rms}}$. Evaluating $\delta u / \delta b_A$ at $k_{\text{d1}} = 1$ appears to estimate the measured value of $k_{\text{Hall}}$ reasonably well, as shown in Fig. 2(c). This agreement suggests that scales adjacent to $d_i$ are important in determining where the transition to Hall term dominance of the electric field occurs, which aligns with expectations as Hall dynamics arise from a decoupling of the ion fluid from the plasma dynamics. A similar trend is obtained for $k_{\text{dlin}}$, albeit with a slightly larger scatter.

Further refinements are possible, such as by breaking up the fluctuations $\delta u / \delta b_A$ into components perpendicular and parallel to the mean field $B_0$. We do not find that any combination of components produces significantly better agreement, indicating that there are no preferred fluctuation directions, which are solely controlling the MHD–Hall dynamics in these turbulent intervals. Indeed, many of the intervals have large fluctuation amplitudes relative to the background magnetic field and seemingly near isotropic fluctuations.\(^3\) Based on the observed values of $\delta u / \delta b_A$ evaluated at $k_{\text{d1}} = 1$, as well as those feeding into $k_{\text{dlin}}$, $\delta u / \delta b_A$ is empirically consistent with 0.5 for the intervals examined in this study. This indicates that, for this selection of magnetosheath plasma data, $k_{\text{Hall}} d_i \sim 0.5$ and, thus, the spatial scale associated with the Hall scale is around $2d_i$. Using this empirical result in the dimensional analysis is also fairly effective in predicting the measured $k_{\text{Hall}}$ across all intervals.

The scatter in Fig. 2(c) could be due to several sources, such as the approximations used to obtain the dimensional analysis expression for $k_{\text{Hall}}$, inexact measurements of $k_{\text{Hall}}$ due to variability in the curve of the power spectrum, or artificial enhancements in the velocity measurements as they approach the noise floor. One of the principle simplifications of the dimensional analysis expression is that scale-dependent alignment between different vector quantities is neglected. Investigating alignment properties between $\delta u$, $\delta b_A$, and $\delta b$, and subsequently including characteristic values of these into the dimensional analysis expressions, yielded no noticeable improvements, implying that they are not directly feeding into the location of $k_{\text{Hall}}$. We estimate the noise level of FPI variables using the methods described in Gershman et al.\(^4\) This involves generating a zero-mean timeseries of white noise, where at each time step, the noise level is a randomly generated value from a normal distribution, which has a standard deviation equal to the FPI error provided by MMS. Where curves are reaching the noise floor, we do not include the range of scales where the spectral curve appears to be contaminated by noise.
B. Relative amplitude of electron pressure and Hall terms

Below $k_{Hall}$, the spectrum of $E$ is dominated by a combination of $E_{Hall}$ and $E_{pe}$. The ratio $E_{pe}/E_{Hall}$ represents the proportion of the fluctuation energy arising from the non-ideal term $E_{pe}$ compared to $E_{Hall}$. This ratio is shown as a function of scale in Fig. 3(a) for an example interval, in which it can be seen that the value is close to constant across a wide range of $k_{np} > 1$. In some cases, there may be a slight increasing trend. Appealing to the dimensional expression in Eq. (5), the scale-dependent part of $E_{pe}/E_{Hall}$ is related to $(\delta P_{e}/\bar{P}_{e})/(\delta B/\bar{B}_{0})$, which is plotted as a function of scale in Fig. 3(b) and is also found to be approximately constant as a function of scale in the sub-ion range. This ratio has been rescaled by the constant factors $\beta_{e}/2$ and $B_{0}/(|B|)$. This is in line with the expectation from Eq. (5) that there should be no overall scale-dependence of the ratio, i.e., that magnetic and density fluctuations have the same power-law as per theoretical expectations in the nearly incompressible limit.\cite{45,82}

Comparable behavior has been found in the solar wind\cite{83} and hybrid kinetic simulations, where similar power laws have been reported for $B$ and $n$ at sub-ion scales, which is consistent with similar scaling between $B$ and $P_{e}$ if the fluctuations are isothermal. However, the presence of temperature fluctuations may lead to a more subtle relationship between the pressure and magnetic field.

The factor $B_{0}/(|B|)$, which is accounting for variations in the magnitude of $B$, is a necessary factor to account for, at least in the magnetosheath. This term encodes how much the fluctuations perturb the magnitude of the magnetic field, ranging from around $10^{-1}$ to 1 in our dataset. Beyond the inclusion of the magnetic field strength fluctuation, another factor to consider is that there are different ways to quantify the electron pressure fluctuation $\delta P_{e}$. For example, one could consider fluctuations in the isotropized pressure, $p_{e} = \text{Tr}[\bar{P}_{e}]/3$. However, this does not take into account individual or unbalanced fluctuations in the diagonal terms that may be giving rise to electron temperature anisotropy, as such it only accounts for variations in the overall amount of pressure, as opposed to variations in the amount of anisotropy. We compute the power spectrum of each of the diagonal terms and sum them to give the scale-dependent fluctuation $\delta P_{e}$. In principle, the off diagonal components could also contribute to the dynamics, but we find that including the off diagonal parts of $P_{e}$ into $\delta P_{e}$ does not improve agreement, implying that these do not significantly contribute to $E_{pe}/E_{Hall}$.

To quantify $E_{pe}/E_{Hall}$, we choose the average value of the smoothed measured ratio in log-space between $k_{np} = 1$ and $kR_{sc} = 1$. Figure 3(c) shows the agreement between the measured $E_{pe}/E_{Hall}$ and the dimensional analysis expression in Eq. (5) (obtained from its spectrum in the same manner as the measured value) across the range of magnetosheath intervals investigated by this study. The general trend shows good agreement when considering the sub-ion scales. This shows that it is necessary to include $B_{0}/(|B|)$ into the expression for $E_{pe}/E_{Hall}$ in the magnetosheath and that one needs to take into account the fluctuations in the diagonal components of the electron pressure tensor, in order to include the anisotropy fluctuations. We performed a simple analysis of the estimated uncertainties for this plot, which we found to be of the order of $\sim 0.1$ for both axes. This implies that the slight uptick observed in the spectra of $E_{pe}/E_{Hall}$ and the dimensional expression [Figs. 3(a) and 3(b), respectively]. Additionally, the three left most points in the scatterplot seem to be

FIG. 3. (a) The ratio of the power spectra of $E_{pe}$ and $E_{Hall}$ in interval “110” (2016-12-08/10:16:52–10:21:02). The scales smaller than $R_{sc}$, where the estimation of $E_{pe}$ is expected to be unreliable, have been faded. (b) The square root of the ratio between the electron pressure spectrum and the magnetic spectrum rescaled in accordance with Eq. (5). The power spectra of the normalized variables, $\delta P_{e}/\bar{P}_{e}$ (using all three diagonal components of the electron pressure fluctuation) and $\delta B/\bar{B}_{0}$, are computed for each spacecraft individually and then averaged. (c) The relationship between measured $E_{pe}/E_{Hall}$ and the value estimated from Eq. (5), both evaluated between $k_{np} = 1$ and $kR_{sc} = 1$. The diagonal dashed line corresponds to agreement, and the example interval “110” is surrounded by a black circle.
systematically enhanced compared to the dimensional analysis estimate, which potentially indicates that the computation of $E_B$ is hitting a noise level.

**C. Nonlinear/linear contributions to $E_{MHD}$ and $E_{Hall}$**

Just as the $E$ dynamics as described by generalized Ohm’s law can be split into different terms corresponding to separate physical effects, so too can each term be decomposed into contributions from nonlinear and linear components. Equations (6)–(14) give expressions for the relationships between the nonlinear and linear components as a function of turbulence conditions and plasma parameters.

Figure 4 shows an example of the ratios of power spectra of nonlinear and linear terms contributing to $E_{MHD}$, $E_{Hall}$ and $E_{Hall} = -v \times B$, where we compute the nonlinear and linear components of $-v \times B$ separately. Above the ion scales (in our case, between $k_i$ and $\rho_i$), the spectrum of every ratio is constant as a function of scale and centered around $\delta B_{rms}/B_0$, indicating that there is a balance between the linear and nonlinear terms in the system. However, interestingly, the ratio between the two terms is set by the strength of $\delta B_{rms}/B_0$ across all scales, which is weighted toward the behavior of the largest-scale fluctuations. At smaller scales, $R_{MHD}^{NL/L}$ diverges from $\delta B_{rms}/B_0$ to become increasingly nonlinear. This occurs due to dealignment of $\delta B$ and $\delta u$ at smaller scales, whereas alignment between $B_0$ and $\delta u$ does not vary with scale, leading to a relative strengthening of the nonlinear term. The dealignment may be occurring due to $E_{Hall}$ causing non-MHD modifications to the magnetic field at scales below $k_{Hall}$. We evaluate $R_{MHD}^{NL/L}$ at length scales between $k_{Hall}$ and $\rho_i$ in the range where the ratio of nonlinear to linear components is unaffected by the Hall effect. $R_{Hall}^{NL/L}$, on the other hand, continues to remain constant as a function of scale, in such a way that the combination of $E_{MHD}$ and $E_{Hall}$, $-v \times B$, also remains approximately constant as a function of scale from the correlation length of the correlation length down to the smallest observable scales. As such, the dominant dynamics in generalized Ohm’s law at any given scale appear to arrange themselves such that they maintain a balance between the linear and nonlinear contributions.

From Sec. III, the relative scaling of nonlinear to linear MHD and Hall terms both reduce to $\delta B/B_0$ within the dimensional analysis treatment. Figure 4(b) shows $R_{MHD}^{NL/L}$ averaged over the scale range $k_{Hall} \leq k < k_{Hall} + 1$ to $k = k_{Hall} \rho_i = 1$ as compared to $\delta B_{rms}/B_0$ for all the intervals, and Fig. 4(c) shows $R_{Hall}^{NL/L}$ averaged over the scale range $k_{Hall} \leq k < k_{Hall} + 1$ compared to $\delta B_{rms}/B_0$ for all intervals. In all cases, the ratios appear to be roughly consistent with the behavior shown in the example interval and clearly scale with the ratio $\delta B_{rms}/B_0$. A scaling with $\delta B/B_0$ is in agreement with Eqs. (6) and (7), where in this case the ratio of large-scale magnetic field fluctuations to average background field, $\delta B_{rms}/B_0$, is determining the relative scaling of nonlinear to linear contributions. Since the constant ratio of nonlinear and linear terms is being set by $\delta B_{rms}/B_0$, there are both intervals that are strongly dominated by the nonlinear dynamics and intervals that are dominated by the linear dynamics even while maintaining a balance between them. The spread in the scatter trends may be associated with how well the estimate of the mean value of the ratio has converged given the variability in the ratio, which has a greater impact on the MHD term because a smaller range of scales means that fewer data points are used to calculate the average.
D. Nonlinear and linear terms contributing to $E_{p_e}$

$R_{NL/L}^{E_{p_e}}$ is a more complex quantity to evaluate because $E_{p_e}$ consists of two linear terms as well as a nonlinear term. When the same dimensional approach as in Sec. IV C—using the rms as the characteristic fluctuation amplitude—is naively applied to $R_{NL/L}^{E_{p_e}}$, the dimensional estimation fails to account for measured values in all 60 intervals. The discrepancy is shown in Fig. 5, in which it can readily be observed that all scatter points are above the agreement line, indicating that the dimensional estimation is underestimating the measured value. It is also interesting to note that every interval has a measured $R_{NL/L}^{E_{p_e}} < 1$, showing that the electron pressure divergence term is dominated by its linear components for every magnetosheath interval that we studied, in contrast to $R_{NL/L}^{P_{\text{Hall}}}$ and $R_{NL/L}^{P_{\text{MHD}}}$ This discrepancy extends to the ratio of nonlinear to individual linear terms (not shown) given by Eqs. (12) and (13), implying that the underestimation is not coming from correlation between the two linear terms in the denominator of $R_{NL/L}^{E_{p_e}}$. Additionally, when the fluctuation amplitudes are evaluated at characteristic scales, such as $k_{R_e} = 1$ and $k_{R_i} = 1$, or when they are averaged between $k_{R_e} = 1$ and $k_{R_i} = 1$, agreement is still not found. As a result, we conclude that there is not a simple dimensional expression to predict the relative scaling of nonlinear to linear parts of $E_{p_e}$ as a function of plasma conditions.

Since we are unable to describe $R_{NL/L}^{E_{p_e}}$ using simple dimensional estimates, we instead break down the contributions to this term and qualitatively compare the behavior of each. The electric field terms that contribute to $E_{p_e}$ are given by Eqs. (8)–(11), the power spectra of which are shown in Fig. 6(a). We find that the ratio $E_{\text{MHD}}/E_{\text{ST}}$ and the ratio $E_{\text{MHD}}/E_{\text{ST}} = 1$, or when they are averaged $k_{R_e} = 1$ and $k_{R_i} = 1$, agreement is still not found. As a result, we conclude that there is not a simple dimensional expression to predict the relative scaling of nonlinear to linear parts of $E_{p_e}$ as a function of plasma conditions.

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linear terms precludes a simple dimensional estimation from being employed on this term. It has been demonstrated that the linear contributions to $E_{p}$ are dominating the term, so any estimate for the relative scaling of nonlinear to linear parts may be sensitive to the relative dynamics of the dominant contributions. It is interesting that the dimensional technique does not work on $E_{nast}/E_{st}$, even though the ratio is constant as a function of scale.

Figure 6(b) shows power spectra of the linear and nonlinear components of $P_{c}$ for an example interval, which corresponds to the variables whose divergences are associated with components of the electron pressure term. The combination of the two linear variables is dominant at all accessible wavenumbers, which is the same for all intervals and in agreement with our observations that $E_{p}$ is dominated by linear components. The trends are broadly the same as those of the different contributions to $E_{p}$, as illustrated in Fig. 6(a). The linear term $n_{0}\delta T_{e}$ becomes progressively larger relative to $\delta n T_{e,0}$ at sub-ion scales. This suggests that the temperature fluctuations are increasingly more important than the density fluctuations at small scales, which indicates that our likewise findings for $E_{nast}$ and $E_{st}$ are not solely arising due to differences between the divergence of each fluctuating variable.

V. DISCUSSION

The relationships demonstrated in Sec. IV convey only part of the complex interdependence of the electrodynamics with the turbulence properties and ambient plasma conditions within the magnetosheath. We have shown that $k_{Halt} \sim 1/d_{i} \times (\delta u/\delta b_{A})$, which means that the length scale where the Hall fluctuations become dominant, $k_{Halt}$, scales with $1/d_{i}$ and the ratio of velocity to magnetic fluctuations. The value of $\delta u/\delta b_{A}$ is a scale-dependent quantity that is influenced by an excess of magnetic energy at MHD scales, before beginning to drop off as velocity decouples when the Hall effect takes over. The fact that large scale $\delta u/\delta b_{A}$ does not control $k_{Halt}$ suggests that the diverging $\delta u$ and $\delta b_{A}$ spectra at sub-ion scales may be playing a role in the dynamics. It is interesting that $k_{Halt}$ is not best described using $(\delta u/\delta b_{A})_{lbreak}$, as this implies that the relative fluctuation amplitude at scales other than those adjacent to $k_{Halt}$ is feeding into the location of the change in dynamics. This nonlocality in spectral space could be explored in future research.

Additionally, there may be an underlying dependence of $\delta u/\delta b_{A}$ itself on the ambient plasma conditions. This means $k_{Halt}$ has a combination of explicit dependence on plasma parameters as well as underlying dependencies coming from $\delta u/\delta b_{A}$. If the fluctuations were Alfvénic, or if there was no dependence of $\delta u/\delta b_{A}$ on plasma conditions, one would expect $k_{Halt} \sim d_{i}^{-1}$. However, we observe that $k_{Halt} \sim d_{i}^{-1.2}$. We then investigated the underlying dependences of $\delta u/\delta b_{A}$ on other parameters, finding the relationship $\delta u/\delta b_{A} \sim (\delta B_{rms}/B_{0})^{-0.21}$ (not shown here). This shows that the fluctuations in velocity and magnetic field become closer to being Alfvénic for intervals with smaller magnetic field fluctuations compared to the background, which is consistent with small amplitude fluctuations being more wave-like. To summarize, in the case of intervals with smaller background fluctuations, $k_{Halt}$ approaches $1/d_{i}$. A future investigation may wish to extend this trend to intervals with very small $\delta B_{rms}/B_{0}$ to see if $k_{Halt}$ becomes smaller than $1/d_{i}$, as well as check whether $\delta B$ contributions from other length scales exhibit stronger control over the behavior.

When checking $\delta B_{rms}/B_{0}$ against other parameters, we uncovered a scaling of $\beta_{I} \sim (\delta B_{rms}/B_{0})^{-0.33}$, which folds into the expression for $k_{Halt}$ as

$$k_{Halt} \sim \beta_{I}^{-0.14} \times (1/d_{i}).$$ (15)

Since $\beta_{I} = \rho_{i}^{2}/d_{i}^{2}$, we find $k_{Halt} \sim \rho_{i}^{-0.28} d_{i}^{-0.72}$, which introduces a dependence on $\rho_{i}$ that was not originally apparent from the dimensional analysis. It would be interesting to identify intervals with a wider range of $\beta_{I} \leq 1$ to test this relationship further.

The (normalized) spatial Hall scale, $l_{Halt}/d_{i} \sim \beta_{I}^{1/4}$, can be compared to an empirical estimate for the B spectral break scale, $l_{breaking} \sim \beta_{I}^{1/3}$.

In our dataset, $\beta_{I}$ ranges from 0.81 to 243.03. Across this range, the curve of $l_{break}/d_{i}$ as a function of $\beta_{I}$ is not similar to that of $l_{Halt}/d_{i}$. Qualitatively, $l_{break}/d_{i}$ drops off faster than $l_{Halt}/d_{i}$, although $l_{break}/d_{i}$ is smaller than $l_{Halt}/d_{i}$ at very small $\beta_{I}$. However, the extent of the relationship between $l_{Halt}$ and $l_{break}$ has not been fully investigated here and may form the basis of a future study. Furthermore, there may be a more complex relationship between $k_{Halt}$ and the break scale.

We have shown that the value of $(E_{p}/E_{Halt})_{sub-ion}$ has a functional dependence on $\beta_{I}/2$, $(\delta P_{e}/P_{e,0})/(\delta B_{rms}/B_{0})_{sub-ion}$, and $B_{0}/\langle B \rangle$. We find that the latter two quantities have direct dependences on $\beta = \beta_{I} + \beta_{E}$, with $(\delta P_{e}/P_{e,0})/(\delta B_{rms}/B_{0})_{sub-ion} \sim \beta^{-0.66}$ [Fig. 7(a)] and with $B_{0}/\langle B \rangle$ featuring two populations with different relationships to $\beta$ [Fig. 7(b)]. Since we have that

$$\frac{(\langle B \rangle)}{B_{0}} = \sqrt{1 + \frac{2\beta_{E}}{B_{0}} + \frac{(\delta B_{rms} / B_{0})^{2}}{B_{0}^{2}}},$$ (17)

we find that there is a population with relatively small $\delta B_{rms}/B_{0}$ in which $B_{0}/\langle B \rangle$ is independent of $\beta$ and $\sim 1$. Another population corresponding to large $\delta B_{rms}/B_{0}$ has $B_{0}/\langle B \rangle \sim \beta^{-0.33}$. Combining the dependencies revealed in Figs. 7(a) and 7(b), two power-law scalings governing the dependence of $E_{p}/E_{Halt}$ with $\beta_{I}$ emerge, as shown in Fig. 7(c): $(\delta P_{e}/P_{e,0})/(\delta B_{rms}/B_{0})_{sub-ion} \times B_{0}/\langle B \rangle \sim \beta^{-0.75}$ for small-fluctuation intervals and $(\delta P_{e}/P_{e,0})/(\delta B_{rms}/B_{0})_{sub-ion} \times B_{0}/\langle B \rangle \sim \beta^{-1.07}$ for large-fluctuation intervals. Folding these relationships into the original expression for $(E_{p}/E_{Halt})_{sub-ion}$ gives

$$(E_{p}/E_{Halt})_{sub-ion} \sim \begin{cases} \beta_{I}^{-0.75}, & \text{small } \delta B_{rms}/B_{0} \\ \beta_{I}^{-1.07}, & \text{large } \delta B_{rms}/B_{0}. \end{cases}$$ (18)

When the observed values of $(E_{p}/E_{Halt})_{sub-ion}$ are plotted against this expression (not shown), good agreement is found.

It would be desirable to extend this research to a much wider dynamic range of $E_{p}/E_{Halt}$ (if this exists) and $\beta_{I}$ in order to further explore the trends of both populations. In particular, regions with low fluctuation amplitudes and high beta to confirm whether the $\beta^{-0.75}$ extends to large $\beta$ alongside the $\beta^{-1.07}$ trend. Additionally, identifying regions with strong fluctuation amplitudes and low beta would be useful as it would show whether the $\beta^{-1.07}$ trend plateaus for small $\beta_{I}$ in accordance with the trend of the other population. Such intervals may be unlikely to occur in the magnetosheath, as when $\delta B_{rms}/B_{0}$ is large this implies small $B_{0}$, which must then be countered by an appropriately small $T_{e,0}$ to generate small $\beta$. However, in our dataset, we
observe that variation in $\beta$ is typically driven by changes in $B_0^2$, which covers $\sim 2.3$ orders of magnitude, whereas $T_e$ only varies over $\sim 1$ order of magnitude. In addition to extending the trends, our findings indicate that it may be difficult to identify magnetosheath intervals where $E_{P_e} \geq E_{Hall}$. This means that, given $\beta = \beta_e + \beta_i \geq \beta_i$ and, in our dataset, $\beta_i/\beta_e > 1$, neither population is expected to produce $E_{P_e}/E_{Hall} > 1$ in the turbulent magnetosheath. Even in the unlikely case that $\beta \approx \beta_i$ in the magnetosheath, for $E_{P_e}/E_{Hall} > 1$, one would need to identify an interval with $\beta_e \sim 10^4$ (for small $\delta B_{rms}/B_0$) or $\beta_i \approx 0$ (for large $\delta B_{rms}/B_0$).

The functional dependence of $(\delta P_{\parallel}/P_{e\theta})/(\delta B/B_0)$ on $\beta$ noted here can be compared to the expectation from linear KAW theory, which is often invoked in theoretical descriptions of kinetic scale plasma turbulence. Imposing the assumption of isothermal $T_e$ typically justifiable in space plasmas [e.g., Ref. 45], the relative amplitude of scalar isotropic electron pressure to magnetic fluctuations is given by

$$\frac{\delta P_{\parallel}/n_0 k_B T_{e,0}}{\delta B/B_0} = \frac{\delta n/|n_0|}{\delta B/B_0} \sim \left( \frac{\beta^2}{2} + \frac{1}{2} \right)^{-1/2}.$$  \hspace{1cm} (19)

We find that this expression underestimates the measured value for the whole dataset (Fig. 7), in agreement with the findings of Stawarz et al.\textsuperscript{37} Equation (19) is a factor of $\sim 3$ smaller than the observed value of $(\delta P_{\parallel}/P_{e\theta})/(\delta B/B_0)$ for small $\beta$, dropping off faster than $(\delta P_{\parallel}/P_{e\theta})/(\delta B/B_0)$ with larger $\beta$, as shown in Fig. 7(a).

There may be several factors that lead to the disagreement between Eq. (19) and the observations. One possibility could be that the dynamics are KAW-like, but the assumptions that go into Eq. (19), such as isothermal and isotropic electrons, are violated. The results of Sec. IV D suggest that the temperature fluctuations are playing a role in the dynamics in the magnetosheath, thus violating the isothermal approximation—an assumption which is commonly made in theoretical descriptions of plasma turbulence.\textsuperscript{45} Furthermore, electron distributions are observed to have anisotropy in the temperature within the magnetosheath ($T_{e,i}/T_{e,0}$ varies between 0.69 and 1.03 for our range of intervals), and taking into account the fluctuations associated with this anisotropy was necessary when comparing $E_{P_e}$ to $E_{Hall}$. Another possibility is that a different wave mode may be relevant, for example, whistler waves. Boldyrev et al.\textsuperscript{38} derive an expression for $(\delta n/|n_0|)^2/(\delta B/B_0)^2$ in the case of whistler waves

$$\left( \frac{\delta n}{n_0} \right)^2 \left( \frac{\delta B}{B_0} \right)^2 = \frac{1}{2} \frac{k^4_{\parallel}}{2 k^2_{\perp}} c^2.$$ \hspace{1cm} (20)

which is expected to be small since the denominator of the expression contains $k^4_{\parallel}$, which is typically large. Finally, it may be that the strong nonlinearity observed in many of the intervals is fundamentally altering the dynamics and that fully nonlinear solutions are needed to account for the observed relationship. In particular, the scaling of $E_{P_e}/E_{Hall}$ as $\sim \beta_e/\beta$ for large $\delta B_{rms}/B_0$ suggests that structures in near pressure balance may be playing a significant role in the dynamics.

VI. CONCLUSIONS

The spectral properties of generalized Ohm’s law terms are evaluated in a statistical sense by use of a large dataset of MMS measurements in the magnetosheath. Characteristics of the relative importance of $R_{Hall}$, $E_{Hall}$, and $E_{P_e}$ are identified, namely, the crossover between MHD and Hall dynamics, $k_{Hall}$, the ratio of electron pressure to Hall contributions, $E_{P_e}/E_{Hall}$, and the ratio of nonlinear to linear contributions from each term. We confirm expectations that $k_{Hall}$ typically occurs around the ion scales and that $(E_{P_e}/E_{Hall})_{\text{MHD}}$ is approximately constant as a function of scale. $R_{Hall}^{\text{MHD}}$ is constant above the ion scales, $R_{Hall}^{\text{MHD}}$ is constant across all accessible scales, and these ratios demonstrate that a scale-by-scale balance between the linear and nonlinear terms is achieved that can be dominated by either the nonlinear or linear terms. Simulations have suggested that the ratio may have a complex anisotropic distribution in $k$-space,\textsuperscript{80,86} which could be explored further in future studies. $E_{P_e}$ is dominated by linear components, typically a mixture of contributions from $\delta n$ and $\delta T_r$ at large scales, than by contributions from $\delta T_e$ at small scales.

We show how $k_{Hall}$, $E_{P_e}/E_{Hall}$, $R_{Hall}^{\text{MHD}}$, and $R_{Hall}^{\text{NL}}$ depend on the properties of the turbulence and ambient plasma conditions. This is in agreement with work by Franci et al.,\textsuperscript{34} which showed that properties of the turbulent cascade are mainly dependent on ambient plasma conditions rather than the type of large-scale driver. $k_{Hall}$ most clearly

![Figure 7](image-url)
depends on $1/d_i$ and $\delta u_0/\delta b_A$ evaluated at the ion inertial length. At sub-ion scales, $E_P/E_{Hall}$ depends on $P_i(\delta P_i/\delta b_A)/(\delta b_i/\delta b_0)$ averaged over the sub-ion scales, with all three of the anisotropic diagonal components of $P_i$ included into $\delta P_i$, and $B_0/(\langle |B| \rangle)$ at scales larger than the ion scales, $R_{Hall}^{NL}$. On the other hand, the $R_{Hall}^{NL}$ is constant and goes as $\delta B_{rms}/B_0$, indicating that the balance between the mean field and large-scale magnetic fluctuations controls the relative nonlinearity to linearity of these terms.

We find that several of the key properties of the turbulent fluctuations, namely, $\delta u_0/\delta b_A$, $\delta P_i/\delta b_i$, and $B_0/(\langle |B| \rangle)$, have underlying dependences on plasma conditions, such that $k_{Hall}$ and $E_P/E_{Hall}$ are expressible solely as empirical functions of ambient plasma parameters. An evaluation of the Hall scale gives $k_{Hall} \sim B_0^{0.14} \times (1/d_i)$. For the relative amplitude of electron pressure to Hall fluctuations, the relationship $(\delta P_e/P_e)/\delta b_i \sim B_0^{-0.08}$ is uncovered. Such a scaling is not consistent with the isothermal linear KAW approximation. Indeed, we find that temperature fluctuations are playing a significant role in generating the electric fields. Additionally, two populations of $B_0/(\langle |B| \rangle)$ are identified, corresponding to small and large $\delta B_{rms}/B_0$, respectively. When combined, the two populations feed into $(E_P/E_{Hall})_{sub-ion}$ corresponding to $\beta_i/\beta_i^{75}$ (small $\delta B_{rms}/B_0$) and $\beta_i/\beta_i^{25}$ (large $\delta B_{rms}/B_0$), respectively.

The results of this work highlight a number of questions that should be explored in future studies. Exploring how the findings of this study compare to different plasma systems, covering an extrapolation beyond the extrema of the intervals used here as well as completely different parameter regimes, would give useful insight into the applicability and limitations of our results. Future works may wish to formally investigate the relationship between the Hall scale and the scale at which the break in the magnetic field spectrum occurs. Another further study may aim to incorporate temperature fluctuations into the KAW treatment of $(\delta P_e/P_e)/\delta b_i$. Furthermore, it would be interesting to investigate the extent to which the different terms directly contribute to energy conversion via $j \cdot E$, especially in the context of the relative amplitude of $E_P$ to $E_{Hall}$, so that the relationship between different terms and dissipation may be uncovered. It would be interesting to explore whether the results in this paper are linked to systematic variations with distance from the bow shock, or more generally with position within the magnetosheath, as a result of recently driven turbulence evolving as it develops. Similarly, it would be interesting to examine whether the results in this paper are linked to systematic variations with upstream solar wind conditions. Finally, simple dimensional analysis expressions did not agree with observations of the relative scaling of linear and nonlinear contributions to $E_P$. It would be interesting to uncover why dimensional analysis methods break down for this term, and not others, which may give insights into the nature of the interplay between the two linear terms which jointly dominate the $E_P$ dynamics.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Harry C. Lewis: Conceptualization (equal); Formal analysis (equal); Investigation (lead); Methodology (equal); Writing – original draft (lead); Writing – review & editing (equal). Christopher Russell: Data curation (equal). Per-Arne Lindqvist: Data curation (equal). Julia E. Stavarz: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (supporting); Methodology (equal); Supervision (lead); Writing – review & editing (equal). Luca Franci: Investigation (supporting); Writing – review & editing (supporting). Lorenzo Matteini: Writing – review & editing (supporting). Kristopher G. Klein: Investigation (supporting); Writing – review & editing (supporting). Chadi Salem: Writing – review & editing (supporting). James Leo Burch: Data curation (equal). Robert E. Ergun: Data curation (equal). Barbara L. Giles: Data curation (equal).

DATA AVAILABILITY

The data used in this study are publicly available through the MMS Science Data Center (https://lasp.colorado.edu/mms/sdc/public/).

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