Some Inequalities Between General Randić-Type Graph Invariants

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Abstract

The Randić-type graph invariants are extensively investigated vertex-degree-based topological indices and have gained much prominence in recent years. The general Randić and zeroth-order general Randić indices are Randić-type graph invariants and are defined for a graph $G$ with vertex set $V$ as $R_{\alpha}(G) = \sum_{\upsilon_i \sim \upsilon_j} (d_{\upsilon_i}d_{\upsilon_j})^{\alpha}$ and $Q_{\alpha}(G) = \sum_{\upsilon_i \in V} d_{\upsilon_i}^{\alpha}$ respectively, where $\alpha$ is an arbitrary real number, $d_{\upsilon_i}$ denotes the degree of a vertex $\upsilon_i$ and $\upsilon_i \sim \upsilon_j$ represents the adjacency of vertices $\upsilon_i$ and $\upsilon_j$ in $G$. Establishing relationships between two topological indices holds significant importance for researchers. Some implicit inequality relationships between $R_{\alpha}$ and $Q_{\alpha}$, have been derived so far. In this paper, we establish explicit inequality relationships between $R_{\alpha}$ and $Q_{\alpha}$. Also, we determine linear inequality relationships between these graph invariants. Moreover, we obtain some new inequalities for various vertex-degree-based topological indices by the appropriate choice of $\alpha$.

1 Introduction

In this paper, we consider a simple finite graph $G = (V, E)$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E$, where the quantities $n = |V|$ and $m = |E|$ are known as the order and the size of $G$, respectively. If $n > 1$, then $G$ is called a nontrivial graph. The ceiling function $\lceil \frac{n}{2} \rceil$ would round $\frac{n}{2}$ to the smallest integer greater than or equal to $\frac{n}{2}$ whereas the floor function $\lfloor \frac{n}{2} \rfloor$ would round $\frac{n}{2}$ to the largest integer less than or equal to $\frac{n}{2}$. For a given vertex $v_i \in V$, the neighborhood of $v_i$ is denoted by $N(v_i)$ and defined as $N(v_i) = \{v_j \in V : v_i \sim v_j\}$, where $v_i \sim v_j$ represents the adjacency of vertices $v_i$ and $v_j$ in $G$. For $v_i \in V$, the degree of the vertex is defined as $d_i = |N(v_i)|$. Among all vertices of $G$, the maximum degree is given by $\Delta$ and the minimum degree is given by $\delta$. Without loss of generality, the degree sequence $(d_i) = (d_1, d_2, \cdots, d_n)$ of the vertices in $G$ is organized as $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$. If $d_i = \delta = \Delta$ for each vertex $v_i$ in $G$, we call it a regular graph. For a vertex $v_i$, denote by $S_i = \sum_{v_j \sim v_i} d_j$. It is obvious that $\delta^2 = \min_{v_i \in V} \{S_i\}$ and $\Delta^2 = \max_{v_i \in V} \{S_i\}$.

A chemical (or molecular) graph can frequently be used to represent the structure of a molecule.

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Chemical graphs play a pivotal role in understanding and representing the structural intricacies of molecules, thereby serving as a fundamental tool in the realm of chemistry. These graphs, composed of vertices representing atoms and edges denoting chemical bonds, provide a visual abstraction that aids in deciphering the three-dimensional arrangement of atoms in a compound. The chemical importance of graphs lies in their ability to elucidate molecular properties, reactivity patterns, and overall structural characteristics critical for predicting a substance’s behavior. A plethora of literature exists, delving into the development and application of various graph-based approaches in chemistry [25]. Graph theory has proven invaluable in medicinal chemistry, material science, and computational chemistry, offering insights into molecular relationships, reaction mechanisms, and the rational design of novel compounds [26].

Graph theory has contributed to the development of chemistry by providing a variety of mathematical tools such as topological indices. A graph invariant that is calculated from the parameters of a chemical graph, is declared a topological index (TI) if it correlates with some molecular property. TIs are the conclusive results of a mathematical and logical procedure that maps the chemical phenomena hidden inside a molecule’s symbolic representation into a useful value, and they have been shown to be useful in modelling varied physicochemical characteristics reflected by QSAR and QSPR calculations [6,12].

Milan Randić, a chemist, proposed a degree-based topological index, called the Randić index [22] which is useful for measuring the degree of branching in the carbon-atom skeleton of saturated hydrocarbons. This index is represented by $R$ and is defined as follows:

$$R = R(G) = \sum_{v_i \sim v_j} (d_id_j)^{-\frac{1}{2}}$$

Randić proved that this index is significantly associated with a variety of physicochemical features of alkanes, including boiling points, enthalpy of formation, surface areas, chromatographic retention times, and so on [10,17]. Eventually $R$ became one of the most well-known molecular descriptors, with two books [13] and [15], several reviews and a plethora of research articles devoted to it. Some bounds of this index have been studied in [2]. Bollobás and Erdős [4] extended $R$ by substituting an arbitrary real number $\alpha$ for the exponent $-\frac{1}{2}$. This graph invariant is called the general product-connectivity index or the general Randić index [18], represented by $R_\alpha$:

$$R_\alpha = R_\alpha(G) = \sum_{v_i \sim v_j} (d_id_j)^\alpha.$$  

Kier and Hall [14] put forward the zeroth-order Randić index, represented by $^0R$. The explicit
formula of $0^R$ is
\[ 0^R = 0^R(G) = \sum_{v_i} d_i^{-\frac{1}{2}}. \]

Eventually, Li and Zheng [16] proposed zeroth-order general Randić index by replacing the fraction $-\frac{1}{2}$ by an arbitrary real number $\alpha$ different from 0 and 1, denoted by $Q_\alpha$:
\[ Q_\alpha = Q_\alpha(G) = \sum_{v_i} d_i^{\alpha}. \]

This index is also studied under the name first general Zagreb index [11]. Moreover, it may be noted that $Q_2$ and $R_2$ are also studied under the names first Zagreb index $M_1$ [7] and second Zagreb index $M_2$ [8], respectively. The AutoGraphiX (conjecture-generating computer method) proposed [5] that the Zagreb indices are generally related to the inequality $M_2(G)/m \geq M_1(G)/n$ for a connected graph $G$ with order $n$ and size $m$. Though there exist graphs for which it does not hold [9], it is true for numerous classes of graphs [1,20,21,23].

The investigation of relationships between two topological indices remains an intriguing and attractive problem for researchers. Liu and Gutman [19] derived the implicit inequalities between $R_\alpha$ and $Q_\alpha$ for $\alpha > 0$ and $\alpha < 0$. Later, Zhou and Vukičević [27] established the inequalities between $R_\alpha$, $Q_\alpha$, $Q_{2\alpha}$ and $Q_{2\alpha+1}$. In this paper, we make a step forward by deriving the explicit relationships between $R_\alpha$ and $Q_\alpha$ for $\alpha > 0$ and $\alpha < 0$. Also, we obtain linear inequalities between $R_\alpha$ and $Q_\alpha$ for $\alpha > 0$ and $\alpha < 0$, where those $\alpha < 0$ that holds some condition on the order of graph. Moreover, we obtain new inequality between $M_2(G)/m$ and $M_1(G)/n$ for any graph $G$ with order $n$ and size $m$. Further, we determine new inequality between $R(G)/m$ and $0^R(G)/n$.

2 Some known results

In this section we review some known results that will be used in our main results.

Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ be positive real numbers such that for $1 \leq i \leq n$, it holds that $p \leq p_i \leq P$ and $q \leq q_i \leq Q$. Then,
\[ \left| n \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i \right| \leq \tau(n) (P - p) (Q - q), \tag{1} \]
where $\tau(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$. Further, equality attains if and only if $p_1 = p_2 = \cdots = p_n$ and $q_1 = q_2 = \cdots = q_n$ [3].

Rodríguez et al. [24] established the following relationships between $Q_\alpha$ and $Q_{\alpha+1}$.
Let $G$ be a nontrivial graph having the parameters $n$ and $m$. Then, for $\alpha > 0$,

$$Q_{\alpha+1}(G) \geq \frac{2m}{n} Q_{\alpha}(G)$$  \hspace{1cm} (2)

and for $\alpha < 0$,

$$Q_{\alpha+1}(G) \leq \frac{2m}{n} Q_{\alpha}(G).$$  \hspace{1cm} (3)

Equality attains in each case if and only if $G$ is regular.

Also, Rodríguez et al. [24] derived the following relation between $Q_{\alpha}$ and $Q_{2\alpha}$.

If $G$ is a nontrivial graph with the parameters $n$, $\delta$ and $\Delta$, then for $\alpha < 0$,

$$Q_{2\alpha}(G) \leq \frac{1}{4n} \left[ \left( \frac{\Delta}{\delta} \right)^{\alpha} + \left( \frac{\delta}{\Delta} \right)^{\alpha} + 2 \right] Q_{\alpha}^{2}(G).$$  \hspace{1cm} (4)

Further, equality attains if and only if $G$ is regular.

Liu and Gutman [19] derived the following implicit quadratic inequality between $R_{\alpha}$ and $Q_{\alpha}$.

If $G$ is a nontrivial graph with the parameters $n$, $\delta$ and $\Delta$, then for $\alpha > 0$,

$$R_{\alpha}(G) \leq \frac{1}{2} Q_{\alpha}(G) \left[ \left( 1 - \frac{1}{n} \right) Q_{\alpha}(G) + (\Delta - n + 1)\delta^{\alpha} \right]$$  \hspace{1cm} (5)

and the equality is achieved if and only if $G$ is regular.

Also, Liu and Gutman [19] established the following inequality between $R_{\alpha}$, $Q_{\alpha}$, $Q_{\alpha+1}$ and $Q_{2\alpha}$.

If $G$ is a nontrivial graph having the parameters $n$ and $\delta$, then for $\alpha < 0$,

$$R_{\alpha}(G) \geq \frac{1}{2} \left[ Q_{\alpha}^{2}(G) - (n - 1)\delta^{\alpha} Q_{\alpha}(G) + \delta^{\alpha} Q_{\alpha+1}(G) - Q_{2\alpha}(G) \right].$$  \hspace{1cm} (6)

Further, equality attains if and only if $G$ is regular.

### 3 Main results

**Lemma 1.** Let $G$ be a nontrivial graph, then for any real number $\alpha$,

$$Q_{\alpha+1}(G) = \sum_{i=1}^{n} S_i(\alpha),$$  \hspace{1cm} (7)

where $S_i(\alpha) = \sum_{v \in N(v_i)} d_i^{v^\alpha}.$

**Proof.**

$$Q_{\alpha+1}(G) = \sum_{i=1}^{n} d_i^{\alpha+1} = \sum_{i=1}^{n} d_i d_i^{\alpha} = d_1 d_1^{\alpha} + d_2 d_2^{\alpha} + \cdots + d_n d_n^{\alpha}$$

$$= \underbrace{d_1^{\alpha} + \cdots + d_1^{\alpha}}_{d_1 \text{ times}} + \underbrace{d_2^{\alpha} + \cdots + d_2^{\alpha}}_{d_2 \text{ times}} + \cdots + \underbrace{d_n^{\alpha} + \cdots + d_n^{\alpha}}_{d_n \text{ times}}$$
By rearranging with respect to the sum of degrees of neighbor vertices of each vertex \( v_i \), we have

\[ Q_{\alpha+1}(G) = \sum_{i=1}^{n} \sum_{v_j \in N(v_i)} d_j^\alpha. \]

By setting \( S_i(\alpha) = \sum_{v_j \in N(v_i)} d_j^\alpha \), the required result follows.

**Lemma 2.** Let \( G \) be a nontrivial graph, then for any real number \( \alpha \),

\[ R_\alpha(G) = \frac{1}{2} \sum_{i=1}^{n} d_i^\alpha S_i(\alpha), \quad (8) \]

where \( S_i(\alpha) = \sum_{v_j \in N(v_i)} d_j^\alpha \).

**Proof.**

\[
R_\alpha(G) = \frac{1}{2} \sum_{v_i \sim v_j} 2d_i^\alpha d_j^\alpha \\
= \frac{1}{2} \left[ d_1^\alpha \sum_{v_j \in N(v_1)} d_j^\alpha + d_2^\alpha \sum_{v_j \in N(v_2)} d_j^\alpha + \cdots + d_n^\alpha \sum_{v_j \in N(v_n)} d_j^\alpha \right] = \frac{1}{2} \sum_{i=1}^{n} d_i^\alpha \sum_{v_j \in N(v_i)} d_j^\alpha.
\]

By taking \( S_i(\alpha) = \sum_{v_j \in N(v_i)} d_j^\alpha \), the required result follows.

In the following Theorem, we derive the left and right explicit inequalities between \( R_\alpha \) and \( Q_\alpha \) for \( \alpha > 0 \) and \( \alpha < 0 \) respectively.

**Theorem 1.** Let \( G \) be a nontrivial graph with order \( n \), size \( m \), minimum vertex-degree \( \delta \) and maximum vertex-degree \( \Delta \). Then, the following left and right inequalities hold for \( \alpha > 0 \) and \( \alpha < 0 \) respectively:

\[
-\phi(m,n,\alpha) + \left( \frac{Q_\alpha(G)}{n} \right)^2 \leq \frac{R_\alpha(G)}{m} \leq \left( \frac{Q_\alpha(G)}{n} \right)^2 + \phi(m,n,\alpha), \quad (9)
\]

where \( \phi(m,n,\alpha) = \frac{\tau(n)}{2mn} (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) \) and \( \tau(n) = n \left[ \frac{n}{2} \right] (1 - \frac{1}{n} \left[ \frac{n}{2} \right]) \). Further, each equality holds if and only if \( G \) is a regular graph.

**Proof.** We choose \( p_i = d_i^\alpha \) and \( q_i = S_i(\alpha) \) in inequality (1), then for \( \alpha \geq 0 \), \( \delta^\alpha \leq d_i^\alpha \leq \Delta^\alpha \) and \( \delta^{2\alpha} \leq S_i(\alpha) \leq \Delta^{2\alpha} \) and for \( \alpha \leq 0 \), \( \Delta^\alpha \leq d_i^\alpha \leq \delta^\alpha \) and \( \Delta^{2\alpha} \leq S_i(\alpha) \leq \delta^{2\alpha} \), where \( i = 1, 2, \cdots, n \).

Note that for any real number \( \alpha \), \( (\Delta^\alpha - \delta^\alpha) (\Delta^{2\alpha} - \delta^{2\alpha}) = (\delta^\alpha - \Delta^\alpha) (\delta^{2\alpha} - \Delta^{2\alpha}) \). Then, for any real number \( \alpha \), inequality (1) takes the form

\[
\left| n \sum_{i=1}^{n} d_i^\alpha S_i(\alpha) - \sum_{i=1}^{n} d_i^\alpha \sum_{i=1}^{n} S_i(\alpha) \right| \leq \tau(n) (\Delta^\alpha - \delta^\alpha) (\Delta^{2\alpha} - \delta^{2\alpha}),
\]
where \( \tau(n) = n \left[ \frac{n}{2} \right] (1 - \frac{1}{n} \left[ \frac{n}{2} \right]) \).

From (7) and (8), we have

\[
|2nR_\alpha(G) - Q_\alpha(G)Q_{\alpha+1}(G)| \leq \tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha).
\]

This implies that

\[
-\tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) \leq 2nR_\alpha(G) - Q_\alpha(G)Q_{\alpha+1}(G) \leq \tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha).
\]

This gives

\[
-\tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) + Q_\alpha(G)Q_{\alpha+1}(G) \leq 2nR_\alpha(G)
\]

\[
\leq Q_\alpha(G)Q_{\alpha+1}(G) + \tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha).
\]

By using (2) with \( \alpha > 0 \) and (3) with \( \alpha < 0 \) in the left and right inequalities respectively, we have

\[
-\tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) + \frac{2m}{n} (Q_\alpha(G))^2 \leq 2nR_\alpha(G)
\]

\[
\leq \frac{2m}{n} (Q_\alpha(G))^2 + \tau(n) (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha).
\]

By taking \( \phi(m, n, \alpha) = \frac{\tau(n)}{2mn} (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) \), the required inequality (9) follows.

Since equality attains in (1) if and only if \( p_1 = p_2 = \cdots = p_n \) and \( q_1 = q_2 = \cdots = q_n \). This gives that equality attains in (9) if and only if \( d_1^\alpha = d_2^\alpha = \cdots = d_n^\alpha \) and \( S_1(\alpha) = S_2(\alpha) = \cdots = S_n(\alpha) \). Also, \( d_1^\alpha = d_2^\alpha = \cdots = d_n^\alpha \) and \( S_1(\alpha) = S_2(\alpha) = \cdots = S_n(\alpha) \) implies \( d_1 = d_2 = \cdots = d_n \) and \( S_1 = S_2 = \cdots = S_n \). This recommends that each equality in (9) attains if and only if \( G \) is a regular graph.

In the following corollary, we derive the linear inequality between \( Q_\alpha \) and \( R_\alpha \) for any positive real number \( \alpha \).

**Corollary 1.** Let \( G \) be a nontrivial graph having order \( n \), size \( m \), minimum vertex-degree \( \delta \) and maximum vertex-degree \( \Delta \) with \( n(n-1) \neq 2m \). Then, for \( \alpha > 0 \) we have

\[
R_\alpha(G) \geq \frac{m}{n(n-1) - 2m} \left[(n-\Delta-1)\delta^\alpha Q_\alpha(G) - n(n-1)\phi(m, n, \alpha)\right],
\]

where \( \phi(m, n, \alpha) = \frac{\tau(n)}{2mn} (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha) \) and \( \tau(n) = n \left[ \frac{n}{2} \right] (1 - \frac{1}{n} \left[ \frac{n}{2} \right]) \). Moreover, equality attains if and only if \( G \) is a regular graph.
Proof. From inequality (5) with $\alpha > 0$, we have

$$R_\alpha(G) \leq \frac{n(n-1)}{2} \left( \frac{Q_\alpha(G)}{n} \right)^2 + \frac{1}{2} (\Delta - n + 1) \delta^\alpha Q_\alpha(G). \tag{11}$$

Also, from left inequality (9), we get

$$\left( \frac{Q_\alpha(G)}{n} \right)^2 \leq \frac{R_\alpha(G)}{m} + \phi(m,n,\alpha), \tag{12}$$

where $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn} (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha)$ and $\tau(n) = n \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right)$.

By using inequality (12) in inequality (11), we have

$$R_\alpha(G) \leq \frac{n(n-1)}{2} \left( \frac{R_\alpha(G)}{m} + \phi(m,n,\alpha) \right) + \frac{1}{2} (\Delta - n + 1) \delta^\alpha Q_\alpha(G), \tag{13}$$

After simplifying (13) and then rearranging, we get the desired inequality (10).

Since the equality attains in both inequality (5) and left inequality (9) if and only if $G$ is a regular graph. Therefore, equality in (10) holds if and only if $G$ is a regular graph. \qed

Lemma 3. For $\alpha < 0$, it is easy to observe that

$$Q_{\alpha+1}(G) \geq \delta Q_\alpha, \tag{14}$$

where equality attains if and only if $G$ is a regular graph.

In the upcoming corollary, we derive the linear inequality between $R_\alpha$ and $Q_\alpha$ for any negative real number $\alpha$ which satisfies some condition on the order of graph $G$.

Corollary 2. Let $G$ be a nontrivial graph having order $n$, size $m$, minimum vertex-degree $\delta$ and maximum vertex-degree $\Delta$. Then, for $\alpha < 0$ and $n > \lambda/4$,

$$R_\alpha(G) \leq \frac{m}{n(4n-\lambda) - 8m} \left[ 4n^\alpha(n-\delta-1)Q_\alpha(G) + n(4n-\lambda) \psi(m,n,\alpha) \right], \tag{15}$$

where $\lambda = \lambda(\alpha) = \left( \frac{\Delta}{\delta} \right)^\alpha + \left( \frac{\delta}{\Delta} \right)^\alpha + 2$, $\phi(m,n,\alpha) = \frac{\tau(n)}{2mn} (\Delta^\alpha - \delta^\alpha)^2 (\Delta^\alpha + \delta^\alpha)$ and $\tau(n) = n \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right)$. Further, equality attains if and only if $G$ is a regular graph.

Proof. From inequalities (4) and (14) with $\alpha < 0$, the inequality (6) becomes

$$R_\alpha(G) \geq \frac{1}{2} \left[ 1 - \frac{1}{4n} \left( \left( \frac{\Delta}{\delta} \right)^\alpha + \left( \frac{\delta}{\Delta} \right)^\alpha + 2 \right) \right] Q_\alpha^2(G) - (n-\delta-1) \delta^\alpha Q_\alpha(G).$$

By taking $\lambda = \lambda(\alpha) = \left( \frac{\Delta}{\delta} \right)^\alpha + \left( \frac{\delta}{\Delta} \right)^\alpha + 2$ and rearranging, we have

$$R_\alpha(G) \geq \frac{1}{2} \left[ n \left( n - \frac{\lambda}{4 \alpha} \right) \left( \frac{Q_\alpha(G)}{n} \right)^2 - (n-\delta-1) \delta^\alpha Q_\alpha(G) \right].$$
Also, from right inequality (9) with $\alpha < 0$ and $n > \frac{1}{4}$, we get

$$R_{\alpha}(G) \geq \frac{n(4n - \lambda)}{8} \left[ \frac{R_{\alpha}(G)}{m} - \phi(m, n, \alpha) \right] - \frac{(n - \delta - 1)}{2} \delta_{\alpha} Q_{\alpha}(G),$$

(16)

where $\phi(m, n, \alpha) = \frac{\tau(n)}{2mn}(\Delta^\alpha - \delta^\alpha)(\Delta^\alpha + \delta^\alpha)$ and $\tau(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$.

After simplifying (16), we achieve the desired inequality (15).

Since equality attains in right inequality (9) and each of the inequalities (4), (14) and (6) if and only if $G$ is a regular graph. This implies that equality in (15) attains if and only if $G$ is a regular graph.

In the following corollary, we get a new inequality between $M_2(G)/m$ and $M_1(G)/n$ for any graph $G$ with order $n$ and size $m$ by taking $\alpha = 2$ in the left inequality (9).

**Corollary 3.** Let $G$ be a nontrivial graph with order $n$ and size $m$, then

$$\frac{M_2(G)}{m} \geq \left( \frac{M_1(G)}{n} \right)^2 - \phi(m, n),$$

(17)

where $\phi(m, n) = \frac{\tau(n)}{2mn}(\Delta^4 - \delta^4)(\Delta^2 - \delta^2)$ and $\tau(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$. Further, each equality holds if and only if $G$ is a regular graph.

In the following corollary, we obtain a new inequality between $R(G)/m$ and $0^0 R(G)/n$ for any graph $G$ with order $n$ and size $m$ by setting $\alpha = -1/2$ in the right inequality (9).

**Corollary 4.** Let $G$ be a nontrivial graph having order $n$ and size $m$, then

$$\frac{R(G)}{m} \leq \left( \frac{0^0 R(G)}{n} \right)^2 + \psi(m, n),$$

(18)

where $\psi(m, n) = \frac{\tau(n)(\Delta - \delta)(\sqrt{\Delta} - \sqrt{\delta})}{2mn(\Delta \delta)^{\frac{3}{2}}}$ and $\tau(n) = n \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right)$. Moreover, each equality holds if and only if $G$ is a regular graph.

**4 Conclusions**

A major contribution of this paper lies in the derivation of explicit inequality relationships between the general Randić and zeroth-order general Randić indices. Furthermore, the paper goes beyond the implicit relationships and determines linear inequality relationships between the general Randić and zeroth-order general Randić indices, providing a more comprehensive framework for their comparison. We would like to conclude this paper by proposing the following open problem:
Open Problem 1. Drive the linear inequality between the general Randić index $R_\alpha$ and zeroth-order general Randić index $Q_\alpha$ for any negative real number $\alpha$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


