Some Inequalities Between General Randić-Type Graph Invariants

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Abstract

The Randić-type graph invariants are extensively investigated vertex-degree-based topological indices and have gained much prominence in recent years. The general Randić and zeroth-order general Randić indices are Randić-type graph invariants and are defined for a graph G with vertex set V as $R_\alpha(G) = \sum_{v_i \sim v_j} (d_i d_j)^\alpha$ and $Q_\alpha(G) = \sum_{v_i \in V} d_i^\alpha$ respectively, where α is an arbitrary real number, d_i denotes the degree of a vertex v_i and $v_i \sim v_j$ represents the adjacency of vertices v_i and v_j in G. Establishing relationships between two topological indices holds significant importance for researchers. Some implicit inequality relationships between R_α and Q_α, have been derived so far. In this paper, we establish explicit inequality relationships between R_α and Q_α. Also, we determine linear inequality relationships between these graph invariants. Moreover, we obtain some new inequalities for various vertex-degree-based topological indices by the appropriate choice of α.

1 Introduction

In this paper, we consider a simple finite graph $G = (V, E)$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set E, where the quantities $n = |V|$ and $m = |E|$ are known as the order and the size of G, respectively. If $n > 1$, then G is called a nontrivial graph. The ceiling function $\lceil \frac{n}{2} \rceil$ would round $\frac{n}{2}$ to the smallest integer greater than or equal to $\frac{n}{2}$ whereas the floor function $\lfloor \frac{n}{2} \rfloor$ would round $\frac{n}{2}$ to the largest integer less than or equal to $\frac{n}{2}$. For a given vertex $v_i \in V$, the neighborhood of v_i is denoted by $N(v_i)$ and defined as $N(v_i) = \{v_j \in V : v_i \sim v_j\}$, where $v_i \sim v_j$ represents the adjacency of vertices v_i and v_j in G. For $v_i \in V$, the degree of the vertex is defined as $d_i = |N(v_i)|$. Among all vertices of $\mathbb{G}$, the maximum degree is given by Δ and the minimum degree is given by δ. Without loss of generality, the degree sequence $(d_i) = (d_1, d_2, \cdots, d_n)$ of the vertices in G is organized as $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$. If $d_i = \delta = \Delta$ for each vertex v_i in $\mathbb{G}$, we call it a regular graph. For a vertex v_i, denote by $S_i = \sum\limits_{v_i \sim v_j} d_j$. It is obvious that $\delta^2 = \min\limits_{v_i \in V} \{S_i\}$ and $\Delta^2 = \max\limits_{v_i \in V} \{S_i\}$.

A chemical (or molecular) graph can frequently be used to represent the structure of a molecule.

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Chemical graphs play a pivotal role in understanding and representing the structural intricacies of molecules, thereby serving as a fundamental tool in the realm of chemistry. These graphs, composed of vertices representing atoms and edges denoting chemical bonds, provide a visual abstraction that aids in deciphering the three-dimensional arrangement of atoms in a compound. The chemical importance of graphs lies in their ability to elucidate molecular properties, reactivity patterns, and overall structural characteristics critical for predicting a substance's behavior. A plethora of literature exists, delving into the development and application of various graph-based approaches in chemistry [25]. Graph theory has proven invaluable in medicinal chemistry, material science, and computational chemistry, offering insights into molecular relationships, reaction mechanisms, and the rational design of novel compounds [26].

Graph theory has contributed to the development of chemistry by providing a variety of mathematical tools such as topological indices. A graph invariant that is calculated from the parameters of a chemical graph, is declared a topological index (TI) if it correlates with some molecular property. TIs are the conclusive results of a mathematical and logical procedure that maps the chemical phenomena hidden inside a molecule's symbolic representation into a useful value, and they have been shown to be useful in modelling varied physicochemical characteristics reflected by QSAR and QSPR calculations [6, 12].

Milan Randić, a chemist, proposed a degree-based topological index, called the Randić index [22] which is useful for measuring the degree of branching in the carbon-atom skeleton of saturated hydrocarbons. This index is represented by R and is defined as follows:

$$R = R(G) = \sum_{v_i \sim v_j} (d_i d_j)^{-\frac{1}{2}}$$

Randić proved that this index is significantly associated with a variety of physicochemical features of alkanes, including boiling points, enthalpy of formation, surface areas, chromatographic retention times, and so on [10, 17]. Eventually R became one of the most well-known molecular descriptors, with two books [13] and [15], several reviews and a plethora of research articles devoted to it. Some bounds of this index have been studied in [2]. Bollobás and Erdös [4] extended R by substituting an arbitrary real number α for the exponent $-\frac{1}{2}$. This graph invariant is called the general product-connectivity index or the general Randić index [18], represented by R_α:

$$R_\alpha = R_\alpha(G) = \sum_{v_i \sim v_j} (d_i d_j)^{\alpha}.$$

Kier and Hall [14] put forward the zeroth-order Randić index, represented by ${}^0R$. The explicit formula of ${}^0R$ is

$$ {}^0R = {}^0R(G) = \sum_{v_i} d_i^{-\frac{1}{2}}. $$

Eventually, Li and Zheng [16] proposed zeroth-order general Randić index by replacing the fraction $-\frac{1}{2}$ by an arbitrary real number α different from 0 and 1, denoted by Q_α:

$$ Q_\alpha = Q_\alpha(G) = \sum_{v_i} d_i^\alpha $$

This index is also studied under the name first general Zagreb index [11]. Moreover, it may be noted that Q_2 and R_2 are also studied under the names first Zagreb index M_1 [7] and second Zagreb index M_2 [8], respectively. The AutoGraphiX (conjecture-generating computer method) proposed [5] that the Zagreb indices are generally related to the inequality $M_2(G)/m \geq M_1(G)/n$ for a connected graph G with order n and size m. Though there exist graphs for which it does not hold [9], it is true for numerous classes of graphs [1, 20, 21, 23].

The investigation of relationships between two topological indices remains an intriguing and attractive problem for researchers. Liu and Gutman [19] derived the implicit inequalities between R_α and Q_α for $\alpha > 0$ and $\alpha < 0$. Later, Zhou and Vukičević [27] established the inequalities between R_α, Q_α, $Q_{2\alpha}$ and $Q_{2\alpha+1}$. In this paper, we make a step forward by deriving the explicit relationships between R_α and Q_α for $\alpha > 0$ and $\alpha < 0$. Also, we obtain linear inequalities between R_α and Q_α for $\alpha > 0$ and $\alpha < 0$, where those $\alpha < 0$ that holds some condition on the order of graph. Moreover, we obtain new inequality between $M_2(G)/m$ and $M_1(G)/n$ for any graph G with order n and size m. Further, we determine new inequality between $R(G)/m$ and ${}^0R(G)/n$.

2 Some known results

In this section we review some known results that will be used in our main results.

Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ be positive real numbers such that for $1 \leq i \leq n$, it holds that $p \leq p_i \leq P$ and $q \leq q_i \leq Q$. Then,

$$ \left| n \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i \right| \leq \tau(n) \left(P - p\right) \left(Q - q\right), \tag{1} $$

where $\tau\left(n\right) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil\right)$. Further, equality attains if and only if $p_1 = p_2 = \cdots = p_n$ and $q_1 = q_2 = \cdots = q_n$ [3].

Rodríguez et al. [24] established the following relationships between Q_α and $Q_{\alpha+1}$.

Let G be a nontrivial graph having the parameters n and m. Then, for $\alpha > 0$,

$$Q_{\alpha+1}(G) \geq \frac{2m}{n} Q_\alpha(G) \tag{2}$$

and for $\alpha < 0$,

$$Q_{\alpha+1}(G) \leq \frac{2m}{n} Q_\alpha(G). \tag{3}$$

Equality attains in each case if and only if G is regular.

Also, Rodríguez et al. [24] derived the following relation between Q_α and $Q_{2\alpha}$.

If G is a nontrivial graph with the parameters n, δ and Δ, then for $\alpha < 0$,

$$Q_{2\alpha}(G) \leq \frac{1}{4n} \left[\left(\frac{\Delta}{\delta}\right)^\alpha + \left(\frac{\delta}{\Delta}\right)^\alpha + 2 \right] Q_\alpha^2(G). \tag{4}$$

Further, equality attains if and only if G is regular.

Liu and Gutman [19] derived the following implicit quadratic inequality between R_α and Q_α.

If G is a nontrivial graph with the parameters n, δ and Δ, then for $\alpha > 0$,

$$R_\alpha(G) \leq \frac{1}{2} Q_\alpha(G) \left[\left(1 - \frac{1}{n}\right) Q_\alpha(G) + (\Delta - n + 1)\delta^\alpha \right] \tag{5}$$

and the equality is achieved if and only if G is regular.

Also, Liu and Gutman [19] established the following inequality between R_α, Q_α, $Q_{\alpha+1}$ and $Q_{2\alpha}$.

If G is a nontrivial graph having the parameters n and δ, then for $\alpha < 0$,

$$R_\alpha(G) \geq \frac{1}{2} \left[Q_\alpha^2(G) - (n-1)\delta^\alpha Q_\alpha(G) + \delta^\alpha Q_{\alpha+1}(G) - Q_{2\alpha}(G) \right]. \tag{6}$$

Further, equality attains if and only if G is regular.

3 Main results

Lemma 1. *Let G be a nontrivial graph, then for any real number α,*

$$Q_{\alpha+1}(G) = \sum_{i=1}^{n} S_i(\alpha), \tag{7}$$

where $S_i(\alpha) = \sum_{v_j \in N(v_i)} d_j^\alpha$.

Proof.

$$\begin{aligned} Q_{\alpha+1}(G) &= \sum_{i=1}^{n} d_i^{\alpha+1} = \sum_{i=1}^{n} d_i d_i^\alpha = d_1 d_1^\alpha + d_2 d_2^\alpha + \cdots + d_n d_n^\alpha \\ &= \underbrace{d_1^\alpha + d_1^\alpha + \cdots + d_1^\alpha}_{d_1 \text{ times}} + \underbrace{d_2^\alpha + d_2^\alpha + \cdots + d_2^\alpha}_{d_2 \text{ times}} + \cdots + \underbrace{d_n^\alpha + d_n^\alpha + \cdots + d_n^\alpha}_{d_n \text{ times}} \end{aligned}$$

By rearranging with respect to the sum of degrees of neighbor vertices of each vertex v_i, we have

$$Q_{\alpha+1}(G) = \sum_{i=1}^{n} \sum_{v_j \in N(v_i)} d_j^{\alpha}.$$

By setting $S_i(\alpha) = \sum\limits_{v_j \in N(v_i)} d_j^{\alpha}$, the required result follows. □

Lemma 2. *Let G be a nontrivial graph, then for any real number α,*

$$R_{\alpha}(G) = \frac{1}{2} \sum_{i=1}^{n} d_i^{\alpha} S_i(\alpha), \tag{8}$$

where $S_i(\alpha) = \sum\limits_{v_j \in N(v_i)} d_j^{\alpha}$.

Proof.

$$\begin{aligned} R_{\alpha}(G) &= \frac{1}{2} \sum_{v_i \sim v_j} 2 d_i^{\alpha} d_j^{\alpha} \\ &= \frac{1}{2} \left[d_1^{\alpha} \sum_{v_j \in N(v_1)} d_j^{\alpha} + d_2^{\alpha} \sum_{v_j \in N(v_2)} d_j^{\alpha} + \cdots + d_n^{\alpha} \sum_{v_j \in N(v_n)} d_j^{\alpha} \right] = \frac{1}{2} \sum_{i=1}^{n} d_i^{\alpha} \sum_{v_j \in N(v_n)} d_j^{\alpha}. \end{aligned}$$

By taking $S_i(\alpha) = \sum\limits_{v_j \in N(v_i)} d_j^{\alpha}$, the desired result follows. □

In the following Theorem, we derive the left and right explicit inequalities between R_{α} and Q_{α} for $\alpha > 0$ and $\alpha < 0$ respectively.

Theorem 1. *Let G be a nontrivial graph with order n, size m, minimum vertex-degree δ and maximum vertex-degree Δ. Then, the following left and right inequalities hold for $\alpha > 0$ and $\alpha < 0$ respectively:*

$$-\phi(m,n,\alpha) + \left(\frac{Q_{\alpha}(G)}{n} \right)^2 \leq \frac{R_{\alpha}(G)}{m} \leq \left(\frac{Q_{\alpha}(G)}{n} \right)^2 + \phi(m,n,\alpha), \tag{9}$$

where $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn} \left(\Delta^{\alpha} - \delta^{\alpha} \right)^2 \left(\Delta^{\alpha} + \delta^{\alpha} \right)$ and $\tau(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$. Further, each equality holds if and only if G is a regular graph.

Proof. We choose $p_i = d_i^{\alpha}$ and $q_i = S_i(\alpha)$ in inequality (1), then for $\alpha \geq 0$, $\delta^{\alpha} \leq d_i^{\alpha} \leq \Delta^{\alpha}$ and $\delta^{2\alpha} \leq S_i(\alpha) \leq \Delta^{2\alpha}$ and for $\alpha \leq 0$, $\Delta^{\alpha} \leq d_i^{\alpha} \leq \delta^{\alpha}$ and $\Delta^{2\alpha} \leq S_i(\alpha) \leq \delta^{2\alpha}$, where $i = 1, 2, \cdots, n$. Note that for any real number α, $\left(\Delta^{\alpha} - \delta^{\alpha} \right) \left(\Delta^{2\alpha} - \delta^{2\alpha} \right) = (\delta^{\alpha} - \Delta^{\alpha})(\delta^{2\alpha} - \Delta^{2\alpha})$. Then, for any real number α, inequality (1) takes the form

$$\left| n \sum_{i=1}^{n} d_i^{\alpha} S_i(\alpha) - \sum_{i=1}^{n} d_i^{\alpha} \sum_{i=1}^{n} S_i(\alpha) \right| \leq \tau(n) \left(\Delta^{\alpha} - \delta^{\alpha} \right) \left(\Delta^{2\alpha} - \delta^{2\alpha} \right),$$

where $\tau(n) = n\left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil\right)$.

From (7) and (8), we have

$$|2nR_\alpha(G) - Q_\alpha(G)Q_{\alpha+1}(G)| \leq \tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right).$$

This implies that

$$-\tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right) \leq 2nR_\alpha(G) - Q_\alpha(G)Q_{\alpha+1}(G) \leq \tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right).$$

Tihs gives

$$\begin{aligned} -\tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right) + Q_\alpha(G)Q_{\alpha+1}(G) &\leq 2nR_\alpha(G) \\ &\leq Q_\alpha(G)Q_{\alpha+1}(G) + \tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right). \end{aligned}$$

By using (2) with $\alpha > 0$ and (3) with $\alpha < 0$ in the left and right inequalities respectively, we have

$$\begin{aligned} -\tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right) + \frac{2m}{n}\left(Q_\alpha(G)\right)^2 &\leq 2nR_\alpha(G) \\ &\leq \frac{2m}{n}\left(Q_\alpha(G)\right)^2 + \tau(n)\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right). \end{aligned}$$

By taking $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn}\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right)$, the required inequality (9) follows.

Since equality attains in (1) if and only if $p_1 = p_2 = \cdots = p_n$ and $q_1 = q_2 = \cdots = q_n$. This gives that equality attains in (9) if and only if $d_1^\alpha = d_2^\alpha = \cdots = d_n^\alpha$ and $S_1(\alpha) = S_2(\alpha) = \cdots = S_n(\alpha)$. Also, $d_1^\alpha = d_2^\alpha = \cdots = d_n^\alpha$ and $S_1(\alpha) = S_2(\alpha) = \cdots = S_n(\alpha)$ implies $d_1 = d_2 = \cdots = d_n$ and $S_1 = S_2 = \cdots = S_n$. This recommends that each equality in (9) attains if and only if G is a regular graph. □

In the following corollary, we derive the linear inequality between Q_α and R_α for any positive real number α.

Corollary 1. *Let G be a nontrivial graph having order n, size m, minimum vertex-degree δ and maximum vertex-degree Δ with $n(n-1) \neq 2m$. Then, for $\alpha > 0$ we have*

$$R_\alpha(G) \geq \frac{m}{n(n-1) - 2m}\left[(n - \Delta - 1)\delta^\alpha Q_\alpha(G) - n(n-1)\phi(m,n,\alpha)\right], \qquad (10)$$

where $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn}\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right)$ and $\tau(n) = n\left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil\right)$. Moreover, equality attains if and only if G is a regular graph.

Proof. From inequality (5) with $\alpha > 0$, we have

$$R_\alpha(G) \le \frac{n(n-1)}{2}\left(\frac{Q_\alpha(G)}{n}\right)^2 + \frac{1}{2}(\Delta - n + 1)\delta^\alpha Q_\alpha(G). \tag{11}$$

Also, from left inequality (9), we get

$$\left(\frac{Q_\alpha(G)}{n}\right)^2 \le \frac{R_\alpha(G)}{m} + \phi(m,n,\alpha), \tag{12}$$

where $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn}(\Delta^\alpha - \delta^\alpha)^2(\Delta^\alpha + \delta^\alpha)$ and $\tau(n) = n\left\lceil \frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$.

By using inequality (12) in inequality (11), we have

$$R_\alpha(G) \le \frac{n(n-1)}{2}\left[\frac{R_\alpha(G)}{m} + \phi(m,n,\alpha)\right] + \frac{1}{2}(\Delta - n + 1)\delta^\alpha Q_\alpha(G), \tag{13}$$

After simplifying (13) and then rearranging, we get the desired inequality (10).

Since the equality attains in both inequality (5) and left inequality (9) if and only if G is a regular graph. Therefore, equality in (10) holds if and only if G is a regular graph. □

Lemma 3. *For $\alpha < 0$, it is easy to observe that*

$$Q_{\alpha+1}(G) \ge \delta Q_\alpha, \tag{14}$$

where equality attains if and only if G is a regular graph.

In the upcoming corollary, we derive the linear inequality between R_α and Q_α for any negative real number α which satisfies some condition on the order of graph G.

Corollary 2. *Let G be a nontrivial graph having order n, size m, minimum vertex-degree δ and maximum vertex-degree Δ. Then, for $\alpha < 0$ and $n > \lambda/4$,*

$$R_\alpha(G) \le \frac{m}{n(4n-\lambda) - 8m}\left[4\delta^\alpha(n - \delta - 1)Q_\alpha(G) + n(4n - \lambda)\psi(m,n,\alpha)\right], \tag{15}$$

where $\lambda = \lambda(\alpha) = \left(\frac{\Delta}{\delta}\right)^\alpha + \left(\frac{\delta}{\Delta}\right)^\alpha + 2$, $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn}(\Delta^\alpha - \delta^\alpha)^2(\Delta^\alpha + \delta^\alpha)$ and $\tau(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$. Further, equality attains if and only if G is a regular graph.

Proof. From inequalities (4) and (14) with $\alpha < 0$, the inequality (6) becomes

$$R_\alpha(G) \ge \frac{1}{2}\left[\left[1 - \frac{1}{4n}\left(\left(\frac{\Delta}{\delta}\right)^\alpha + \left(\frac{\delta}{\Delta}\right)^\alpha + 2\right)\right]Q_\alpha^2(G) - (n - \delta - 1)\delta^\alpha Q_\alpha(G)\right].$$

By taking $\lambda = \lambda(\alpha) = \left(\frac{\Delta}{\delta}\right)^\alpha + \left(\frac{\delta}{\Delta}\right)^\alpha + 2$ and rearranging, we have

$$R_\alpha(G) \ge \frac{1}{2}\left[n\left(n - \frac{\lambda}{4}\right)\left(\frac{Q_\alpha(G)}{n}\right)^2 - (n - \delta - 1)\delta^\alpha Q_\alpha(G)\right].$$

Also, from right inequality (9) with $\alpha < 0$ and $n > \frac{\lambda}{4}$, we get

$$R_\alpha(G) \geq \frac{n(4n-\lambda)}{8}\left[\frac{R_\alpha(G)}{m} - \phi(m,n,\alpha)\right] - \frac{(n-\delta-1)}{2}\delta^\alpha Q_\alpha(G), \tag{16}$$

where $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn}\left(\Delta^\alpha - \delta^\alpha\right)^2\left(\Delta^\alpha + \delta^\alpha\right)$ and $\tau(n) = n\left\lceil \frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$.

After simplifying (16), we achieve the desired inequality (15).

Since equality attains in right inequality (9) and each of the inequalities (4), (14) and (6) if and only if G is a regular graph. This implies that equality in (15) attains if and only if G is a regular graph. □

In the following corollary, we get a new inequality between $M_2(G)/m$ and $M_1(G)/n$ for any graph G with order n and size m by taking $\alpha = 2$ in the left inequality (9).

Corollary 3. *Let G be a nontrivial graph with order n and size m, then*

$$\frac{M_2(G)}{m} \geq \left(\frac{M_1(G)}{n}\right)^2 - \phi(m,n), \tag{17}$$

where $\phi(m,n) = \frac{\tau(n)}{2mn}\left(\Delta^4 - \delta^4\right)\left(\Delta^2 - \delta^2\right)$ and $\tau(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$. Further, each equality holds if and only if G is a regular graph.

In the following corollary, we obtain a new inequality between $R(G)/m$ and ${}^0R(G)/n$ for any graph G with order n and size m by setting $\alpha = -1/2$ in the right inequality (9).

Corollary 4. *Let G be a nontrivial graph having order n and size m, then*

$$\frac{R(G)}{m} \leq \left(\frac{{}^0R(G)}{n}\right)^2 + \psi(m,n), \tag{18}$$

where $\psi(m,n) = \frac{\tau(n)(\Delta-\delta)\left(\sqrt{\Delta}-\sqrt{\delta}\right)}{2mn(\Delta\delta)^{\frac{3}{2}}}$ and $\tau(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$. Moreover, each equality holds if and only if G is a regular graph.

4 Conclusions

A major contribution of this paper lies in the derivation of explicit inequality relationships between the general Randić and zeroth-order general Randić indices. Furthermore, the paper goes beyond the implicit relationships and determines linear inequality relationships between the general Randić and zeroth-order general Randić indices, providing a more comprehensive framework for their comparison. We would like to conclude this paper by proposing the following open problem:

Open Problem 1. *Drive the linear inequality between the general Randić index R_α and zeroth-order general Randić index Q_α for any negative real number α.*

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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