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Network revenue management game in the railway industry: Stackelberg equilibrium, global optimality, and mechanism design

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ABSTRACT

Many countries have adopted the vertical separation governance structure in the railway freight industry over the past decades. Under this governance structure, an Infrastructure Manager (IM), which might be an independent company or a government agency, sells train itineraries to Freight Operating Companies (FOCs). After purchasing the itineraries, a FOC will have the rights to run trains on the designated paths at the designated times and thus can provide transport service to shippers. In the process, an IM needs to determine a list of prices for their train itineraries; and a FOC needs to determine which train itineraries to purchase to serve uncertain customer demands based on the IM's price list. This study considers the interaction between an IM and a FOC as a network-based Stackelberg game. Our study first formulates a bi-level optimisation model to determine the equilibrium prices that the IM would charge to maximise its own profits unilaterally without collaboration. A method involving gradient and local search has been developed to solve the bi-level model. Secondly, an inverse optimisation model is proposed to determine the prices leading to global optimality. A Fenchel cutting plane-based algorithm is developed to solve the inverse optimisation model. Thirdly, a subsidy contract is designed for the game to coordinate the players' decisions. A two-layer gradient search method is developed to determine the optimal subsidy rate. Numerical cases based on the UK rail freight industry data are provided to validate the models and algorithms.

1. Introduction

To avoid monopolies and increase competition in the rail freight market, thus improving the efficiency and service quality provided by the railway network (Alexandersson & Rigas 2013; Laroche & Guihéry 2013; Nash, Nilsson, & Link, 2013a), the vertical separation of railway infrastructure ownership and operation has been implemented in many countries such as the UK, Sweden, Netherland, Romania and Germany (Nash, Nilsson, & Link, 2013b). Vertical separation involves two types of independent entities: Infrastructure Manager (IM), which provides tracks, signalling, bridges, tunnels and station, and Freight Operating Company (FOC), which operates freight service using the infrastructure provided by IM. After the implementation of vertical separation, IMs and FOCs no longer belong to a single company. The pricing for track access between them became a new and outstanding problem, which significantly affects not only IMs' and FOCs' profitability but also the utilisation of entire railway systems.

Franchising and auction are two ways a government may choose to regulate the pricing between IMs and FOCs in a vertically separated system. Auction is a widely used method in allocating scarce resources. However, adopting auctions in selling train itineraries in vertically separated systems is arguable (Affuso, 2003), and therefore franchising is thus commonly adopted in practice, e.g., in the UK. The issue with a franchising system is that an IM may take advantage of being the leader in designing the tariff to unilaterally maximise its profits without caring about FOCs' profits.

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According to our industrial visits to the UK rail freight industry, the interface between different players in the rail freight system is very complex. Currently, FOCs can only purchase itineraries from the only IM. Typically, the IM will first allocate the network capacity to the passenger trains, and FOCs must bid for the left limited freight capacity. The timetable needs to be determined a few months ago, although contingency arrangements are possible in some rare cases. In the bidding process, the three players in the freight system are taking different roles. The IM has the absolute right to make decisions on prices and capacity allocations, while the FOCs are the followers of the IM, and the government is not involved.

To prevent the UK IM from being the leader in setting up the tariff to achieve unlimited benefits, which would no doubt damage the health of the freight system, the UK government applies a cap on the IMs' profits. It remains unclear whether using the cap is effective in practice. This is because, as a countermeasure, an IM may choose to its cost and attempt to gain additional profits. In such a battle between the government and an IM, the government is often not in a good position as the government has great difficulty in ascertaining an IM's actual costs due to the complexity of railway system operations. Hence, the current pricing mechanism where FOCs are completely excluded is not really as effective as expected.

In recognition of this issue, this study aims to propose novel mathematical models to capture the complicated relationships between IMs and FOCs and design a coordinated pricing mechanism for the vertically separated railway system that adopts a franchising system to allocate train itineraries. In our research, we do not assume that FOCs own any infrastructure. Therefore, the pricing problem between vertically integrated rail companies, e.g., in the US, is outside our research scope.

In developing the model to investigate the pricing mechanism, a stylised railway system that adopts the vertical separation governance structure is considered a three-echelon service network supply chain comprising an IM, a FOC, and end customers. The whole pricing process between the IM and the FOC is regarded as a Stackelberg (leader–follower) game. As the leader, the IM's decision is the price tariff for train itineraries on their network. As the follower, the FOC must design a service network based on the IM's tariff and shippers' orders, which is a typical network design problem.

The unique feature of the problems is that the 'product' sold across the supply chain is freight train itineraries, which makes it different from the traditional Stackelberg game that usually does not involve networks and assumes a homogeneous product with a single price. Consequently, the problem investigated here is more challenging. Due to the network effect, many integer decision variables have to be used in the models, making it a non-differential game. Further, the uncertainty in customer demands in the gaming model will also be considered, and thus stochastic programming will have to be applied to handle the uncertainty.

This study will analyse three aspects of the game: 1) Stackelberg equilibrium solution, which can lead to the maximum profits the IM can achieve unilaterally. A bilevel optimisation model will be developed to identify the solution at Stackelberg equilibrium, and a stochastic gradient search and local search will be employed to solve the model. 2) System optimal solution, which can maximise the system profits – the total profits of the IM and the FOC. An inverse programming model will be developed to obtain the system's optimal solution. Due to the integer variables involved, a Fenchel cutting plane method (Boyd, 1994) will be used to solve the model. 3) Mechanism design. A subsidy contract will be proposed to coordinate the network-based supply chain. A two-layer gradient search method is proposed to determine the optimal subsidy rate.

The study makes contributions to both theory and practice. From the theory perspective, we propose a framework to investigate the properties and mechanism design of network-based Stackelberg games characterised by many non-differentiable integer decision variables. Supply chain contract design is an active research area that is thought to originate from Pasternack (1985) and Cachon (2003), and a large number of papers have been published over the past decades in this field. These studies are mainly based on differentiable games with a relatively limited number of decision variables, e.g., 1 or 2, as explained in the textbook by Snyder and Shen (2019). Most studies use the Newton-Leibniz formula to determine the optimal contract design by calculating the first-order and second-order derivatives. However, the differentiation methods widely used in the extant supply contract coordination studies cannot be applied to the network revenue management game discussed in the paper. This is because the IM–FOC game is NP-complete and involves many non-differentiable integral or binary decision variables. In this study, we will use inverse programming and gradient search methods to tackle this issue and analyse the network revenue management game. From the perspective of practice, our research will be helpful for governments to regulate the relationship between infrastructure managers/owners and freight operating companies under the government structure of vertical separation.

The remainder of this paper is organised as follows. In Section 2, the relevant research in the existing literature is discussed and analysed. Section 3 defines the research problem and formulates the decision-making models for an IM and a FOC, respectively. In Section 4, the IM - FOC game is formulated as a bilevel optimisation model, and a gradient search algorithm is developed to obtain the solution at Stackelberg equilibrium. In Section 5, the global optimality of the game will be investigated. The optimal prices that can lead to global optimality are designed. Section 6 proposes a subsidy contract and a double-layer gradient search-based solution method. Numerical case studies are given in Section 7 to validate our models and algorithms. Conclusions and discussions are presented in the last section.

2. The current state of the art

The state-of-the-art review will focus on two aspects: 1) railway network revenue management; and 2) railway network design problem. The former relates to our research question on how an IM should design a tariff for charging a FOC, and the latter relates to the research question of how a FOC should design a freight service network based on an IM's tariff and customer demands.

2.1. Railway network revenue management

In recent years, network revenue management has become an active research field, drawing increasing attention from academia. This topic has been studied extensively in the literature in the field of airline management (Grauberger & Kimms 2016a; 2016b; Hosseinalifam et al. 2016; Huang & Lin 2014; Li et al. 2016). In the most recent research, scholars developed nonlinear models to simultaneously optimise price and quantity and investigated competitors' behaviours considering the horizontal and vertical competition in the air transportation network (Grauberger & Kimms 2016a, 2016b). In addition to air transportation, this issue has also been explored in the road transportation context. Labbe et al. (1998) developed a bilevel model to optimise highway tolls on a multi-commodity transportation network. However, these research outcomes cannot be directly applied to the context of the railway freight industry due to the difference in the operation of these transport modes. Particularly, compared to air and road transportation, rail freight is more flexible in terms of capacity utilisation and traffic speed /route (Crevier, Cordeau, & Savard, 2012). Armstrong and Meissner (2010) observed a lack of study in rail network revenue management after reviewing 18 relevant papers on both passenger and freight rail revenue management. They found that the existing research covers the issues including capacity allocation, service differentiation, and booking horizon, but importantly network revenue management in rail freight industry has been overlooked, which is a clear research gap.

In the limited number of studies relating to rail network revenue management, the majority of them focus on rail passenger transportation where seat control is a focus, e.g., Ben-khedher et al. (1998), Ciancimino et al. (1999), Kraft, Srikar, and Phillips (2000), Zhang, Ma, and Zhang (2017). In the studies relating to the pricing problem in rail freight, it is common in existing studies that only a single stakeholder is considered. For instance, Kraft (2002) adopted a bid price approach to schedule railway shipment delivery times for railway operating companies. Li and Tayur (2005) developed a medium-term pricing model for inter-model transportation operators. Gorman (2001) and Gorman (2005) investigated the pricing problems of a railway company, Burlington Northern and Santa Fe Railway (BNSF), in the USA.

The conflicting interests of multiple stakeholders and the application of game theory in rail network revenue receive little attention, although it has been suggested that cooperation between multiple stakeholders can improve profits (Li et al., 2016). An early study involving both multiple stakeholders and track pricing was carried out by Harker and Hong (1994). They used pricing as a lever to coordinate the train scheduling between different divisions in a railway company. Variational inequalities are used to model the game. Crevier et al. (2012) have considered the joint optimisation of pricing and capacity for rail freight using a bi-level model. Both Harker and Hong's (1994) and Crevier et al. (2012)'s studies were to make pricing decisions concerning capacities which can be modelled as continuous variables, and the calculation of duals in their models have been used. This research is different from the two studies in the following aspects: 1) the bilevel model in this study has binary integer variables in the lower-level problem, which makes the existing duals based solution method invalid for our problem. A new solution method for the IM–FOC game has been developed. 2) this research conducted a further study on identifying the solution for global optimality and 3) this research will also design the mechanism for the network revenue game.

2.2. Network design

Rail network design is a traditional research topic in operations research and transportation science. Similar to the literature in network revenue management, rail network design in the published studies were mainly conducted for a single stakeholder, e.g. Infrastructure Manager. Crainic and Rousseau (1986) developed a modelling framework to optimise the network design process for multimode, multi-commodity freight transportation problems and provided an algorithm to solve the model. It has been noted that some research in the literature focused on different angles of network design in different railway system settings. e.g. Pazour et al. (2010) presented models for high-speed rail freight distribution network design in US railway system settings. This study mainly focuses on a policymaker's perspective. Lulli et al. (2011) proposed a customised mathematical model to design an Italian rail service network. Lin et al. (2012) formulated models to optimise the freight train connection problem in a large-scale railway network in China. A simulated annealing algorithm was applied in the optimisation process. Murali et al. (2016) developed a decision tool for train planners to select the best route in terms of travel time. A capacity constraint for the train movement was considered. This tool used integer programming to model the network's capacity, and a genetic algorithm was used to solve the model. However, It can be observed that there are very limited studies considering the conflicting interests of multiple stakeholders in network design in the railway freight sector.

To further clarify our contributions to the existing body of literature, a summary of the most relevant research is given in Table 1. When analysing the literature, the following aspects were considered: 1) if the proposed model is deterministic or stochastic; 2) if the number of stakeholders is 1 or more than 1 where game theory was applied; 3) if the model involves integer variables or non-differential game; 4) if the topological structure of a network is considered.

To summarise, different from the existing research, in this work, a stochastic model was developed for the railway network that adopts a vertical integration governance structure. In this model, the demand is a statistical variable, and multiple stakeholders including the IM, the FOC and end customers, will be considered. All the variables in the model are integers; hence the Stackelberg game is non-differential.

In light of the above discussion, the novelties of this study are:

(1) A vertically separated railway freight system is considered a three-tier service supply chain consisting of an IM, a FOC and end customers. The 'products' passed on the supply chain are a service network consisting of many itineraries.
(2) A network-based Stackelberg game is developed to investigate the conflicting interest of an IM, a FOC and end customers in designing the optimal pricing strategy for the IM and the optimal freight service network for the FOC.

Table 1
A summary of the relevant literature.

Existing studies	Deterministic or stochastic	Stakeholder number	Integer or continuous	Network considered
(Crainic & Rousseau, 1986)	D	1	I	√
(Pazour, Meller, & Pohl, 2010)	D	1	I	√
(Crevier, Cordeau, & Gilles Savard, 2012)	D	Multiple	C	√
(Crainic, 2000)	D	1	I	√
(Lin et al., 2012)	D	1	I	√
(Lulli, Pietropaoli, & Ricciardi, 2011)	D	1	I	√
(Marcotte, Savard, & Zhu, 2009)	D	Multiple	C	√
(Murali et al., 2016)	D	1	I	√
(Harker & Hong, 1994)	D	2	C	X
Our study	**S**	**Multiple**	**I**	**√**

(3) Both the equilibrium solution and the global optimal solution for the game models are obtained.
(4) Also, a subsidy mechanism is designed to achieve the global optimality of the entire system through profit sharing.

3. Problem formulation

The notations used in the study will be defined first in the section. Text-based problem descriptions will then be presented. Based on the notations and the text-based problem description, mathematical formulation for the problem will be given at the end of the section.

3.1. Notations

Set:

$\mathcal{P}$	The set of paths with each linking a pair of origin and destination stations
$\mathcal{I}_n$	The set of itineraries on Path n
$\mathcal{D}$	The set of periods over the planning horizon
$\mathcal{J}$	The set of jobs(customer orders)
Ω	The set of sample processes of customer demands
$\mathcal{L}$	The set of sections on the rail network
$\mathcal{K}$	The set of train stations

Index:

j	A transportation task from customers
n	A path between a pair of origin and destination train stations
i	An itinerary
d	A day(or a period)
ξ	A sample process of customer demands in the planning horizon
k	A station on the network
l	A section between two consecutive stations

Parameters:

T_{ndj}	The volume of wagons required for serving transport task j on day d on path n
r_{ndj}	The revenue a FOC can obtain for transporting one wagon for fulfilling task j on day d on path n
C_{ni}	The maximum number of wagons that the train serving itinerary i can carry on path n
$O(\cdot)$	An indicator function which indicates the origin of a job or a train
$D(\cdot)$	An indicator function which indicates the destination of a job or a train
a_{ni}	The fixed operational cost of the train serving itinerary i on path n
V_n	The IM's variable cost per mile per wagon on Path n
S_n	The travel distance between the pair of origin and destination station on path n
Q^l	The maximum number of trains that can be accommodated at section$l$per day
Q^k	The handling capacity in wagons at Station k
V_n'	The FOC's variable operational cost on path n per mile per wagon
δ_{ni}^l	Binary input data. 1 indicates that section l is used by the train serving itinerary i on path n; otherwise, 0
γ_{ni}^k	Binary input data. 1 indicates that station k is visited by the train serving itinerary i on path n; otherwise, 0
N	The number of samples in Ω
A	A pre-defined integer indicates the allowed maximum local search times.
L	A pre-defined nonnegative number, which is the step length of the inner gradient search
r	A pre-defined number denoting the value of adjustment for each local search iteration
Z	A very large positive integer
Δ	A nonnegative small number

Variables:

f_{ni}	Binary variable. 1 if itinerary i on path n is purchased; otherwise, 0;
$x_{ndij}(\xi)$	binary variable dependent on a sample process of demand ξ . 1 if a job j is served by the train serving itinerary i on day d on path n; otherwise, 0
p_{ni}	the price that the IM charges the FOC for running the train serving itinerary i on path n
w	The subsidy per wagon provided by the government to the IM
$S(w^k)$	The system profit when the subsidy is w^k
$S(w^{k'})$	The system profit when the subsidy is $w^{k'}$
AE	The total subsidy provided by the government

3.2. Problem description

Suppose a stylised railway freight system comprising an IM and a FOC adopts a vertical separation governance structure. The IM, as the owner or its representative of the railway network, operates all the tracks, signalling, bridges, tunnels and stations apart from rolling stock. The FOC is responsible for preparing and running rolling stock. The IM also creates timetables that specify when a train arrives at and departs from a station. In the study, we define an ***itinerary*** as the journey a train makes from its origin station at a designated time to its destination station. An itinerary differs from a train ***path*** that refers to a physical segment of a rail track. Multiple itineraries may be made on the same path.

The IM is the monopolist infrastructure provider providing itineraries for freight and passenger transportation operators. The FOC needs to purchase itineraries from the IM to run their trains based on a tariff agreed upon earlier, then start to provide freight transport service to shippers. Shippers must pay a fee to the FOC for the service according to the agreed transportation rates. The IM, the FOC, and the shippers form a three-tier service supply chain. The product associated with the supply chain is a network-based transport service instead of physical products that a conventional supply chain handles. This makes the research different from the common revenue management problems.

The assumptions to be adopted in formulating the problem are as follows:

Assumption 1. The capacity required by passenger services has been pre-determined.

Assumption 2. The pricing information for end customers is known.

Assumption 3. Only direct transportation services are considered.

Assumption 1 is proposed based on the common practice that passenger trains normally have priority in capacity/path acquisition over freight trains.

Due to the existence of roads and other transportation modes, the prices that the FOCs can charge shippers are relatively stable. Therefore, we proposed Assumption 2 in the study.

Assumption 3 is due to the development of rail freight. In the UK, due to the decline of the rail freight industry before the 1980s, large marshalling or classification yards have been closed and dismantled, e.g., Tinsley Marshalling Yard in Sheffield (Rhodes, 2016). Nowadays, there are no hump facilities, and only limited flat-shunting facilities are left in the UK. The percentage of wagon transhipments is low.

The pricing process is considered a dynamic Stackelberg (Leader–Follower) game that involves an IM and a FOC as the decision-makers. The game can be divided into two phases.

3.3. Phase 1: the leader (the IM)'s decision

In phase 1, as the Leader, the IM first designs a tariff in which the price for each itinerary is specified at each time period. As the IM is a monopolist who owns almost all the railway infrastructure, theoretically, the IM can charge any prices they wish. However, there are some practical constraints. If the IM charges low prices, their profit may be low, or they may even incur a loss from selling the railway itineraries; if the prices are too high, the number of itineraries they can sell to the FOC may be low, and the revenue they can gain may be low as well.

The IM's objective function is to maximise its expected profit, which can be formulated as follows,

$$\text{Max}\,Z(p,f,x) = \sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni} - \sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f_{ni} - \frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x_{ndij}(\xi)\cdot V_n\cdot S_n \quad (1)$$

The first term on the right-hand side (RHS) of Eq. (1) represents the revenue generated by selling itineraries to the FOC. The second term represents the total fixed cost of the sold itineraries. The third term is the IM's total expected variable costs, reflecting the track wear from transporting shippers' cargoes. This cost is proportional to the number of customer demands served, the distance between an Origin station and Destination Station (O–D pair), the average variable cost per wagon per train-mile V, and the mileage in the network between the O–D pair in path n, S_n.

The IM's decision to be made is the price tariff. However, the maximisation of IM's profit depends on f_{ni}, i.e., the FOC's itinerary purchasing plan and the order fulfilment plan $x_{ndij}(\xi)$, which are decisions made by the FOC in the second phase.

3.4. Phase 2: the follower (the FOC)'s decision

The FOC needs to design a service network and a customer order fulfilment plan to maximise its profit. More specifically, given the IM's tariff, the operating cost for each itinerary, and the unit revenue generated for fulling demands, the FOC needs to make decisions on which itineraries they should purchase, how to fulfil customer demands, i.e., which customer orders should be accepted or rejected; and which itineraries should be used to serve which orders.

The Phase 2 gaming process is considered a two-stage stochastic programming model. The first stage is to purchase itineraries from the IM, and the second stage is to fulfil stochastic customer demands using purchased itineraries. The stochastic factor considered in the model is customer demand. The sample Average Approximation method (Kleywegt, Shapiro, & Homem-de-Mello, 2002) is applied to handle the uncertain factor. The objective function of Phase 2 decision-making is to maximise the expected profits of the FOC as formulated in Eq. (2).

$$\text{Max}\ Y(f,x) = -\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni} + \frac{1}{N}F(x,\xi) \quad (2)$$

The first term on the right-hand side (RHS) of Eq. (2) represents the acquisition cost of itineraries from the IM, which is also the fee that the FOC needs to pay the IM to obtain the right to access track. The second term represents the expected revenue from freight service operation, which is the main focus of the second decision-making stage.

The profit obtained from running the freight service is formulated in the following equation.

$$F(x,\xi) = \sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot r_{ndj}\cdot T_{ndj} - \sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot V'_n\cdot S_n\cdot T_{ndj} \quad (3)$$

The first item in Eq. (3) is the revenue generated from order fulfilment, and the second item is the FOC's operational costs, including fuel or energy consumption cost, crew cost and all the other costs related to travel distance and loading status (the number of loaded wagons). By plugging Eq. (3) into Eq. (2), we can obtain the complete objective function of the FOC.

$$\text{Max}\ Y(f,\ x) = -\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni} + \frac{1}{N}\sum_{\xi\in\Omega}\left[\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot r_{ndj}\cdot T_{ndj} - \sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot V'_n\cdot S_n\cdot T_{ndj}\right] \quad (4)$$

The FOC must consider satisfying customer demands in the second stage subject to the purchased service network. The FOC's constraints are formulated as follows.

Constraint 1:

$$x_{ndij}(\xi) \le f_{ni}\ \forall n,d,i \quad (5)$$

The transportation task j on day d in path n can only be allocated to itinerary i when itinerary i has been purchased.

Constraint 2:

$$\sum_{n\in P}\sum_{i\in I_n} x_{ndij}(\xi) \le 1\ \forall d,j \quad (6)$$

No more than 1 itinerary will be needed to serve job j.

Constraint 3:

$$\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot T_{ndj} \le C_{ni}\ \forall n,d,i \quad (7)$$

This is the capacity constraint. The total amount of rail cars required by all tasks allocated to itinerary i cannot exceed the capacity of itinerary i in path n.

Constraint 4:

$$x_{ndij}(\xi) = 0\ \forall\ \{i,\ j\mid O(j)\ \ne\ O(i),\ D(j)\ne\ D(i)\},\ \forall n,\ d \quad (8)$$

Order j cannot be allocated to itinerary i when j cannot be covered by itinerary i geographically.

Constraint 5:

When providing train itineraries to the FOC, the IM is constrained by the capacity at a section and the handling capacity at a station. Hence, we have

$$\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} f_{ni}\cdot\delta^l_{ni} \le Q^l\ \forall\ l\in\mathcal{L} \quad (9)$$

$$\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} f_{ni}\cdot\gamma^k_{ni}\cdot C_{ni} \le Q^k\ \forall\ k\in\ \mathcal{K} \quad (10)$$

$$\sum_{j\in\mathcal{J}} x_{ndij}\cdot T_{ndj} \le C_{ni}\quad \forall n,d,i \quad (11)$$

$$x_{ndij}\in\{0,1\},\ \ f_{ni}\in\{0,1\}\ \forall\ n,\ \ i$$

Proposition 1. *The FOC's decision-making problem is NP-complete.*

Proof. We consider a simplified case where a FOC aims to determine the itineraries f_{ni} that they need to purchase from the IM, but the number of itineraries to be purchased is subject to rail network capacity, such as section capacity Q^l and station handling capacity Q^k. We further assume that the customer orders the FOC receives happen to be the maximum number of wagons, C_{ni}, and thus we do not need to make decisions on how to fulfil orders, and the revenue generated from an itinerary is constant. The FOC's problem is reduced to purchasing itineraries subject to capacity constraints. The simplified case is a Knapsack problem, which is a known NP-complete problem. However, the FOC's decision-making problem is more difficult than the simplified case, e.g., their order fulfilment decision $x_{ndij}(\xi)$ needs to be made for uncertain demands.

Proposition 2. *The IM–FOC's game is NP-complete.*

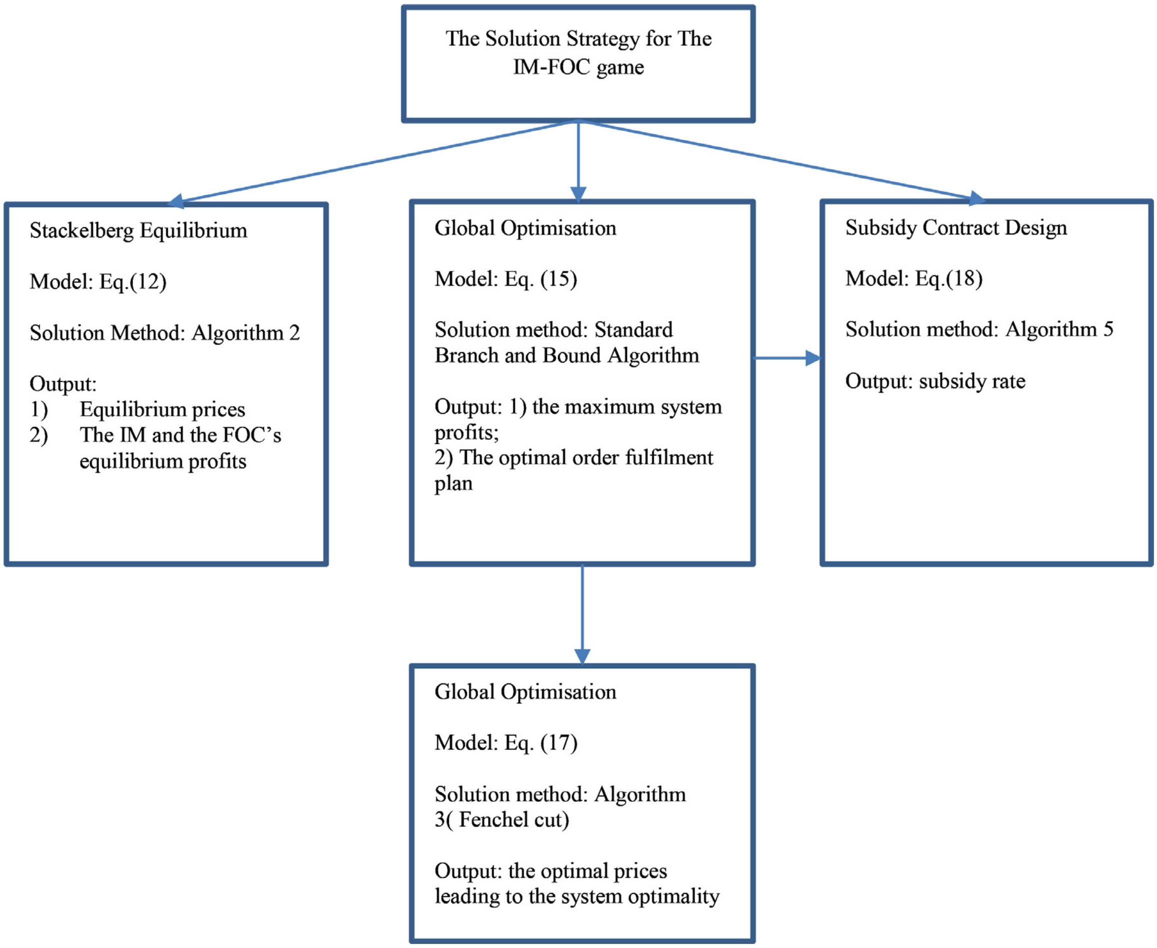

Fig. 1. An overview of the algorithmic framework.

Algorithm 1
Computing the IM's profits for a given price vector.

Step 1: substitute the given price vector into the lower level model in Eq. (11) – the FOC's model;
Step 2: solve the model using standard integer programming methods, and obtain the optimal decisions the FOC will make, $x^*_{ndij}(\xi)$ and f^*_{ni};
Step 3: substitute $x^*_{ndij}(\xi)$, f^*_{ni}, and the price vector into the objective function of the upper level model, and the profit of IM, $Z(p,\ f, x)$, can be obtained.

Proof. for every price tariff that an IM proposes to sell its itineraries, a FOC must solve an NP-complete problem. Therefore, the IM–FOC game is also NP-complete.

3.5. Solution strategy

In this study, we will develop algorithms to obtain the solutions that can achieve Stackelberg equilibrium and global optimisation of the IM–FOC game. Further, we will design a subsidy mechanism to achieve the coordination of the network service supply chain. Fig. 1 provides a flowchart to visualise the entire algorithmic framework. To investigate the properties of Stackelberg equilibrium, a bi-level optimisation model is formulated in Eq. (12), and a gradient search algorithm(Algorithm 2) is developed to obtain the equilibrium prices and the corresponding profits for the IM and the FOC, respectively. To identify the maximum profits that the entire supply chain(or the system) can achieve, a centralised model under perfect collaboration is formulated in Eq. (15). The model

Algorithm 2
A Gradient search based algorithm for solving integer bilevel optimisation model.

Step1: Initialisation.
Set a counter M to record the times of local search attempts, $M = 0$;
a counter k to record the number of steps the algorithm runs, $k = 0$
the initial price vector 0, i.e., $\boldsymbol{C}^0 =\ 0$;
the Optimal_Profit_So_Far $= -Z$ (Z is a very big nonnegative number),
the Optimal_Price $= \boldsymbol{C}^0$;
step length $= L$, (L is a pre-defined nonnegative number)
the maximum number of local search attempts = A (A is a pre-defined number)
Step 2: if $M <$ A, continue; otherwise, stop.
Step 3: Computing Z^k with $\boldsymbol{C^k}$ using Algorithm 1.
Step 3.1 if $Z^k >$ Optimal_Profit_So_Far
Optimal_Price $= \boldsymbol{C^k}$; Optimal_Profit_So_Far $= Z^k$; $k = k+1$
Calculate gradient of $\boldsymbol{C^k}$ using Eq. (13) and Algorithm 1,
$\boldsymbol{C^{k+1}} = \boldsymbol{C^k} + L * grad(\boldsymbol{C^k})$; $M = 0$;
Go to **Step 4**;
otherwise, go to **Step 3.2**
Step 3.2: Local Search
$\boldsymbol{C^{k+1}} = \boldsymbol{C^k} + S$; ($S$ is a predefined step); $M = M+1$;
Step 4: Goto **Step 2**.

is a standard Mixed Integer Linear programming model and thus can be solved using commercial optimisation software with Branch and Bound algorithm. The system optimisation model, Eq. (15), does not contain pricing information as it is developed based on the assumption of perfect collaboration. Hence, we develop an inverse programming model as shown in Eq. (17) to obtain the prices that can lead to system optimisation. A Fenchel cut based algo-

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Algorithm 3
A cutting plane algorithm for inverse MILP.

Step 1: Initialisation. Set a counter of steps, k, and let $k \leftarrow 0$; and $\boldsymbol{d}^{(k)} \leftarrow \boldsymbol{c}$,
Step 2: Substitute the price vector $\boldsymbol{d}^{(k)}$ into $P_{K,}$ solve the FOC's model P_K, and obtain its optimal solution $x^{(k)} = \{f_{ni}^{(k)} | (\forall n, i);\ x_{ndij}^{(k)}(\xi) | (\forall n, d, i, j, \xi)\}$.
Add the following cut to inv_P$_K$,
$d^{T(k)}x^0 \le d^{T(k)}x^{(k)}$
$\sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} \mathrm{d}_{ni}^{(k)} \cdot f_{ni}^* - \frac{1}{N}(\sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{J}} x_{ndij}^*(\xi) \cdot r_{ndj} \cdot T_{ndj}$
$- \sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{J}} x_{ndij}^*(\xi) \cdot V_n' \cdot S_n \cdot T_{ndj}) \le \sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} \mathrm{d}_{ni}^{(k)} \cdot f_{ni}^{(k)}$
$- \frac{1}{N}(\sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{J}} x_{ndij}^{(k)} \cdot r_{ndj} \cdot T_{ndj}$
$- \sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum\limits_{j\in\mathcal{J}} x_{ndij}^{(k)}(\xi) \cdot V_n' \cdot S_n \cdot T_{ndj})$
Solve inv_P$_K$, and obtain the optimal solution $\boldsymbol{d}^*$, $\boldsymbol{d}^k \leftarrow \boldsymbol{d}^*$;
$k \leftarrow k+1$;
Step 3: if $d^{T(k)}x^0 \le d^{T(k)}x^{(k)}$, go to Step 2; otherwise, Stop.

Algorithm 4
Computing the system profits for a given subsidy.

Step 1: for a given w, apply Algorithm 2 to solve model (19), and obtain $p_{ni}^{*\prime}$, $x_{ndij}^*(\xi)'$ and $f_{ni}^{*\prime}$, the profit of the FOC, $Y^{*\prime}(f, x)$
Step 2: substitute $x_{ndij}^*(\xi)'$, $f_{ni}^{*\prime}$, the subsidy w, and the price vector $p_{ni}^{*\prime}$ into the objective function of the upper level model in Eq. (19), and the profit of IM, $Z^{*\prime}(p, f, x)$, can be obtained;
Step 3: the total profit of the freight system $\boldsymbol{S}$ equals the profit of the FOC, $Y^{*\prime}(f, x)$ plus the profit of IM, $Z^{*\prime}(p, f, x)$ minus the total subsidy provided by the government **AE =** $\frac{1}{N} \sum_{\xi\in\Omega} \sum\limits_{n\in\mathcal{P}} \sum\limits_{i\in\mathcal{I}_n} \sum\limits_{d\in\mathcal{D}} \sum\limits_{j\in\mathcal{J}} T_{ndj} \cdot x_{ndij}^*(\xi) \cdot w$.

Algorithm 5
Double-layer gradient search algorithm.

Step1: Initialization
Set a counter T to record the times of local search attempts, $T = 0$;
a counter k to record the number of steps the algorithm runs, $k = 0$
the initial subsidy 0, i.e., $\boldsymbol{w}^0 = 0$;
the Optimal_System-Profit_So_Far $= -Z$ (Z is a very big nonnegative number),
the Optimal_Subsidy $= \boldsymbol{w}^0$;
step length $= L$, (L is a pre-defined nonnegative number)
the maximum number of local search attempts = A (A is a pre-defined number)
Step 2: if $T <$ A, continue; otherwise, stop.
Step 3: Computing $S(w^k)$ with $\boldsymbol{w^k}$ using Algorithm 4.
Step 4
Step 4.1: if $S(w^k) >$ Optimal_System-Profit_So_Far; continue 4.2; otherwise, go to Step 4.3
Step 4.2:
Optimal_Subsidy $= \boldsymbol{w^k}$;
Optimal_System-Profit_So_Far $= S^k$; $k = k+1$;
Compute $grad(\boldsymbol{w^k})$, using Eq. (20), (21) and Algorithm 4;
$\boldsymbol{w^{k+1}} = \boldsymbol{w^k} + L * grad(\boldsymbol{w^k})$; $T = 0$; Goto Step 3;
Step 4.3: Local Search
$\boldsymbol{w^{k+1}} = \boldsymbol{w^k} + r \times rand()$; ($r$ is a predefined step, rand() is a random number between -1 and 1);
$T = T+1$; Goto Step 2;

---

rithm (Algorithm 3) is developed to solve the inverse programming model. One of the inputs required by Algorithm 3 is the optimal solution obtained through solving the system optimisation model Eq. (15). To incentivise the IM and the FOC to make decisions that can achieve system optimality rather than forcing them to accept externally imposed prices such as the prices obtained through the inverse programming model, a subsidy contract is developed. The IM–FOC game with a subsidy is re-formulated in Eq. (18), and a double-layer gradient search algorithm (Algorithm 5) is developed. It is worth mentioning that the maximum system profit obtained through solving Eq. (15) is required in Algorithm 5 to measure the convergency.

4. Stackelberg equilibrium of the IM–FOC game

As discussed above, the dynamic gaming process between the IM and the FOC can be formulated as a bilevel optimisation model. In this section, the explicit form of the bilevel optimisation model is given, and then a solution method is developed for it.

4.1. A bilevel optimisation model for the IM–FOC game

Based on Eqs. (1)–(10), we can obtain the complete bilevel model below:

Upper Level :

$$\mathrm{Max}Z(p,\ f, x) = \sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} p_{ni} \cdot f_{ni} - \sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} a_{ni} \cdot f_{ni} - \frac{1}{N} \sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} \sum_{d\in\mathcal{D}} \sum_{j\in\mathcal{J}} T_{ndj} \cdot x_{ndij}(\xi) \cdot V_n \cdot S_n$$

Subject to : $p_{ni} \ge 0\ \forall n, i$

For a given p_{ni}, solves

Lower Level :

$$\mathrm{Max}\,Y(f, x) = -\sum_{n\in\mathcal{P}} \sum_{i\in\mathcal{I}_n} p_{ni} \cdot f_{ni} + \frac{1}{N}\left(\sum_{\xi\in\Omega} \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{J}} x_{ndij}(\xi) \cdot\ r_{ndj} \cdot T_{ndj} - \sum_{n\in\mathcal{P}} \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{J}} x_{ndij}(\xi) \cdot V_n' \cdot S_n \cdot T_{ndj}\right)$$

Subject to :

$$\begin{aligned} &Eqs.\ (5) - (11) \\ &x_{ndij}(\xi) \in \{0, 1\} \qquad \forall n, d, i, j \\ &f_{ni} \in \{0, 1\} \qquad \forall n, i \end{aligned} \tag{12}$$

All the variables in the lower level of the above bilevel model are binary integer variables. Therefore the lower level model has no dual form and cannot be solved using the existing dual based methods. An Approximate Gradient Search based method is developed to solve the model.

4.2. Approximate gradient search based algorithm for the IM–FOC game

As the binary integer variables are contained within the FOC's model, and there is a complex interaction between the upper and lower level models, the gradient of the IM's profit function can be approximately estimated. Let $\boldsymbol{C^k} = (c_1^k, c_2^k, \cdots, c_i^k, \cdots c^k_{\prod\limits_{n\in P} |\mathcal{I}_n|})$ denote the price vector, Z^k the IM's profit corresponding to $\boldsymbol{C^k}$; $\frac{dZ_i^k}{d\boldsymbol{C_i^k}}$ the change of the IM's profit with regard to a small change of c_i^k that is an element in $\boldsymbol{C^k}$ denoting the selling price of itinerary i, at the k^{th} iteration. The gradient of the IM's function at $\boldsymbol{C^k}$, $grad(\boldsymbol{C_k})$, at the k^{th} iteration can be defined as,

$$grad\left(\boldsymbol{C^k}\right) = \frac{dZ^k}{d\boldsymbol{C^k}} = \left(\frac{dZ_1^k}{d\boldsymbol{C_1^k}}, \frac{dZ_2^k}{d\boldsymbol{C_2^k}}, \cdots, \frac{dZ_i^k}{d\boldsymbol{C_i^k}}, \cdots \frac{dZ^k_{\prod_{n\in P} |\mathcal{I}_n|}}{d\boldsymbol{C^k_{\prod_{n\in P} |\mathcal{I}_n|}}}\right) \tag{13}$$

To compute $grad(\boldsymbol{C^k})$, it is essential to compute $\frac{dZ_i^k}{d\boldsymbol{C_i^k}}$, which is an element in the $grad(\boldsymbol{C^k})$. Let $Z_i^{k'}$ denote the IM's profit when the price vector is set as $\boldsymbol{C_i^{k'}} = (c_1, c_2, \cdots, c_i + \Delta, \cdots c_{\prod\limits_{n\in P} |\mathcal{I}_n|})$, where Δ is a nonnegative small number, we can have

$$\frac{dZ_i^k}{d\boldsymbol{C_i^k}} = \frac{Z_i^{k'} - Z_i^k}{\Delta} \tag{14}$$

Z_i^k and $Z_i^{k'}$ can be obtained by solving the lower level model in Eq. (11) for the given price vectors $\boldsymbol{C_k^i}$ and $\boldsymbol{C_k^{i'}}$, respectively. The detailed steps for calculating Z_n^k or $Z_n^{k'}$ are described above (Algorithm 1).

At some point, the IM's profit under a new price vector may be no better than the previous optimal profit. Before stopping the algorithm, a local search procedure will be performed. A random step length will be added to the price vector to generate a new one, i.e., $C_{k+1} = C_k + S \times rand()$, where S is the length of search step, rand() is a random number in the interval [−1,1]. If there is no improvement after searching a certain amount of times, the algorithm will stop. The local search method helps reduce the chance of the calculation stopping at a local optimal price (profit).

The algorithm that combines Gradient Search and Local Search is described as follows.

5. System optimisation of the IM–FOC game

In Section 4, an algorithm is designed to obtain the optimal non-cooperative prices and the maximum profits the IM can obtain unilaterally. However, the prices optimal for the IM may not be optimal for the entire freight service supply chain. In this section, the way to achieve system optimisation will be explored. The section first considers an ideal situation where the IM and the FOC are deemed a single virtual organisation, i.e., having perfect collaboration. Then it considers how a vertically separated railway system should set up the prices to make the system profits the same as that in the ideal case.

5.1. Perfect collaboration

Under perfect collaboration, the IM and the FOC are treated as a single company, and no fees are applied to each other; thus, the system's profit can be maximised. The perfect collaboration will lead to the following standard Integer Programming model.

$$\begin{aligned}
\text{Max}\, Z = & -\sum_{n}\sum_{i} a_{ni}\cdot f_{ni} + \frac{1}{N}\Bigg(\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot r_{ndj}\cdot T_{ndj} \\
& -\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x_{ndij}(\xi)\cdot V_n\cdot S_n \\
& -\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}(\xi)\cdot V_n'\cdot S_n\cdot T_{ndj}\Bigg) \\
& Eqs.\ (5)-(11) \\
& x_{ij} = 0 \qquad \forall\{i, j | O(j)\neq O(i), D(j)\neq D(i)\} \\
& x_{ndij}\in\{0,1\} \qquad \forall n, d, i, k \\
& f_{ni}\in\{0,1\} \qquad \forall n, i
\end{aligned} \tag{15}$$

5.2. Prices leading to system optimisation

The perfect collaboration case described above does not exist in a vertically separated railway management system. In practice, the IM must charge the FOC to recover its costs and generate profit. The optimal price the IM should charge the FOC to achieve system optimisation under the perfect collaboration will be identified in what follows. The technique to be adopted is Inverse Linear Programming (ILP) (Toint 2013; Tayyebi & Aman 2016; You, Chow, & Ritchie, 2016). The idea of ILP is to make a known feasible solution to be optimal by changing the coefficients of objective function appropriately. In this study, we will make the optimal solution for the entire system optimal for the FOC by changing the prices charged by the IM. The classical solution method (Ahuja & Orlin, 2001) that requests the dual form of the lower model cannot be applied to our problem. This is because the proposed problem has binary variables in the lower level model and has no duals. This section will introduce the formulation of an inverse programming model that can identify the optimal prices leading to system optimisation. Then, the Fenchel cutting plane algorithm(Boyd, 1994) will be developed to solve the model.

By solving the perfect collaboration model in Eq. (14), the FOC's purchasing plan $f_{ni}^*(\forall n, i)$ and the optimal order fulfilment plan $x_{ndij}^*(\forall\ n, d, i, j)$ are obtained. The problem of inverse programming in this study is, for a given set of $f_{ni}^*(\forall n, i)$ and $x_{ndij}^*(\forall\ n, d, i, j)$ obtained in Eq. (14), how to determine the prices $p_{ni}(\forall n, i)$. This ensures that the FOC will make the same decision under the governance structure of vertical separation.

To simplify the narrative, let P_k denote the FOC's model at the k^{th} iteration, Inv_P_k denotes a specific inverse form of the model at the k^{th} iteration. There might be many different sets of $p_{ni}(\forall n, i)$ which can make the given set of $f_{ni}^*(\forall n, i)$ and $x_{ndij}^*(\forall\ n, d, i, j)$ obtained in the model described in Eq. (14) to be the optimal solution for the FOC's model. The L1 norm is used to limit the choices. By complying with the L1 norm, only a single set $p_{ni}(\forall n, i)$ will be selected. Let $\boldsymbol{c} = \{c_{ni}|\forall n, i\}$ denote a known initial price vector; $\boldsymbol{d} = \{d_{ni}|\forall n, i\}$ the price vector that can make a feasible set of prices, $\boldsymbol{x}^0 = \{f_{ni}^*|\forall n, i; x_{ndij}^*|\forall\ n, d, i, j\}$, to be optimal as discussed above; $\boldsymbol{x^{(k)}}$ another feasible solution of prices at the k^{th} iteration. It should be noted that $\boldsymbol{d} = \{d_{ni}|\forall n, i\}$ is the decision variable in the inverse programming model.

The problem Inv_P_k can be formulated as,

$$\begin{aligned}
&(\text{Inv_P}_\text{k})\text{Z}(\boldsymbol{d}) = min\left|\mathbf{c}-\mathbf{d}\right| \\
&\qquad d^T x^0 \leq d^T x^{(k)}
\end{aligned} \tag{16}$$

It should be noted that $\boldsymbol{d} = \{d_{ni}|\forall n, i\}$ are the decision variables in the inverse programming model.

The explicit form of $d^T x^0$ and $d^T x^{(k)}$ is given below in Eq. (17)

$$\begin{aligned}
d^T x^0 = & \sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} \text{d}_{ni}^{(k)}\cdot f_{ni}^* - \frac{1}{N}\Bigg(\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}^*(\xi)\cdot r_{ndj}\cdot T_{ndj} \\
& -\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}^*(\xi)\cdot V_n'\cdot S_n\cdot T_{ndj}\Bigg) \\
d^T x^{(k)} = & \sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} \text{d}_{ni}^{(k)}\cdot f_{ni}^{(k)} - \frac{1}{N}\Bigg(\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}^{(k)}\cdot r_{ndj}\cdot T_{ndj} \\
& -\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}^{(k)}(\xi)\cdot V_n'\cdot S_n\cdot T_{ndj}\Bigg)
\end{aligned} \tag{17}$$

In Eq. (15), the L1 norm can be linearised by introducing a positive auxiliary vector $\boldsymbol{\theta} = \{\theta_{ni}\geq 0|\forall n, i\}$. By substituting x^0 with $f_{ni}^*(\forall n, i)$ and $x_{ndij}^*(\forall\ n, d, i, j)$ obtained in the model described in Eqs. (14) and (15) can be re-formulated as,

$$\begin{aligned}
&(\text{Inv_P}_\text{k}) \qquad \text{Z}\left(\boldsymbol{d^{(k)}}\right) = \min\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\theta_{ni}^{(k)} \\
&\text{Subject to} \\
&\text{c}_{ni} - \text{d}_{ni}^{(k)} \leq \theta_{ni}^{(k)} \quad \forall n, i \\
&\text{d}_{ni}^{(k)} - \text{c}_{ni} \leq \theta_{ni}^{(k)} \quad \forall n, i \\
&d^T x^0 \leq d^T x^{(k)}
\end{aligned} \tag{18}$$

Note in the above formulation, the variables associated with the superscript (k) will be updated at each iteration.

Based on the formulation of inverse programming, an algorithm is developed to obtain the optimal price, which can lead to global optimality.

$d^{T(k)}x^0$, $d^{T(k)}x^{(k)}$ in Algorithm 3 can be calculated using Eq. (17).

In the above algorithm, $\boldsymbol{c}$ theoretically can be set as any value. However, in our experiment, a zero vector for $\boldsymbol{c}$ was set. This means we seek a price vector that is close to 0 rather than any other possible vectors. This complies with the idea of providing a minimal price to the end customer to ensure the competitiveness of rail freight in the freight market.

6. Subsidy contract design

The prices obtained in Section 5.2 can lead to the system optimisation of the game. However, a third party, e.g., the government, is needed to force the IM and the FOC to accept the prices. In this section, a subsidy contract will be introduced. If the IM and the FOC comply with the subsidy contract, system optimisation can be achieved.

The subsidy contract contains the following four steps:

(1) The government first decides a subsidy rate, w, for each customer order to be served.
(2) The IM makes decisions on the itinerary prices, taking into consideration the subsidy rate w unilaterally, like what they are doing in practice now.
(3) The FOC decides the itinerary purchasing plan and the customer order fulfilment plan.
(4) Charge the IM and the FOC from their increased profits to recover the subsidy.

The key challenge for implementing the subsidy contract is to determine the subsidy rate w. In the following, algorithms will be developed to obtain the rate. We first introduce the following proposition.

Proposition 3. *The IM will not set up a price tariff that can lead to global optimality if the subsidy rate of the contract is lower than a lower bond,*

$$w \geq \frac{B}{\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'}$$

where, $B=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^*{}_{ni}\cdot f^*{}_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^*{}_{ni}-\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)\cdot V_n\cdot S_n-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^{*\prime}_{ni}\cdot f^{*\prime}_{ni}+\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^{*\prime}_{ni}+\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)'\cdot V_n\cdot S_n$

Proof. Without applying the subsidy contract, the IM's profit can be calculated by:

$$Z\left(p^*,\ f^*,x^*\right)=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^*_{ni}\cdot f^*_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^*_{ni} -\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)\cdot V_n\cdot S_n$$

Where $p^*,\ f^*,x^*$ are the optimal solution at the Stakelberge Equilibrium without any contract.

If the proposed contract is accepted, the IM's profit would be:

$$Z'\left(p^{*\prime},\ f^{*\prime},x^{*\prime}\right)=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^{*\prime}_{ni}\cdot f^{*\prime}_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^{*\prime}_{ni} -\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'\cdot V_n\cdot S_n +\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'\cdot \mathrm{w}$$

And $p^{*\prime},\ f^{*\prime},x^{*\prime}$ are the optimal solution of the IM-FOC game with applying the subsidy contract.

To stimulate the IM to accept this contract, their profit after applying this mechanism must be guaranteed more than or at least no less than the profit in the scenario without any contract. That is:

$$\begin{aligned}
&Z'\left(p^{*\prime},\ f^{*\prime},x^{*\prime}\right)\geq Z\left(p^*,\ f^*,x^*\right)\\
&\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^{*\prime}_{ni}\cdot f^{*\prime}_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^{*\prime}_{ni}\\
&-\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'\cdot V_n\cdot S_n\\
&+\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'\cdot \mathrm{w}\\
&\geq\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^*{}_{ni}\cdot f^*{}_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^*{}_{ni}\\
&-\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)\cdot V_n\cdot S_n
\end{aligned}$$

Then,

$$w \geq \frac{B}{\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}\left(\xi\right)'}$$

where, $B=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^*_{ni}\cdot f^*_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^*_{ni}-\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)\cdot V_n\cdot S_n-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^{*\prime}_{ni}\cdot f^{*\prime}_{ni}+\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^{*\prime}_{ni}+\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)'\cdot V_n\cdot S_n$

Therefore, in the designed contract, the subsidy rate, w, is equal to or bigger than the above inequation RHS value is the primary requirement.

According to the subsidy contract and Proposition 1, the IM–FOC game under the subsidy contract can be re-formulated as follows:

$$\begin{aligned}
&\textbf{Upper Level}:\\
&\mathrm{Max}\, Z\left(p,\ f,x\right)\\
&=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f_{ni}\\
&-\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x_{ndij}\left(\xi\right)\cdot V_n\cdot S_n\\
&+\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x_{ndij}\left(\xi\right)\cdot w\\
&\text{Subject to}:\quad p_{ni}\geq 0\forall n,i\\
&\text{Foragiven } p_{ni}\text{, solves}\\
&\textbf{Lower Level}:\\
&\mathrm{Max}\, Y\left(f,x\right)=-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni}\\
&+\frac{1}{N}\left(\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}\left(\xi\right)\cdot\ r_{ndj}\cdot T_{ndj}\right.\\
&\left.-\sum_{n\in\mathcal{P}}\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{J}} x_{ndij}\left(\xi\right)\cdot V'_n\cdot S_n\cdot T_{ndj}\right)\\
&\text{Subject to}:\\
&x_{ndij}\leq f_{ni}\quad \forall n,d,i,j\\
&\sum_{n\in\mathcal{P}}\sum_{i\in I_n} x_{ndij}\leq 1\quad \forall n,d,j\\
&\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} f_{ni}\cdot\delta^l_{ni}\leq Q^l\quad \forall\ l\in\mathcal{L}
\end{aligned}$$

$$\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} f_{ni}\cdot\gamma_{ni}^k\cdot C_{ni}\le Q^k \quad \forall\, k\in\ \mathcal{K}$$
$$\sum_{j\in\mathcal{J}} x_{ndij}\cdot T_{ndj}\le C_{ni}\quad \forall n,d,i$$
$$w\ge \frac{B}{\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)'}$$
$$where,\ B=\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p^*_{ni}\cdot f^*_{ni}-\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f^*_{ni} -\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x^*_{ndij}(\xi)\cdot V_n\cdot S_n -\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} p_{ni}\cdot f_{ni}+\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n} a_{ni}\cdot f_{ni} +\frac{1}{N}\sum_{\xi\in\Omega}\sum_{n\in\mathcal{P}}\sum_{i\in\mathcal{I}_n}\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{J}} T_{ndj}\cdot x_{ndij(\xi)}\cdot V_n\cdot S_n$$
$$x_{ij}=0\quad \forall\{i,j|O(j)\ne O(i),D(j)\ne D(i)\}$$
$$x_{ndij}\in\{0,1\}\quad \forall n,d,i,j$$
$$f_{ni}\in\{0,1\}\quad \forall n,i \tag{19}$$

The objective function of the model (19) is the IM's profit including the subsidy received. In the constraint, the lower bound of the subsidy rate is added, which can ensure that the IM will be beneficial after joining the contract.

To solve w, the gradient search method will be applied twice. The outer-level gradient search method is used to identify the government's optimal subsidy per wagon. The inner-level gradient search is used to solve the bi-level model for a given subsidy.

Let w^k denote the subsidy per wagon the IM receives from the government; $S(w^k)$ the freight system profit at the Stackelberg equilibrium corresponding to the subsidy per wagon w^k from the government; $\frac{d\ S(w^k)}{dw^k}$ the change of the system profit with regard to a small change of w^k, at the k^{th} iteration. The gradient of the system profit at w^k, $grad(w_k)$, at the k^{th} iteration can be defined as,

$$grad\left(w^k\right)=\frac{d\ S\left(w^k\right)}{dw^k} \tag{20}$$

To compute $grad(w^k)$, let $S(w^{k'})$ denote the system profit corresponding to another given subsidy per wagon $w^{k'}=w^k+\Delta$ from the government, where Δ is a nonnegative small number, we can have

$$\frac{dS\left(w^k\right)}{dw^k}=\frac{S\left(w^{k'}\right)\ -S\left(w^k\right)}{\Delta} \tag{21}$$

For the given subsidy w^k and $w^{k'}$, the optimal profit $S(w^{k'})$, $S(w^k)$ can be obtained by the above algorithm (Algorithm 4).

To solve the optimal subsidy w^*, the following double-layer gradient search based algorithm is proposed.

The outputs of the double-layer gradient search algorithm include the optimal value of subsidy per wagon and the freight system profit, the IM's profit and the FOC's profit under this subsidy.

Remarks: in the current practice, the government, e.g., in the UK, subsidise IMs to lower the selling prices of itineraries. However, IMs may request governments to increase the amount of subsidy as they claim that insufficient subsidy may cause a loss in selling itineraries. In this study, we propose the idea of subsidising IMs based on the customer orders that FOCs receive. This is conducive to shifting the service supply chain from Stackelberg equilibrium to system optimisation.

7. Numerical cases

In this section, two case studies are provided. In each case study, three scenarios, i.e. non-cooperative Stackelberg equilibrium, global optimality and subsidy contract, will be analysed. The optimal tariff, the optimal subsidy, the maximum system profit and the maximum profit of each player under each scenario will be computed and compared. The expriments were conducted on a Microsoft Surface laptop with an intel 2.5 GHz i7 processor and 16 GB internal memory.

The first case is hypothetical, which has one service path with three trains (itineraries). The second case considers a network with four freight rail stations, multiple paths and multiple itineraries in the UK.

7.1. Case 1

In this case, a freight service between two stations, Donnington and Burton, in the UK, is considered. Three trains operate on the line every day. The input data for the experiment are:

- Demands

	Orders (Amount in carriages, revenue in thousand pounds)					
Day 1	8,15	6,9	6,6	7,6	6,8	7,9
Day 2	9,10	7,12	4,10	7,8	6,9	8,9
Day 3	7,8	1,9	2,7	4,9	5,10	5,15

- The IM's fixed operational cost is £2000 per freight car.
  - ➢ The capacity for all the trains is 30 wagons.
  - ➢ The IM's variable cost is £ 10 per ton-mile;
  - ➢ The distance between Donnington and Burton is 100 miles.

7.1.1. Solutions to the IM–FOC game at equilibrium

Firstly, feed the data into the developed bilevel programming programme in Eq.(11) to calculate the non-cooperative price, which enables the IM to maximise its profit unilaterally. The calculation results are shown in Table 2.

The initial price is set to be (55,55,55). The programme stops when it shows the optimal price is (245,245,245), the IM's profit is 162, and the FOC's profit is 0.33. The calculation result indicates that the IM's profit is relatively high, whereas the FOC's profit is close to zero. Further increasing the optimal price, the FOCs will choose not to operate any line, and the IM's profit will go down. Therefore, the current price (245,245,245) is the best price the IM can charge the FOC in the sense that the IM's profit is maximised unilaterally. It also indicates that, under the price (245,245,245), the game will be at Stackelberg equilibrium.

Fig. 2 shows the convergence of the IM's profit. The computational time required to identify the IM's best profit at the Stackelberg equilibrium is 80 seconds.

7.1.2. Global optimality

To compute the maximum profit that the entire rail freight service supply chain can achieve, the Inverse Linear Programming model and the Fenchel cut based algorithm detailed in Section 5.2 is adopted. The calculation results are shown in Table 3.

In Table 3, it can be observed when the price is set as 58,58,130 for each train, respectively, the corresponding system profit is the same as that under the perfect collaboration. However, when global optimality is achieved, the profit that the IM can obtain is only 7, which is much lower than 162 obtained in the equilibrium status of the game where the IM makes a decision unilaterally to maximise its profits. Interestingly, it can also be found that the system profits at equilibrium, 162.33, is lower than 221.3, which

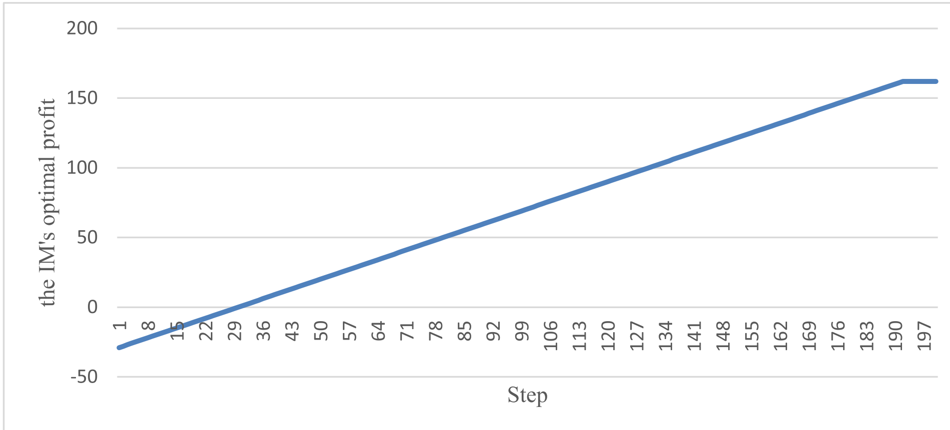

Fig. 2. The converence of the IM's profit to the Stackelberg equilibrium.

Table 2
The IM's and the FCO's profits at Stackelberg equilibrium (in 1000 pounds).

	Initial price	Optimal price	The IM's profit	FOCs profit	System profit
Optimal system	(55,55,55)	***(245,245,245)***	***162***	***0.33***	***162.33***

Table 3
Profit level under different price.

Scenario	Optimal price per train	The IM's profit	The FOC's profit	System profit	Purchasing plan
Perfect collaboration	N/A	N/A	N/A	221.3	1,1,0
Global optimality	58,58,130	7	214.3	221.3	1,1,0

Note that the prices and profits in the table are measured in thousand pounds.

Table 4
Convergence of Fenchel cut based solution algorithm for Inverse programming game.

	Optimal price per train	The IM's profit	The FOC's profit	System profit	Purchasing plan
Cut 1	120,130,130	37	152.3	189.3	1,0,0
Cut 2	120,58,130	−25	214.3	189.3	0,1,0

Table 5
The subsidy contract design (Case 1).

	Subsidy (£)	Optimal price per train	The system profit thousand £	the IM's profit thousand £	The FOC's profit thousand of £
Subsidy contract	22	58,58,130	221.33	162	59.33

is the maximum profits the entire supply chain can obtain. The difference between the two system profits is 221.3−162.3 = 58.97. Table 3 also gives the FOC's optimal itineraries purchasing plan. Since we use a binary variable to denote whether the FOC should purchase an itinerary or not, the value (1,1,0) in the last column in Table 3 suggests that the FOC should purchase the first and the second itinerary.

In this experiment, the convergence of the Fenchel cut based algorithm is also observed. Table 4 shows the converging process. Each row in Table 4 corresponds to a cut/ iteration. It shows the best prices identified for itineraries, the IM's profit, the FOC's profit, the overall system profit, and the FOC's purchasing plan at each iteration. Please note that the FOC's purchasing plan is denoted by binary data. For example,(0,1,0) represents that the FOC should purchase the second itinerary. It has been found that only two cuts or iterations were required, as shown in Table 4, to reach the optimal point. By applying Algorithm 3, Cut 1 is generated first. Compared to the scenario under perfect collaboration, it is easy to find that system profit is less than the targeted perfect collaboration case. This indicates that the algorithm has not converged yet. Cut 2 needs to be generated then. However, it also cannot make the system profit equal to that under the perfect collaboration. After adding Cut 2 into the model described by Eq. (4), the algorithm converged, and the optimal solution in the second row in Table 3 was obtained.

Fig. 3 visualise the convergence steps of the system's profit. The computational time required is 2 seconds.

7.1.3. Subsidy contract design

By applying the two-layer gradient search algorithm, we obtained the optimal subsidy rate of £ 22 per wagon to the IM. Under the optimal rate,

- The system profit can reach £ 221, 330 which is the same as its maximal profit under the global optimisation scenario and better than the equilibrium scenario (£162,330).
- The IM's profit is £162,000, which is the same as its maximal profit under the equilibrium scenario.

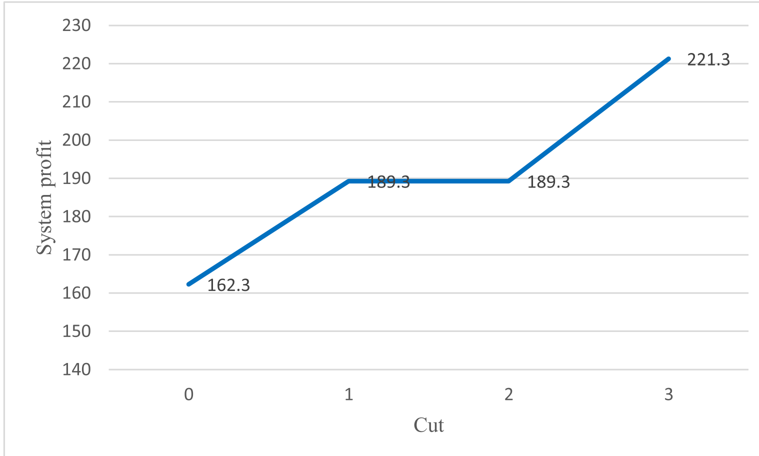

Fig. 3. The convergence process of Fenchel cuts algorithim.

Table 6
Customer order (units).

Felixstowe to Southampton						
Day 1	8	6	6	7	6	7
Day 2	9	7	6	7	6	8
Day 3	7	8	6	9	10	9
Southampton to Mossend						
Day 1	10	6	8	5	8	7
Day 2	12	5	6	5	6	10
Day 3	4	8	7	6	15	12
Felixstow to Mossend						
Day 1	8	4	10	7	6	9
Day 2	12	15	6	7	8	4
Day 3	6	8	12	4	5	10
Mossend to Southampton						
Day 1	10	4	10	8	5	9
Day 2	11	12	9	7	10	4
Day 3	10	6	6	4	12	11
Mossend to Felixstowe						
Day 1	8	6	10	6	8	9
Day 2	12	10	12	9	12	6
Day 3	13	9	5	9	15	10
Southampton to Felixstowe						
Day 1	8	6	10	6	7	12
Day 2	10	12	8	8	9	6
Day 3	10	8	6	6	14	15

Table 7
Unit revenue for each order (£/unit).

Felixstowe to Southampton						
Day 1	390	388	384	385	382	386
Day 2	389	384	389	394	382	384
Day 3	392	395	388	383	388	393
Southampton to Mossend						
Day 1	580	577	582	578	578	584
Day 2	582	572	572	591	588	570
Day 3	581	570	587	588	581	589
Felixstow to Mossend						
Day 1	591	582	587	585	585	589
Day 2	593	594	591	594	590	600
Day 3	593	582	592	591	593	602
Mossend to Southampton						
Day 1	583	587	572	576	577	586
Day 2	586	572	582	583	583	574
Day 3	583	578	585	585	588	591
Mossend to Felixstowe						
Day 1	590	587	582	588	588	584
Day 2	592	592	592	591	600	580
Day 3	590	580	597	598	591	600
Southampton to Felixstowe						
Day 1	391	386	387	384	388	388
Day 2	386	388	386	391	385	385
Day 3	390	385	386	385	386	391

- The FOC's profit is £59, 330 after returning the subsidy to the government. It is significantly higher than the equilibrium scenario when the IM and the FOC are non-cooperate (£ 330).

Fig. 4 shows the convergence of the system's profit. The computational time required is 311 seconds.

7.2. Case 2

A four-node railway network in the UK as shown in Fig. 5 is considered in case 2. The network was chosen from UK national railway network and contains four freight depots: Node 1 - Mossend; Node 2- DRIFT (Daventry International Rail Freight terminal); Node 3 - Southampton; Node 4 - Felixstowe. The distance for each link on the network is 1–2 (330 miles), 2–3 (140 miles), and 2–4 (160 miles). Six paths have been considered: 1–3;1–4; 3–4; 3–1;4–1; and 4–3. For each path, there are three itineraries available for sale. It is assumed that the FOC makes the decision based on three days' operation data.

Table 8
The other data.

	Path 1	Path 2	Path 3	Path 4	Path 5	Path 6
The IM's fixed operational cost (£/unit/mile)	0.11	0.12	0.05	0.05	0.05	0.05
Capacity (units or wagons)	34/17	32/16	32/16	32/16	32/16	32/16
the IM's variable costs (£/unit/mile)	0.05	0.05	0.05	0.05	0.05	0.05
The FOC'S variable costs (£/unit/mile)	1	1	1	1	1	1
Mileage (miles)	300	470	490	470	490	300

Note: Path 1: Felixstowe to Southampton; Path 2: Southampton to Mossend; Path 3: Felixstow to Mossend; Path 4: Mossend to Southampton; Path 5: Mossend to Felixstow; Path 6: Southampton to Felixstowe.

7.2.1. Key data

The railway lines in case 2 are mainly used for transporting containers. Therefore, the number of orders and the relevant costs

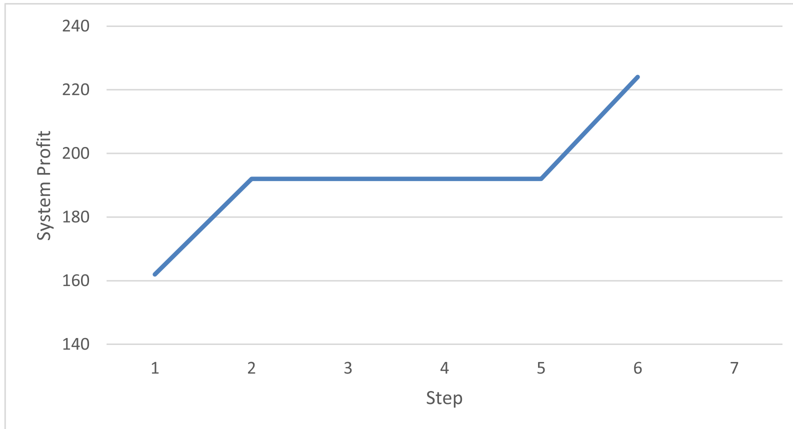

Fig. 4. The convergence of the two-layer gradient search algorithm for the subsidy contract design.

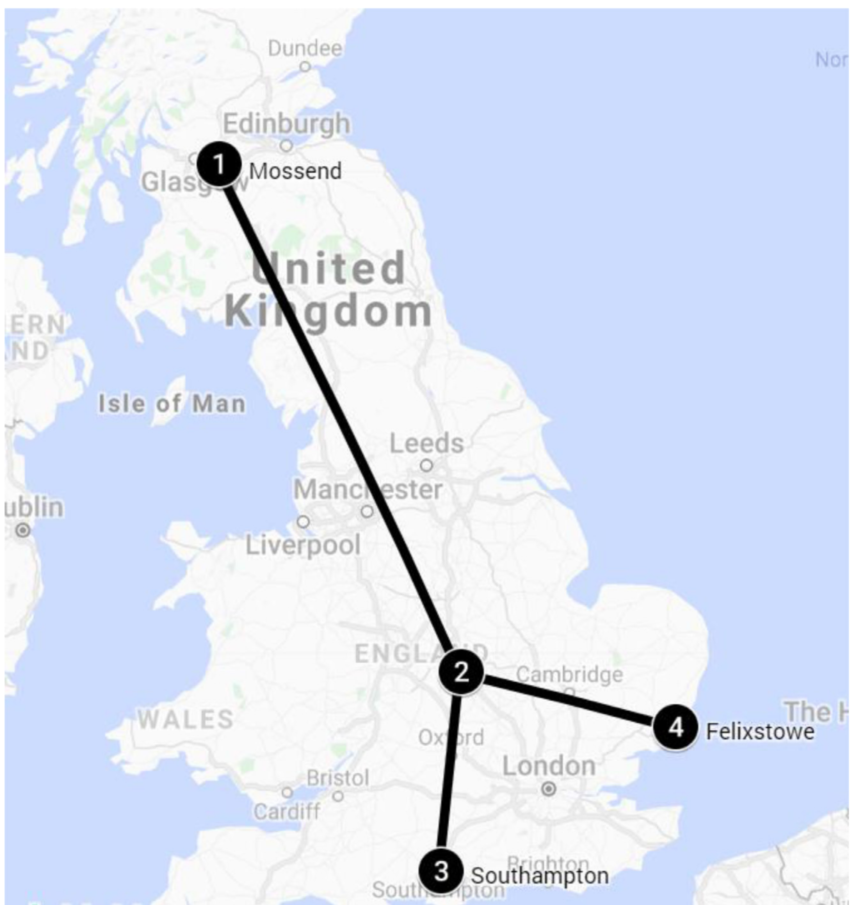

Fig. 5. A four-node railway network in the UK.

are measured by containers (termed Units in UK industrial practice). In the following, the data used in the case are presented, i.e., the orders received, the price the FOC charges shippers and other data such as costs and capacity.

8. Results

By plugging data into the above models and applying our algorithms to solve the models, the solutions for equilibrium and global optimality are obtained, as shown in Tables 9 and 10. It can be observed that, without a contract, the best profits that the IM can achieve is £6087 unilaterally, and the corresponding system profit is £9907.33. If the IM chooses to implement global optimality, its profit will drop to £1193.63, and a higher system profit of £11,147.67 can be achieved. The computational times required to obtain the solutions at the Stackelberg equilibrium and the system's optimal solution are 978 seconds and 12 seconds, respectively.

When the IM receives the optimal subsidy rate of £4 subsidy per wagon from the government, we obtained the calculation results as shown Table 11. The computational time required to calculate the optimal subsidy rate is 23,730 seconds.

Table 9
Stackelberg equilibrium.

Path	Price (£)	System profit(£)	IM's profit(£)	FOCs' profit(£)
Mossend to Southampton	3766, 3206, 3206	9907.33	6087	3820.33
Mossend to Felixstowe	2425, 2425, 2425			
Southampton to Felixstowe	1899, 1899,1899			
Felixstow to Mossend	2466,2466, 2466			
Southampton to Mossend	3532,3788,6420			
Felixstowe to Southampton	1902,1902,1902			
Mossend to Southampton	3766, 3206, 3206			
Mossend to Felixstow	2425, 2425, 2425			

Table 10
System optimisation.

Path	Price (£)	System profit (£)	IM's profit (£)	FOCs' profit (£)
Mossend to Southampton	1859.67,1859.67,2150	11,147.67	1193.63	9954.04
MOSSEND to Felixstowe	2050, 2200, 2200			
Southampton to Felixstowe	1400, 1600, 1600			
Felixstow to Mossend	1505.7,1505.7,2550			
Southampton to Mossend	1629,1629,2200			
Felixstowe to Southampton	1400,1600,1600			
Mossend to Southampton	1859.67,1859.67,2150			
Mossend to Felixstow	2050, 2200, 2200			

Table 11
the subsidy contract design (Case 2).

Path	Subsidy	System profit	The IM's profit	The FOC's profit
Mossend to Southampton	4	10,612	6264	4348
Mossend to Felixstowe	4			
Southampton to Felixstowe	4			
Southampton to Mossend	4			
Felixstow to Mossend	4			
Felixstowe to Southampton	4			

- The system profit will be £10,612, which is close to the profits under the perfect cooperation scenario (£11,147.67) and higher than that under the equilibrium scenario (£9907.33);
- The IM's maximum profit is £6264, which is slightly better than its maximum profit of £6087 under the equilibrium scenario;
- The FOC's profit is £4348. It is significantly higher than the equilibrium scenario when the IM and the FOC are non-cooperative (£3820.33).

9. Conclusions

The study investigates the Stackelberg equilibrium, global optimality, and mechanism design of the network revenue game in the rail freight industry that adopts the vertical separation government structure with uncertain demands. First, the network revenue game model is formulated to investigate the Stackelberg equilibrium, where the leader, the IM, unilaterally makes pricing decisions first, and the FOC makes decisions on purchasing itineraries afterwards. To find the optimal solution of the equilibrium of the network revenue game, a bilevel programming model is formulated, and a combination of gradient search and local search is adopted to solve the bilevel optimisation model. Secondly, the global optimality of the network revenue game is investigated through an inverse optimisation approach. The Fenchel cut based solution method for the specific gaming model is adopted, and the optimal solution which can maximise the profits of the entire service network supply chain has been successfully found. Thirdly, a subsidy contract was designed for the game. By following the contract, the two players will be better off than the Stackelberg equilibrium, and the system optimisation can be achieved.

The study has proposed a framework to investigate the network revenue management game. Our game involves a large number of non-differential integer decision variables due to the network effect, which is different from the differential Stackelberg games involving a single product in the existing literature in the field of supply chain management. Also, our research can be considered as a tool for governments to regulate their rail freight system under the vertical separation governance structure. In the current practice, governments do not have such an analytical tool to make their decision. Our research also makes recommendations to governments that it might be more reasonable to subsidies FOCs instead of IMs to promote the rail freight industry.

Further research could be done by extending the model to include multiple FOCs, who may compete for customer demands horizontally.

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