On the skew characteristics polynomial/eigenvalues of operations on bipartite oriented graphs and applications

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Abstract. Let $\overrightarrow{G}$ be an oriented graph with n vertices and m arcs having underlying graph G. The skew matrix of $\overrightarrow{G}$, denoted by $S(\overrightarrow{G})$ is a $(-1,0,1)$-skew symmetric matrix. The skew eigenvalues of $\overrightarrow{G}$ are the eigenvalues of $S(\overrightarrow{G})$ and its characteristic polynomial is the skew characteristic polynomial of $\overrightarrow{G}$. The sum of the absolute values of the skew eigenvalues is the skew energy of $\overrightarrow{G}$ and is denoted by $E_S(\overrightarrow{G})$. In this paper, we study the skew characteristic polynomial and skew eigenvalues of joined union of oriented bipartite graphs and some of its variations. We show that the skew eigenvalues of the joined union of oriented bipartite graphs and some variations of oriented bipartite graphs is the union of the skew eigenvalues of the component oriented graphs except some eigenvalues, which are given by an auxiliary matrix associated with the joined union. As a special case we obtain the skew eigenvalues of join of two oriented bipartite graphs and the lexicographic product of an oriented graph and an oriented bipartite graph. Some examples of orientations of well-known graphs are presented to highlight the importance of the results. As applications to our result we obtain some new infinite families of skew equienergetic oriented graphs. Our results extend and generalize some of the results obtained in [C. Adiga and B.R. Rakshith, More skew-equienergetic digraphs, Commun. Comb. Optim., 1(1) (2016) 55-71].

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1 Introduction

Let G be a simple graph having n vertices and m edges. The vertex set is $\{v_1, v_2, \ldots, v_n\}$. Let $\overrightarrow{G}$ be a digraph, where edge is assigned arbitrarily a direction. The digraph $\overrightarrow{G}$ is said to be an orientation of G or oriented graph associated with G. The graph G is viewed as the underlying graph of $\overrightarrow{G}$. Let $d_i^+ = d^+(v_i)$ be the out-degree, $d_i^- = d^-(v_i)$ be the in-degree and $d_i = d_i^+ + d_i^-$ be the degree of the vertex $v_i \in V(\overrightarrow{G})$. Let $N_{\overrightarrow{G}}^+(v_i)$ be the set of out-neighbours, $N_{\overrightarrow{G}}^-(v_i)$ be the set of in-neighbours and $N_{\overrightarrow{G}}(v_i) = N_{\overrightarrow{G}}^+(v_i) \cup N_{\overrightarrow{G}}^-(v_i)$ be the set of neighbours of the vertex v_i in $\overrightarrow{G}$. The adjacency matrix $A(G) = (a_{ij})$ of a graph G is a n-square matrix with $a_{ij} = 1$, if there is an edge between the vertices v_i and v_j and $a_{ij} = 0$, otherwise. All the eigenvalues of $A(G)$ are real numbers as it is a real symmetric matrix. The eigenvalues of the matrix $A(G)$ are called eigenvalues (or adjacency eigenvalues) of G and are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The sum of the absolute values of the eigenvalues of G is called energy of G and is denoted by $\mathcal{E}(G)$. That is,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This spectral graph invariant is one among the most studied spectral graph invariants in spectral graph theory because of its applications in mathematical and other sciences. For some recent works on energy of graphs, we refer to [3] and the book [16].

The skew adjacency matrix $S = S(\overrightarrow{G}) = (s_{ij})$ of an oriented graph $\overrightarrow{G}$ is an $n \times n$ matrix with $s_{ij} = 1$ when there is an arc from v_i to v_j, and $s_{ij} = -1$ when there is an arc from v_j to v_i, and $s_{ij} = 0$ otherwise. It is clear that the matrix $S(\overrightarrow{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The characteristic polynomial of $S(\overrightarrow{G})$ is the skew characteristic polynomial of $\overrightarrow{G}$ and is denoted by $P_s(\overrightarrow{G}, x)$. The zeros of the polynomial $P_s(\overrightarrow{G}, x)$ are the eigenvalues of the matrix $S(\overrightarrow{G})$ and are called skew eigenvalues of $\overrightarrow{G}$. The skew spectrum of $\overrightarrow{G}$ is denoted by $Sp_s(\overrightarrow{G})$, which describes the eigenvalues of $S(\overrightarrow{G})$ as well as their multiplicities.

The skew energy of the oriented graph $\overrightarrow{G}$ is called the energy of the matrix $S(\overrightarrow{G})$. It is defined by the following equation.

$$E_s(\overrightarrow{G}) = \sum_{i=1}^{n} |\xi_i|,$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the skew eigenvalues of $\overrightarrow{G}$. This type of spectral invariant appears in the literature with numerous results regarding their bounds and it has abundant connections with the different graph parameters like matching number, vertex covering number and independence number, its connections with the skew rank (the rank of the matrix $S(\overrightarrow{G})$ is called skew rank of $\overrightarrow{G}$). One of the most studied problems in the theory of skew energy is the determination of extremal oriented graphs for $E_s(\overrightarrow{G})$ in a given class of oriented graphs. In fact, due to the hardness of this problem, many researchers have started with a graph G and tried to find the orientations of G which attain the extremal value for $E_s(\overrightarrow{G})$. This problem is the topic of many papers in literature. Some recent examples can be found in [8, 23]. We refer to [5, 10, 11, 14, 15, 19–21, 24] for more development of skew energy theory.

Given a cycle $C_k = u_1u_2\dots u_ku_1$, its sign is signified as $sgn(C_k) = s_{12}s_{23}\dots s_{k-1k}s_{k1}$. Here, s_{ij} means the entry of the skew matrix $S(\overrightarrow{G})$ in the intersection of u_i row and u_j column. If the sign of an even oriented cycle C_k is positive or negative, it is referred to as evenly-oriented or oddly-oriented, respectively. We say $\overrightarrow{G}$ is evenly-oriented if every even cycle in $\overrightarrow{G}$ is evenly-oriented. When $sgn(C_{2k}) = (-1)^k$, the even oriented cycle C_{2k} becomes uniformly oriented.

The rest of the papers is organized as follows. In Section 2, we study the joined union of oriented bipartite graphs and some of its variations. We obtain the skew spectrum of joined union of oriented bipartite graphs and its some of its variations, in terms of the component oriented graphs and an auxiliary matrix determined by the operation. In Section 3, we use the results obtained in Section 2 to obtain the skew spectrum of various families of oriented graphs. As applications to results obtained in Section 2 and 3, we construct various new families of skew equienergetic oriented graphs in Section 4.

2 The skew spectrum of joined union of oriented graphs

Consider an $n \times n$ complex matrix

$$M = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1s} \\ X_{21} & X_{22} & \cdots & X_{2s} \\ \vdots & \vdots & \cdots & \vdots \\ X_{s1} & X_{s2} & \cdots & X_{ss} \end{pmatrix}, \tag{2.1}$$

where X_{ij} is an $n_i \times n_j$ block matrix for $1 \le i, j \le s$ and $n = \sum_{i=1}^{s} n_i$. The element b_{ij} is the average row sum of X_{ij}. We define an $s \times s$ matrix with elements being the average row sums of X_{ij} and we call it the quotient matrix $B = (b_{ij})$. The matrix B becomes a *equitable quotient matrix* when each block X_{ij} has constant row sum. A complex matrix has a connection with *equitable quotient matrix* in terms of its spectrum as below [26].

Lemma 2.1 *The equitable quotient matrix B and the matrix M defined in (2.1) share the same eigenvalues.*

The generalized join (also called joined union) of graphs has different versions of definition. The spectrum of generalized join of graphs in terms of different matrices has been investigated in [9, 18, 24]. The joined union was extended to digraphs in [9]. In [9], the author have discussed the A_α-spectrum of the joined union of diagonalizable digraphs and as applications the A_α-spectrum of various families of digraphs are found. Recently, in [12], the authors defined generalized join of oriented graphs as follows:

Let $\overrightarrow{G}(V, E)$ be an oriented graph of order n and let $\overrightarrow{G}_i(V_i, E_i)$ be oriented graphs of order n_i, where $i = 1, \dots, n$. The *joined union* of the oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \dots, \overrightarrow{G}_n$ with respect to oriented graph $\overrightarrow{G}$ is denoted by $\overrightarrow{G}[\overrightarrow{G}_1, \dots, \overrightarrow{G}_n]$ and is defined as the oriented graph $\overrightarrow{H}(W, F)$

with vertex set $W = \bigcup_{i=1}^{n} V_i$ and arc set

$$F = \bigcup_{i=1}^{n} E_i \cup \Big\{(u, v) \in E(H), \quad \text{whenever} \ \ u \in \overrightarrow{G}_i, v \in \overrightarrow{G}_j \quad \text{and} \ \ v_j \in N^{+}_{\overrightarrow{G}}(v_i)\Big\}.$$

In other words, the joined union is the union of oriented graphs $\overrightarrow{G}_1, \dots, \overrightarrow{G}_n$ together with the arcs (v_{ik}, v_{jl}), where $v_{ik} \in \overrightarrow{G}_i$ and $v_{jl} \in \overrightarrow{G}_j$, whenever (v_i, v_j) is an arc in $\overrightarrow{G}$. Clearly, the usual join of two oriented graphs $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ defined in [22] is a special case of the joined union $\overrightarrow{K}_2[\overrightarrow{G}_1, \overrightarrow{G}_2] = \overrightarrow{G}_1 \to \overrightarrow{G}_2$ where $\overrightarrow{K}_2$ is the oriented graph corresponding to the complete graph of order 2. By taking each of the component in joined union as bipartite oriented graphs, we can define the following variations of the *joined union* of the oriented graphs.

Let $\overrightarrow{G}(V, E)$ be an oriented graph of order n and let $\overrightarrow{G}_i = \overrightarrow{G}_i(V_i, U_i)$, be a bipartite oriented graph with partite sets V_i and U_i, for all $i = 1, 2, \dots, n$. Let $\overrightarrow{H}_1$ be the *joined union* of the oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \dots, \overrightarrow{G}_n$ with respect to oriented graph $\overrightarrow{G}$. That is, $\overrightarrow{H}_1 = \overrightarrow{G}[\overrightarrow{G}_1, \dots, \overrightarrow{G}_n]$. Note that if there is an arc between the vertices v_i and v_j in $\overrightarrow{G}$, then there are arcs between all the vertices of V_i and V_j; between all the vertices of V_i and U_j; between all the vertices of U_i and V_j and between all the vertices of U_i and U_j. Let $\overrightarrow{H}_2$ be the oriented graph obtained from $\overrightarrow{H}_1$ by deleting all the arcs between U_i and V_j and all the arcs between U_i and U_j. Let $\overrightarrow{H}_3$ be the oriented graph obtained from $\overrightarrow{H}_1$ by deleting all the arcs between V_i and V_j and all the arcs between V_i and U_j.

A digraph D is said to Eulerian if the out-degree of any vertex in D is same as its in-degree, that is, $d_i^+ = d_i^-$, for all $v_i \in V(D)$. The following theorem was obtained in [12] and gives the skew spectrum of the joined union of Eulerian oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \dots, \overrightarrow{G}_n$, in terms of the skew spectrum of the component oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \dots, \overrightarrow{G}_n$ and the eigenvalues of an auxiliary matrix determined by the joined union.

Theorem 2.2 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$ having m arcs. Let $\overrightarrow{G}_i$ be Eulerian oriented graph of order n_i having skew characteristic polynomial $P_s(\overrightarrow{G}_i, x)$, where $i = 1, 2, \dots, n$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{G}[\overrightarrow{G}_1, \dots, \overrightarrow{G}_n]$ of order $N = \sum\limits_{i=1}^{n} n_i$ is*

$$P_s(\overrightarrow{G}[\overrightarrow{G}_1, \dots, \overrightarrow{G}_n], x) = \frac{\phi(M, x)}{x^n} \prod_{i=1}^{n} P_s(\overrightarrow{G}_i, x), \tag{2.2}$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 0 & \psi_{12} & \dots & \psi_{1n} \\ \psi_{21} & 0 & \dots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \dots & 0 \end{pmatrix},$$

where $\psi_{ij} = n_j$, if there is an arc from v_i to v_j; $\psi_{ij} = -n_j$, if there is an arc from v_j to v_i and $\psi_{ij} = 0$, if there is no arc between v_i and v_j.

It is clear that Theorem 2.2 is applicable to Eulerian oriented graphs only. However, in the next theorem we will show that for the bipartite oriented digraphs, the condition of being Eulerian can be relaxed.

For $i = 1, 2, \ldots, n$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with partite sets V_i and U_i of same cardinality n_i, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \mathbf{e}_{n_i} = r_i \mathbf{e}_{n_i}$ and $\mathbf{e}$ is the all one column vector.

In the next theorem we determine the skew characteristic polynomial of the joined union of oriented bipartite graphs $\overrightarrow{B}_1, \overrightarrow{B}_2, \ldots, \overrightarrow{B}_n$, in terms of the skew characteristic polynomial of the component oriented graphs and the eigenvalues of an auxiliary matrix determined by the joined union.

Theorem 2.3 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$ having m arcs. For $i = 1, 2, \ldots, n$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \boldsymbol{e}_{n_i} = r_i \boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2, \ldots, n$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of the oriented graph $H_1 = \overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]$ of order $N = 2\sum_{i=1}^{n} n_i$ is*

$$P_s(\overrightarrow{H}_1, x) = \phi(M, x) \prod_{i=1}^{n} \frac{P_s(\overrightarrow{B}_i, x)}{(x^2 + r_i^2)}, \tag{2.3}$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \ldots & \phi_{1n} \\ \phi_{21} & \phi_2 & \ldots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \ldots & \phi_n \end{pmatrix},$$

where $\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}$; $\phi_{ij} = \begin{pmatrix} n_j & n_j \\ n_j & n_j \end{pmatrix}$, if there is an arc from v_i to v_j; $\phi_{ij} = \begin{pmatrix} -n_j & -n_j \\ -n_j & -n_j \end{pmatrix}$, if there is an arc from v_j to v_i and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if there is no arc between v_i to v_j.

Proof. Let $V(\overrightarrow{G}) = \{v_1, \ldots, v_n\}$ be the vertex set of $\overrightarrow{G}$ and let $V(\overrightarrow{B}_i) = \{x_{i1}, \ldots, x_{in_i}, y_{i1}, \ldots, y_{in_i}\}$ be the vertex set of $\overrightarrow{B}_i$, for $i = 1, 2, \ldots, n$. Let $H_1 = \overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]$. Let us label the vertices in H_1 in such a way that the vertices in $\overrightarrow{B}_1$ are labelled first, the vertices of $\overrightarrow{B}_2$ are labelled after the vertices in $\overrightarrow{B}_1$, and so on. With this labelling, the skew matrix of $\overrightarrow{H}_1$ takes the form

$$S(\overrightarrow{H}_1) = \begin{pmatrix} \Gamma_1 & \Gamma_{12} & \ldots & \Gamma_{1n} \\ \Gamma_{21} & \Gamma_2 & \ldots & \Gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n1} & \Gamma_{n2} & \ldots & \Gamma_n \end{pmatrix},$$

where,

$$\Gamma_i = S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}, \qquad \text{for } i = 1, 2, \dots, n,$$

and $\Gamma_{ij} = \begin{pmatrix} A_{n_i\times n_j} & B_{n_i\times n_j} \\ C_{n_j\times n_i} & D_{n_j\times n_i} \end{pmatrix}$, with $A_{n_i\times n_j} = B_{n_i\times n_j} = J_{n_i\times n_j}$ and $C_{n_j\times n_i} = D_{n_j\times n_i} = J_{n_j\times n_i}$, if $(v_i, v_j) \in E(\overrightarrow{G})$; $A_{n_i\times n_j} = B_{n_i\times n_j} = -J_{n_i\times n_j}$ and $C_{n_j\times n_i} = D_{n_j\times n_i} = -J_{n_j\times n_i}$, if $(v_j, v_i) \in E(\overrightarrow{G})$ and $A_{n_i\times n_j} = B_{n_i\times n_j} = 0_{n_i\times n_j}$ and $C_{n_j\times n_i} = D_{n_j\times n_i} = 0_{n_j\times n_i}$, if $(v_i, v_j), (v_j, v_i) \notin E(\overrightarrow{G})$. Note that $J_{n_i\times n_j}$ is the all one matrix of order $n_i \times n_j$ and $0_{n_i\times n_j}$ is the zero matrix of order $n_i \times n_j$.

By assumption $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$ is a bipartite oriented graph for all i with partite sets V_i and U_i of same cardinality n_i and skew matrix, $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0, 1)$-matrix satisfying $X_i\mathbf{e}_{n_i} = r_i\mathbf{e}_{n_i}$. It is easy to verify that ιr_i is an eigenvalue of $S(\overrightarrow{B}_i)$ with corresponding eigenvector $\mathbf{e}_{2n_i} = \begin{pmatrix} -\iota\mathbf{e}_{n_i} \\ \mathbf{e}_{n_i} \end{pmatrix}$. Similarly, we can verify that $-\iota r_i$ is an eigenvalue of $S(\overrightarrow{B}_i)$ with corresponding eigenvector $\begin{pmatrix} \iota\mathbf{e}_{n_i} \\ \mathbf{e}_{n_i} \end{pmatrix}$. Since $S(\overrightarrow{B}_i)$ is a skew symmetric matrix, so it is a diagonalisable matrix with its $2n_i$ eigenvectors forming an orthogonal set. Let λ_{ik} be an eigenvalue of $S(\overrightarrow{B}_i)$ other than $\pm\iota r_i$ with the corresponding eigenvector $X = (s_{i1}, s_{i2}, \dots, s_{in_i}, t_{i1}, t_{i2}, \dots, t_{in_i})^T$ satisfying $\mathbf{e}_{2n_i}^T X = 0$. That is, $-\iota \sum_{k=1}^{n_i} s_{ik} + \sum_{k=1}^{n_i} t_{ik} = 0$. Now, consider the vector $Y = (y_1, y_2, \dots, y_N)^T$, where

$$y_j = \begin{cases} s_{ij} & \text{if } v_{ij} \in V(\overrightarrow{B}_i) \cap V_i \\ t_{ij} & \text{if } v_{ij} \in V(\overrightarrow{B}_i) \cap U_i \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathbf{e}_{2n_i}^T X = 0$ gives that $\Gamma_{ij}X = 0$ and coordinates of the vector Y corresponding to vertices of $\overrightarrow{H}_1$ which are not in $\overrightarrow{B}_i$ are zeros, we have

$$S(\overrightarrow{H}_1)Y = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{ik}X \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_{ik}Y.$$

This shows that Y is an eigenvector of $S(\overrightarrow{H}_1)$ corresponding to the eigenvalue λ_{ik} and so every eigenvalue λ_{ik} (other than $\pm\iota r_i$) of $S(\overrightarrow{B}_i)$ is an eigenvalue of $S(\overrightarrow{H}_1)$. So, using this process we will obtain $\sum_{i=1}^{n} 2n_i - 2n = N - 2n$ eigenvalues of $S(\overrightarrow{H}_1)$. To determine the remaining $2n$ eigenvalues of $S(\overrightarrow{H}_1)$, we use the equitable quotient matrix. The equitable quotient matrix of

$S(\overrightarrow{H}_1)$ is

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_2 & \dots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_n \end{pmatrix},$$

where $\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}$; $\phi_{ij} = \begin{pmatrix} n_j & n_j \\ n_j & n_j \end{pmatrix}$, if $(v_i, v_j) \in E(\overrightarrow{B}_i)$; $\phi_{ij} = \begin{pmatrix} -n_j & -n_j \\ -n_j & -n_j \end{pmatrix}$, if $(v_j, v_i) \in E(\overrightarrow{B}_i)$ and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if $(v_i, v_j), (v_j, v_i) \notin E(\overrightarrow{B}_i)$. Since by Lemma 2.1, the eigenvalues of M are the eigenvalues of $S(\overrightarrow{H}_1)$, the result follows. ■

The *lexicographic product* $G\Big[H\Big]$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge $(a, x)(b, y) \in E(G[H])$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. It is interesting to see that the lexicographic product $G\Big[H\Big]$ can be constructed by joined union $G[G_1, G_2, \dots, G_n]$ where $G_i = H$ for $1 \le i \le n$. Note that in the case $G_i = K_1$ we get $G[K_1, K_1, \dots, K_1] = G$.

If in particular the oriented bipartite graphs $\overrightarrow{B}_1, \overrightarrow{B}_2, \dots, \overrightarrow{B}_n$ in Theorem 2.2 are same, say $\overrightarrow{B}_i = \overrightarrow{B}_1$, for $2 \le i \le n$, then we obtain the following Theorem, which gives the skew spectrum of the joined union $\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_1]$, which represents an orientation of the *lexicographic product* $G\Big[B_1\Big]$.

Theorem 2.4 *Let $\overrightarrow{G}$ be an oriented graph of order $n \ge 2$ having m arcs. Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$, be a bipartite oriented graph with $|V_1| = |U_1| = n_1$, having the skew adjacency matrix $S(\overrightarrow{B}_1) = \begin{pmatrix} 0_{n_1 \times n_1} & X_1 \\ -X_1 & 0_{n_1 \times n_1} \end{pmatrix}$, where X_1 is a $(0,1)$-matrix satisfying $X_1 \boldsymbol{e}_{n_1} = r_1 \boldsymbol{e}_{n_1}$. Let $P_s(\overrightarrow{B}_1, x)$ be the skew characteristic polynomial of $\overrightarrow{B}_1$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_1]$ of order $N = 2nn_1$ is*

$$P_s(\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_1], x) = \phi(M, x) \Big[\frac{P_s(\overrightarrow{B}_1, x)}{(x^2 + r_i^2)} \Big]^n, \tag{2.4}$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_1 & \dots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_1 \end{pmatrix},$$

where $\phi_1 = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}$; $\phi_{ij} = n_1 J_2$, if there is an arc from v_i to v_j; $\phi_{ij} = -n_1 J_2$, if there is an arc from v_j to v_i and $\phi_{ij} = 0_2$, if there is no arc between v_i to v_j, where J_2 is the all one matrix of order 2×2 and 0_2 is the zero matrix of order 2×2.

Proof. If $\overrightarrow{B}_1 = \overrightarrow{B}_2 = \cdots = \overrightarrow{B}_n$, then from Theorem 2.3 the $2n$ eigenvalues of $\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_1]$ are given by the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \ldots & \phi_{1n} \\ \phi_{21} & \phi_1 & \ldots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \ldots & \phi_1 \end{pmatrix},$$

where $\phi_1 = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}$; $\phi_{ij} = \begin{pmatrix} n_1 & n_1 \\ n_1 & n_1 \end{pmatrix}$, if there is an arc from v_i to v_j; $\phi_{ij} = \begin{pmatrix} -n_1 & -n_1 \\ -n_1 & -n_1 \end{pmatrix}$, if there is an arc from v_j to v_i and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if there is no arc between v_i to v_j. With this the result now follows. ■

In the next theorem we determine the skew characteristic polynomial of the oriented graph $\overrightarrow{H}_2$, when the component oriented graphs are bipartite $\overrightarrow{B}_i(V_i, U_i)$ with partite sets of same cardinality n_i, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \mathbf{e}_{n_i} = r_i \mathbf{e}_{n_i}$.

Theorem 2.5 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$ having m arcs. For $i = 1, 2, \ldots, n$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \mathbf{e}_{n_i} = r_i \mathbf{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2, \ldots, n$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{H}_2$ of order $N = 2\sum\limits_{i=1}^{n} n_i$ is*

$$P_s(\overrightarrow{H}_2, x) = \phi(M, x) \prod_{i=1}^{n} \frac{P_s(\overrightarrow{B}_i, x)}{(x^2 + r_i^2)}, \tag{2.5}$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \ldots & \phi_{1n} \\ \phi_{21} & \phi_2 & \ldots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \ldots & \phi_n \end{pmatrix},$$

where $\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}$; $\phi_{ij} = \begin{pmatrix} n_j & n_j \\ 0 & 0 \end{pmatrix}$, if there is an arc from v_i to v_j; $\phi_{ij} = \begin{pmatrix} -n_j & -n_j \\ 0 & 0 \end{pmatrix}$, if there is an arc from v_j to v_i and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if there is no arc between v_i to v_j.

Proof. The proof follows on similar lines as in Theorem 2.3 and is therefore omitted. ■

In the next theorem we determine the skew characteristic polynomial of the oriented graph $\overrightarrow{H}_3$, when the component oriented graphs are bipartite $\overrightarrow{B}_i(V_i, U_i)$ with partite sets of same cardinality n_i, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i\mathbf{e}_{n_i} = r_i\mathbf{e}_{n_i}$.

Theorem 2.6 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$ having m arcs. For $i = 1, 2, \ldots, n$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2, \ldots, n$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{H}_3$ of order $N = 2\sum\limits_{i=1}^{n} n_i$ is*

$$P_s(\overrightarrow{H}_3, x) = \phi(M, x) \prod_{i=1}^{n} \frac{P_s(\overrightarrow{B}_i, x)}{(x^2 + r_i^2)}, \tag{2.6}$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_2 & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_n \end{pmatrix},$$

where $\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}$; $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ n_j & n_j \end{pmatrix}$, if there is an arc from v_i to v_j; $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ -n_j & -n_j \end{pmatrix}$, if there is an arc from v_j to v_i and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if there is no arc between v_i to v_j.

Proof. The proof follows on similar lines as in Theorem 2.3 and is therefore omitted. ■

Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$ and $\overrightarrow{B}_2 = \overrightarrow{B}_2(V_2, U_2)$ be two oriented bipartite graphs of order $2n_1$ and $2n_2$, respectively. Let $\overrightarrow{B} = \overrightarrow{B}_1 \to \overrightarrow{B}_2$ be the join of $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$. Clearly, $\overrightarrow{B} = \overrightarrow{K}_2[\overrightarrow{B}_1, \overrightarrow{B}_2]$. The following consequence of Theorem 2.3, gives the skew spectrum of the join of two oriented bipartite graphs. We note that Theorem 2.7 is Theorem 5 obtained in [2].

Theorem 2.7 *For $i = 1, 2$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{B}_1 \to \overrightarrow{B}_2$ is*

$$P_s(\overrightarrow{B}_1 \to \overrightarrow{B}_2, x) = (x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2r_2^2) \frac{P_s(\overrightarrow{B}_1, x)P_s(\overrightarrow{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)}. \tag{2.7}$$

Proof. The proof follows from Theorem 2.3 by taking $\overrightarrow{G} = \overrightarrow{K}_2$ and using the fact that the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 0 & r_1 & n_2 & n_2 \\ -r_1 & 0 & n_2 & n_2 \\ -n_1 & -n_1 & 0 & r_2 \\ -n_1 & -n_1 & -r_2 & 0 \end{pmatrix}$$

is $x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2r_2^2$. ∎

Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$, $\overrightarrow{B}_2 = \overrightarrow{B}_2(V_2, U_2)$ and $\overrightarrow{B}_3 = \overrightarrow{B}_3(V_3, U_3)$ be three oriented bipartite graphs of order $2n_1, 2n_2$ and $2n_3$, respectively. Let $\overrightarrow{G} = \overrightarrow{B}_1 \to (\overrightarrow{B}_2 \cup \overrightarrow{B}_3)$ be the join of $\overrightarrow{B}_1$ with the union of $\overrightarrow{B}_2$ and $\overrightarrow{B}_3$. It is easy to see that $\overrightarrow{G} = \overrightarrow{K}_{1,2}[\overrightarrow{B}_2, \overrightarrow{B}_1, \overrightarrow{B}_3]$, where $\overrightarrow{K}_{1,2}$ is the orientation of the star graph $K_{1,2}$ with arcs directed from vertex of degree 2. The following consequence of Theorem 2.3, gives the skew spectrum of the oriented graph $\overrightarrow{B}_1 \to (\overrightarrow{B}_2 \cup \overrightarrow{B}_3)$.

Theorem 2.8 *For $i = 1, 2, 3$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \boldsymbol{e}_{n_i} = r_i \boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2, 3$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of the oriented graph $\overrightarrow{B}_1 \to (\overrightarrow{B}_2 \cup \overrightarrow{B}_3)$ is*

$$P_s(\overrightarrow{B}_1 \to (\overrightarrow{B}_2 \cup \overrightarrow{B}_3), x) = \phi(M, x) \frac{P_s(\overrightarrow{B}_1, x) P_s(\overrightarrow{B}_2, x) P_s(\overrightarrow{B}_3, x)}{(x^2 + r_1^2)(x^2 + r_2^2)(x^2 + r_3^2)},$$

where $\phi(M, x)$ is the characteristic polynomial of matrix M given by

$$M = \begin{pmatrix} 0 & r_2 & -n_1 & -n_1 & 0 & 0 \\ -r_2 & 0 & -n_1 & -n_1 & 0 & 0 \\ n_2 & n_2 & 0 & r_1 & n_3 & n_3 \\ n_2 & n_2 & -r_1 & 0 & n_3 & n_3 \\ 0 & 0 & -n_1 & -n_1 & 0 & r_3 \\ 0 & 0 & -n_1 & -n_1 & -r_3 & 0 \end{pmatrix}.$$

Proof. The proof follows from Theorem 2.3 by taking $\overrightarrow{G} = \overrightarrow{K}_{1,2}$, where $\overrightarrow{K}_{1,2}$ is the orientation of star graph $K_{1,2}$ with arcs directed from vertex of degree 2. ∎

Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$ and $\overrightarrow{B}_2 = \overrightarrow{B}_2(V_2, U_2)$ be oriented bipartite graphs with partite sets U_1, V_1, U_2 and V_2, respectively. The join-1 of $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$, denoted by $\overrightarrow{B}_1 j_1 \overrightarrow{B}_2$, is defined in [2] as the oriented graph obtained from $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$ by joining arcs from all the vertices of U_1 to each the vertex of U_2 and V_2. The next Theorem was obtained as part first of Theorem 8 in [2] and gives the skew characteristic polynomial of join-1 $\overrightarrow{B}_1 j_1 \overrightarrow{B}_2$ of the oriented graphs $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$.

Theorem 2.9 *For $i = 1, 2$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of join-1 $\overrightarrow{B}_1 j_1 \overrightarrow{B}_2$ is*

$$P_s(\overrightarrow{B}_1 j_1 \overrightarrow{B}_2, x) = [x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2]\frac{P_s(\overrightarrow{B}_1, x)P_s(\overrightarrow{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)}.$$

Proof. The proof follows from Theorem 2.5 by taking $\overrightarrow{G} = \overrightarrow{K}_2$ and using the fact that the characteristic polynomial of matrix M given by

$$M = \begin{pmatrix} 0 & r_1 & n_2 & n_2 \\ -r_1 & 0 & 0 & 0 \\ -n_1 & 0 & 0 & r_2 \\ -n_1 & 0 & -r_2 & 0 \end{pmatrix}$$

is $x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2$. ■

This shows that Theorem 2.9 is a generalization of the part first of Theorem 8 in [2]. In fact, the operation defined to obtain the oriented graph $\overrightarrow{H}_2$ is actually the generalization of the join-1 operation defined in [2].

Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$ and $\overrightarrow{B}_2 = \overrightarrow{B}_2(V_2, U_2)$ be oriented bipartite graphs with partite sets U_1, V_1, U_2 and V_2, respectively. We define the join-$1'$ of $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$, denoted by $\overrightarrow{B}_1 j_1' \overrightarrow{B}_2$ as the oriented graph obtained from $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$ by joining arcs from all the vertices of V_1 to each vertex of U_2 and V_2. In the next Theorem we obtain the skew characteristic polynomial of join-$1'$, $\overrightarrow{B}_1 j_1' \overrightarrow{B}_2$ of oriented graphs $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$.

Theorem 2.10 *For $i = 1, 2$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. Let $P_s(\overrightarrow{B}_i, x)$, where $i = 1, 2$ be the skew characteristic polynomial of $\overrightarrow{B}_i$. Then the skew characteristic polynomial of join-$1'$, $\overrightarrow{B}_1 j_1' \overrightarrow{B}_2$ is*

$$P_s(\overrightarrow{B}_1 j_1' \overrightarrow{B}_2, x) = [x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2]\frac{P_s(\overrightarrow{B}_1, x)P_s(\overrightarrow{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)}.$$

Proof. The proof follows from Theorem 2.6 by taking $\overrightarrow{G} = \overrightarrow{K}_2$ and using the fact that the characteristic polynomial of matrix M is given by

$$M = \begin{pmatrix} 0 & r_1 & 0 & 0 \\ -r_1 & 0 & n_2 & n_2 \\ 0 & -n_1 & 0 & r_2 \\ 0 & -n_1 & -r_2 & 0 \end{pmatrix}$$

is $x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2$. ■

3 Skew spectrum of some oriented graphs

As applications to the resulted obtained in Section 2, we obtain the skew spectrum of some special classes of oriented graphs.

Let K_n be a complete graph on n vertices. Any orientation of K_n is said to be a tournament. Consider the complete t-partite graph $K_{2n_1,2n_2,\dots,2n_t}$, it is easy to verify that $K_{2n_1,2n_2,\dots,2n_t} = K_t[\overline{K}_{2n_1}, \overline{K}_{2n_2}, \dots, \overline{K}_{2n_t}]$. Let us orient the edges in K_t arbitrarily to obtain the oriented graph $\overrightarrow{K}_t$, then oriented graph $\overrightarrow{K}_t[\overline{K}_{2n_1}, \overline{K}_{2n_2}, \dots, \overline{K}_{2n_t}]$ gives an orientation of the complete t-partite graph $K_{2n_1,2n_2,\dots,2n_t}$, which we denote by $CT(2n_1, 2n_2, \dots, 2n_t)$. In the following result we obtain the skew characteristic polynomial of $CT(2n_1, 2n_2, \dots, 2n_t)$.

Corollary 3.1 *The skew characteristic polynomial of* $CT(2n_1, 2n_2, \dots, 2n_t) = \overrightarrow{K}_t[\overline{K}_{2n_1}, \overline{K}_{2n_2}, \dots, \overline{K}_{2n_t}]$, *where* $2n_1 + 2n_2 + \cdots + 2n_t = N$ *with each* $n_i \geq 1$ *and* $t \geq 2$, *is given by*

$$P_s(CT(2n_1, 2n_2, \dots, 2n_t), x) = x^{N-n}\phi(M, x),$$

where $\phi(M, x)$ *is the characteristic polynomial of the matrix*

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_2 & \dots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_n \end{pmatrix}, \tag{3.8}$$

with $\phi_{ij} = n_j J_2$ *or* $-n_j J_2$, *according to as there is an arc from* v_i *to* v_j *or from* v_j *to* v_i *in* $\overrightarrow{K}_n$ *and* $\phi_i = 0_2$, *for all* i.

Proof. Taking $\overrightarrow{G} = \overrightarrow{K}_n$, a tournament on n vertices and $\overrightarrow{G}_i = \overline{K}_{2n_i}$, an empty graph, for all $i = 1, 2, \dots, n$ in Theorem 2.3 and using the fact that the skew characteristic polynomial of $\overline{K}_{2n_i}$ is $P_s(\overline{K}_{2n_i}, x) = x^{2n_i}$, for all i, the result follows. Note that the empty graph $\overline{K}_{2n_i}$ can be considered as the bipartite graph with partite sets V_i and U_i of same cardinality n_i and the skew adjacency matrix $S(\overline{K}_{2n_i})$ of $\overline{K}_{2n_i}$ given by $S(\overline{K}_{2n_i}) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, which satisfies $X_i \mathbf{e}_{n_i} = 0\mathbf{e}_{n_i}$. ∎

If $n_1 = n_2 = \cdots = n_t = a$, then by Corollary 3.1, it follows that the skew eigenvalues of $CT(2a, 2a, \dots, 2a) = \overrightarrow{K}_t[\overline{K}_{2a}, \overline{K}_{2a}, \dots, \overline{K}_{2a}]$ consists of the eigenvalue 0 with multiplicity $(2a-1)t$ and the t eigenvalues $2a\lambda_1, 2a\lambda_2, \dots, 2a\lambda_t$, where $\lambda_1, \lambda_2, \dots, \lambda_t$ are the skew eigenvalues of tournament $\overrightarrow{K}_t$. This follows from Corollary 3.1 and the fact that the quotient matrix M for the oriented graph $\overrightarrow{K}_t[\overline{K}_{2a}, \overline{K}_{2a}, \dots, \overline{K}_{2a}]$ is $M = S(\overrightarrow{K}_n) \otimes aJ_2$. Now using the Theorem 4.2.12 from [13] the result follows. In fact, if $\overrightarrow{G}$ is any oriented graph of order t, then the skew eigenvalues of oriented graph $\overrightarrow{G}[\overline{K}_{2a}, \overline{K}_{2a}, \dots, \overline{K}_{2a}]$ consists of the eigenvalue 0 with multiplicity $(2a-1)t$ and the t eigenvalues $2a\lambda_1, 2a\lambda_2, \dots, 2a\lambda_t$, where $\lambda_1, \lambda_2, \dots, \lambda_t$ are the skew eigenvalues of $\overrightarrow{G}$.

Taking $\overrightarrow{G} = \overrightarrow{K}_n$, a tournament on n vertices and $\overrightarrow{G}_i = \overrightarrow{K}_{n_i,n_i}$, an orientation of the complete bipartite graph K_{n_i,n_i} with all edges oriented from one partite set to another, for all

$i=1,2,\ldots,n$, in Theorem 2.3 and using the fact that the skew characteristic polynomial of $\overrightarrow{K}_{n_i,n_i}$ is $P_s(\overrightarrow{K}_{n_i,n_i},x)=x^{2n_i-2}(x^2+n_i^2)$, we obtain the following consequence of Theorem 2.3.

Corollary 3.2 *The skew characteristic polynomial of* $\overrightarrow{K}_n[\overrightarrow{K}_{n_1,n_1},\overrightarrow{K}_{n_2,n_2},\ldots,\overrightarrow{K}_{n_n,n_n}]$*, where* $2n_1+2n_2+\cdots+2n_n=N$ *with each* $n_i\geq 1$*, is given by*

$$P_s(\overrightarrow{K}_n[\overrightarrow{K}_{n_1,n_1},\overrightarrow{K}_{n_2,n_2},\ldots,\overrightarrow{K}_{n_n,n_n}],x)=x^{N-2n}\phi(M,x),$$

where $\phi(M,x)$ *is the characteristic polynomial of the matrix*

$$M=\begin{pmatrix}\phi_1 & \phi_{12} & \cdots & \phi_{1n}\\ \phi_{21} & \phi_2 & \cdots & \phi_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{n1} & \phi_{n2} & \cdots & \phi_n\end{pmatrix}, \tag{3.9}$$

with $\phi_{ij}=n_jJ_2$ *or* $-n_jJ_2$*, according to as there is an arc from* v_i *to* v_j *or from* v_j *to* v_i *in* $\overrightarrow{K}_n$ *and* $\phi_i=\begin{pmatrix}0 & n_i\\ -n_i & 0\end{pmatrix}$.

Proof. Note that the skew adjacency matrix $S(\overrightarrow{K}_{n_i,n_i})$ of $\overrightarrow{K}_{n_i,n_i}$ given by $S(\overrightarrow{K}_{n_i,n_i})=\begin{pmatrix}0_{n_i\times n_i} & X_i\\ -X_i & 0_{n_i\times n_i}\end{pmatrix}$, satisfies $X_i\mathbf{e}_{n_i}=n_i\mathbf{e}_{n_i}$. ■

In particular, if $n_1=n_2=\cdots=n_n=a$ then by Corollary 3.2, it follows that the skew eigenvalues of $\overrightarrow{K}_n[\overrightarrow{K}_{a,a},\overrightarrow{K}_{a,a},\ldots,\overrightarrow{K}_{a,a}]$ consists of the eigenvalue 0 with multiplicity $(2a-2)n$ and the $2n$ eigenvalues of the matrix $S(\overrightarrow{K}_n)\otimes aJ_2+\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\otimes I$, where I is the identity matrix of order n and $S(\overrightarrow{K}_n)$ is the skew matrix of $\overrightarrow{K}_n$. In fact, if $\overrightarrow{G}$ is any oriented graph of order n, then the skew eigenvalues of oriented graph $\overrightarrow{G}[\overrightarrow{K}_{a,a},\overrightarrow{K}_{a,a},\ldots,\overrightarrow{K}_{a,a}]$ consists of the eigenvalue 0 with multiplicity $(2a-2)n$ and the $2n$ eigenvalues of the matrix $S(\overrightarrow{G})\otimes aJ_2+\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\otimes I$.

Let $B_i=B_i(V_i,U_i)$ be a n_i-regular bipartite graph with partite sets $|V_i|=|U_i|=n_i$. Let us orient the edges of B_i from V_i to U_i, then it is clear that the resulting oriented graph $\overrightarrow{B}_i$ is evenly-oriented. So, using the fact (see Theorem 5.4 in [15]) that the skew spectrum of $\overrightarrow{B}_i$ is i times the adjacency spectrum of B_i. It follows that the skew eigenvalues of $\overrightarrow{B}_i$ are $\iota\lambda_{i1},\iota\lambda_{i2},\ldots,\iota\lambda_{in}$, where $n_i=\lambda_{i1},\lambda_{i2},\ldots,-n_i=\lambda_{in}$ are the eigenvalues (adjacency eigenvalues) of B_i. Also, it is clear that the skew adjacency matrix $S(\overrightarrow{B}_i)=\begin{pmatrix}0_{n_i\times n_i} & X_i\\ -X_i &)_{n_i\times n_i}\end{pmatrix}$ satisfies $X_i\mathbf{e}_{n_i}=n_i\mathbf{e}_{n_i}$. Therefore, it follows from Theorem 2.3 that the skew spectrum of the oriented graph $\overrightarrow{G}[\overrightarrow{B}_1,\ldots,\overrightarrow{B}_n]$ consists of the eigenvalues $\iota\lambda_{i2},\iota\lambda_{i3},\ldots,\iota\lambda_{in-1}$, for $i=1,2,\ldots,n$, the remaining $2n$ eigenvalues are given by the matrix $M=\begin{pmatrix}\phi_1 & \phi_{12} & \cdots & \phi_{1n}\\ \phi_{21} & \phi_2 & \cdots & \phi_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{n1} & \phi_{n2} & \cdots & \phi_n\end{pmatrix}$ with $\phi_{ij}=n_jJ_2$ or $-n_jJ_2$ or $0J_2$, according to as there is an arc from v_i to v_j or from v_j to v_i or there is no arc between v_i and v_j in $\overrightarrow{G}$ and $\phi_i = \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix}$.

Consider the complete bipartite graphs $K_{a,a}$ and $K_{b,b}$. Let V_1, U_1 and V_2, U_2 be the partite sets of $K_{a,a}$ and $K_{b,b}$. Let us orient the edges of $K_{a,a}$ and $K_{b,b}$ in such a way that all the edges are directed from one partite set to another. Let $\overrightarrow{K}_{a,a}$ and $\overrightarrow{K}_{b,b}$ be the resulting oriented graphs. Since the oriented graphs $\overrightarrow{K}_{a,a}$ and $\overrightarrow{K}_{b,b}$ are evenly-oriented, it follows that their skew spectrum is ι times their adjacency spectrum. Therefore, the skew spectrum of $\overrightarrow{K}_{a,a}$ and $\overrightarrow{K}_{b,b}$ are $\{\pm\iota a, 0^{[2a-2]}\}$ and $\{\pm\iota b, 0^{[2b-2]}\}$, respectively. Moreover, their skew adjacency matrices $S(\overrightarrow{K}_{a,a}) = \begin{pmatrix} 0_{a\times a} & J_{a\times a} \\ -J_{a\times a} & 0_{a\times a} \end{pmatrix}$ and $S(\overrightarrow{K}_{b,b}) = \begin{pmatrix} 0_{b\times b} & J_{b\times b} \\ -J_{b\times b} & 0_{b\times b} \end{pmatrix}$ satisfies $J_{a\times a}\mathbf{e}_a = a\mathbf{e}_a$ and $J_{b\times b}\mathbf{e}_b = b\mathbf{e}_b$, respectively. We have the following consequence of Theorem 2.7, Theorem 2.9 and Theorem 2.10.

Corollary 3.3 *Let $\overrightarrow{K}_{a,a}$ and $\overrightarrow{K}_{b,b}$ be the orientations of the complete bipartite graphs $K_{a,a}$ and $K_{b,b}$ defined above.*
(i) *The skew spectrum of oriented graph $\overrightarrow{K}_{a,a} \to \overrightarrow{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a+2b-4$, the remaining four eigenvalues are the zeros of the polynomial $x^4 + (a^2+b^2+4ab)x^2 + a^2b^2$.*
(ii) *The skew spectrum of oriented graph $\overrightarrow{K}_{a,a} j_1 \overrightarrow{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a+2b-4$, the remaining four eigenvalues are the zeros of the polynomial $x^4 + (a^2+b^2+2ab)x^2 + a^2b^2$.*
(iii) *The skew spectrum of oriented graph $\overrightarrow{K}_{a,a} j_1' \overrightarrow{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a+2b-4$, the remaining four eigenvalues are the zeros of the polynomial $x^4+(a+b)^2x^2+a^2b^2$.*

Let C_n be the cycle of order $n \geq 3$, where n is even. Since C_n is a bipartite graph, let us orient all the edges with direction from one partite set to another and let $\overrightarrow{C}_n$ be the resulting oriented graph. It is shown in [1] that the skew spectrum of the oriented graph $\overrightarrow{C}_n$ is $\{\pm\iota 2, 2\iota\sin\frac{2\pi(j-1)}{n} : j = 1, 2, \ldots, n-2\}$. Let us orient the edges of the cycles C_{2n_1}, C_{2n_2} and C_{2n_3} in such a way that the orientations $\overrightarrow{C}_{2n_1}, \overrightarrow{C}_{2n_2}$ and $\overrightarrow{C}_{2n_3}$ are evenly-oriented, then using Theorem 2.8, we have the following observation, which gives the skew spectrum of the oriented graph $\overrightarrow{C}_{2n_1} \to (\overrightarrow{C}_{2n_2} \cup \overrightarrow{C}_{2n_3})$.

Corollary 3.4 *Let $\overrightarrow{C}_{2n_i}$ be an evenly oriented cycle of order $2n_i$, for $i = 1, 2, 3$. Then the skew spectrum of the oriented graph $\overrightarrow{C}_{2n_1} \to (\overrightarrow{C}_{2n_2} \cup \overrightarrow{C}_{2n_3})$ consists of the eigenvalues $2\iota\sin\frac{2\pi(j-1)}{2n_k}$, for $j = 1, 2, \ldots, 2n_k - 1$, where $k = 1, 2, 3$, the remaining six eigenvalues are given by the matrix M in Theorem 2.8.*

In particular, if $n_1 = n_2 = n_3 = a$, then the skew spectrum of the oriented graph $\overrightarrow{C}_{2a} \to (\overrightarrow{C}_{2a} \cup \overrightarrow{C}_{2a})$ consists of the eigenvalues $2\iota\sin\frac{2\pi(j-1)}{2a}$, for $j = 1, 2, \ldots, 2a-1$, each with multiplicity three and the six zeros of the polynomial $(x^2+4)(x^4+(8+8a^2)x^2+16)$. In fact, if B is a r-regular bipartite graph with partite sets U and V of same cardinality n_1 and $\overrightarrow{B}$ an orientation of B with all edges directed from U to V, then it is clear that the skew adjacency matrix

$S(\overrightarrow{B}) = \begin{pmatrix} 0_{n_1\times n_1} & X \\ -X & 0_{n_1\times n_1} \end{pmatrix}$ of $\overrightarrow{B}$ satisfies $X\mathbf{e}_{n_1} = r\mathbf{e}_{n_1}$. Therefore, using Theorem 2.8 the skew eigenvalues of the oriented graph $\overrightarrow{B} \to (\overrightarrow{B} \cup \overrightarrow{B})$ consists of the eigenvalues $\iota\lambda_2, \dots, \iota\lambda_{n-1}$ each with multiplicity three, where $r = \lambda_1, \lambda, \dots, \lambda_{n-1}, \lambda_n = -r$ are the adjacency eigenvalues of B and the remaining six eigenvalue are given by the zeros of the polynomial $(x^2 + r_1^2)(x^4 + (2r_1^2 + 8n_1^2)x^2 + r_1^4)$.

Let $\overrightarrow{B}_1 = \overline{K}_{2n_1}, \overrightarrow{B}_2 = \overrightarrow{K}_{n_2,n_2}$ and $\overrightarrow{B}_3 = \overrightarrow{C}_{2n_3}$, where $\overline{K}_{2n_1}$ is the empty oriented graph of order $2n_1$, $\overrightarrow{K}_{n_2,n_2}$ is the evenly-oriented complete bipartite oriented graph of order $2n_2$ and $\overrightarrow{C}_{2n_3}$ is the directed cycle. Then from Theorem 2.7 and Theorem 2.8, we obtain the skew spectrum of the oriented graphs $\overline{K}_{2n_1} \to \overrightarrow{K}_{n_2,n_2}$, $\overline{K}_{2n_1} \to \overrightarrow{C}_{2n_3}$, $\overrightarrow{K}_{n_2,n_2} \to \overrightarrow{C}_{2n_3}$, $\overline{K}_{2n_1} \to (\overrightarrow{K}_{n_2,n_2} \cup \overrightarrow{C}_{2n_3})$, $\overrightarrow{K}_{n_2,n_2} \to (\overline{K}_{2n_1} \cup \overrightarrow{C}_{2n_3})$ and $\overrightarrow{C}_{2n_3} \to (\overrightarrow{K}_{n_2,n_2} \cup \overline{K}_{2n_1})$.

Corollary 3.5 **(i)** *The skew spectrum of* $\overline{K}_{2n_1} \to \overrightarrow{K}_{n_2,n_2}$ *consists of the eigenvalue* 0 *with multiplicity* $2n_1 + 2n_2 - 2$ *and the remaining two eigenvalues are* $\pm\iota\sqrt{n_2^2 + 4n_1n_2}$.
(ii) *The skew spectrum of* $\overline{K}_{2n_1} \to \overrightarrow{C}_{2n_3}$ *consists of the eigenvalue* 0 *with multiplicity* $2n_1$, *the eigenvalues* $2\iota\sin\frac{2\pi(j-1)}{2n_3}$, *for* $j = 1, 2, \dots, 2n_3 - 1$ *and the remaining two eigenvalues are* $\pm\iota 2\sqrt{n_1n_3 + 1}$.
(iii) *The skew spectrum of* $\overrightarrow{K}_{n_2,n_2} \to \overrightarrow{C}_{2n_3}$ *consists of the eigenvalue* 0 *with multiplicity* $2n_2 - 2$, *the eigenvalues* $2\iota\sin\frac{2\pi(j-1)}{2n_3}$, *for* $j = 1, 2, \dots, 2n_3 - 1$ *and the remaining four eigenvalues are the zeros of the polynomial* $x^4 + (n_2^2 + 4n_1n_2 + 4)x^2 + 4n_2^2$.
(iv) *The skew spectrum of oriented graph* $\overline{K}_{2n_1} \to (\overrightarrow{K}_{n_2,n_2} \cup \overrightarrow{C}_{2n_3})$ *consists of the eigenvalue* 0 *with multiplicity* $2n_1 + 2n_2 - 4$, *the eigenvalues* $2\iota\sin\frac{2\pi(j-1)}{2n_3}$, *for* $j = 1, 2, \dots, 2n_3 - 1$ *and the remaining six eigenvalues are given by the matrix* M *given by Theorem 2.8 with* $r_1 = 0, r_2 = n_2$ *and* $r_3 = 2$.
(v) *The skew spectrum of oriented graph* $\overrightarrow{K}_{n_2,n_2} \to (\overline{K}_{2n_1} \cup \overrightarrow{C}_{2n_3})$ *consists of the eigenvalue* 0 *with multiplicity* $2n_1 + 2n_2 - 4$, *the eigenvalues* $2\iota\sin\frac{2\pi(j-1)}{2n_3}$, *for* $j = 1, 2, \dots, 2n_3 - 1$ *and the remaining six eigenvalues are given by the matrix* M *given by Theorem 2.8 with* $r_1 = n_2, r_2 = 0, r_3 = 2, n_1 = n_2$ *and* $n_2 = n_1$.
(vi) *The skew spectrum of oriented graph* $\overrightarrow{C}_{2n_3} \to (\overline{K}_{2n_1} \cup \overrightarrow{K}_{n_2,n_2})$ *consists of the eigenvalue* 0 *with multiplicity* $2n_1 + 2n_2 - 4$, *the eigenvalues* $2\iota\sin\frac{2\pi(j-1)}{2n_3}$, *for* $j = 1, 2, \dots, 2n_3 - 1$ *and the remaining six eigenvalues are given by the matrix* M *in Theorem 2.8 with* $r_1 = 2, r_2 = 0, r_3 = n_2, n_1 = n_3, n_2 = n_1$ *and* $n_3 = n_2$.

Similarly, we can obtain the skew spectrum of the oriented graphs $\overrightarrow{C}_{n_1} \to (\overrightarrow{C}_{n_2} \cup \overrightarrow{K}_{n_3})$, $\overrightarrow{K}_{n_3} \to (\overrightarrow{C}_{n_2} \cup \overleftarrow{C}_{n_1})$, $\overrightarrow{C}_{n_1} \to (\overrightarrow{C}_{n_2} \cup K_{n_3})$, $K_{n_3} \to (\overrightarrow{C}_{n_2} \cup \overleftarrow{C}_{n_1})$, etc.

4 Skew equienergetic oriented graphs

In this section, by using the results obtained in Section 2, we construct some new infinite families of non-cospectral skew equienergetic digraphs.

Two oriented graphs D_1 and D_2 are said to be skew equienergetic if they have same skew energy, that is, $E_s(D_1) = E_s(D_2)$. If two oriented graphs are cospectral, then they are trivially skew equienergetic. Therefore, in what follows, we will be interested in finding skew equienergetic non-cospectral oriented graphs. The following problem was proposed in [15] by Li and Lian.

Problem 1 *How to construct families of oriented graphs such that they have equal skew energy, but they do not have the same skew spectra?*

The above problem was addressed by Ramane et al. in [22], Adiga et al. in [2] and Liu et al. in [17]. In [22] the authors have extended the definition of join of graphs to oriented graphs. They obtained the skew spectrum of the join of two oriented graphs $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ with the property that the out-degree and in-degree of each vertex in $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ is same (that is the oriented graphs $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are Eulerian digraphs). Using their results they have constructed some infinite families of non-cospectral skew equienergetic oriented graphs. In [2] the authors have introduced some variations of the join of two oriented graphs for bipartite oriented graphs. They have defined four types of join operations for the bipartite oriented graphs. Using their results they were able to obtain some more infinite families of non-cospectral skew equienergetic oriented graphs. Recently, in [17] the authors have introduced the concept of corona and neighborhood corona of oriented graphs. Using these operations together with join operation they have constructed some new infinite families of non-cospectral skew equienergetic oriented graphs. Recently, the authors [12] have extended the definition of join of two oriented graph by defining the joined union of oriented graphs. They have discussed the skew spectrum of the joined union of oriented Eulerian graphs and as applications they have added some new infinite families of non-cospectral skew equienergetic oriented graphs. Moreover, the results obtained in [22] were obtained as particular cases. In the rest of this section, we aim to construct some new infinite families of non-cospectral skew equienergetic oriented graphs.

The following result gives the skew energy of the joined union $\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_n]$ of oriented bipartite graphs $\overrightarrow{B}_1, \overrightarrow{B}_2, \dots, \overrightarrow{B}_n$.

Theorem 4.1 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$. For $i = 1, 2, \dots n$, let $\overrightarrow{B}_i$ be an oriented bipartite graph with partite sets of same cardinality n_i having the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0,1)$-matrix satisfying $X_i \boldsymbol{e}_{n_i} = r_i \boldsymbol{e}_{n_i}$. Then*

$$E_S(\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_n]) = \sum_{i=1}^{n} E_S(\overrightarrow{B}_i) - 2\sum_{i=1}^{n} r_i + 2\sum_{i=1}^{n} |x_i|,$$

where $\pm \iota x_1, \pm \iota x_2, \dots, \pm \iota x_n$ are the eigenvalues of the matrix M given in Theorem 2.3.

Proof. Since the skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$ of $\overrightarrow{B}_i$ has the property that $X_i \mathbf{e}_{n_i} = r_i \mathbf{e}_{n_i}$. It is easy to verify that $\pm \iota r_i$ is eigenvalue of $S(\overrightarrow{B}_i)$. Let $\iota r_i, \psi_{i2}, \psi_{i3}, \dots, \psi_{i2n_i-1}, -\iota r_i$ be the skew eigenvalues of $\overrightarrow{B}_i$, for $i = 1, 2, \dots, n$. Then, it is clear from Theorem 2.3 that the skew eigenvalues of the oriented graph $\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_n]$ are $\psi_{i2}, \psi_{i3}, \dots, \psi_{i2n_i-1}$, for $i = 1, 2, \dots, n$

and the remaining $2n$ eigenvalues are the eigenvalues of the matrix M. Now, using the definition of skew energy the result follows. ■

Taking in particular $\overrightarrow{B}_1 = \overrightarrow{B}_2 = \cdots = \overrightarrow{B}_n$ in Theorem 4.1, we obtain the skew energy of the oriented graph $\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_1]$.

Corollary 4.2 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$. Let $\overrightarrow{B}_1$ be an oriented bipartite graph with partite sets of same cardinality n_1 having the skew adjacency matrix $S(\overrightarrow{B}_1) = \begin{pmatrix} 0_{n_1\times n_1} & X_1 \\ -X_1 & 0_{n_1\times n_1} \end{pmatrix}$, where X_1 is a $(0,1)$-matrix satisfying $X_1\boldsymbol{e}_{n_1} = r_1\boldsymbol{e}_{n_1}$. Then*

$$E_S(\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_1]) = nE_S(\overrightarrow{B}_1) - 2nr_1 + \sum_{i=1}^{n} |x_i|,$$

where $\pm\iota x_1, \pm\iota x_2, \ldots, \pm\iota x_n$ are the eigenvalues of the matrix $S(\overrightarrow{G}) \otimes n_1 J_2 + \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \otimes I$.

Since the matrix M is determined by the structure of $\overrightarrow{G}$ and the orders n_i of the oriented bipartite graphs $\overrightarrow{B}_i$, for $i = 1, 2, \ldots, n$. We have the following observation from Theorem 4.1.

Corollary 4.3 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$. Let $\overrightarrow{B}_i$ and $\overrightarrow{G}_i$ be oriented bipartite graphs of order n_i, for $i = 1, 2, \ldots, n$ and let $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$ and $S(\overrightarrow{G}_i) = \begin{pmatrix} 0_{n_i\times n_i} & Y_i \\ -Y_i & 0_{n_i\times n_i} \end{pmatrix}$ be their skew adjacency matrices with $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i} = Y_i\boldsymbol{e}_{n_i}$. If the oriented bipartite graphs $\overrightarrow{B}_i$ and $\overrightarrow{G}_i$ are non-cospectral with $E_S(\overrightarrow{B}_i) = E_S(\overrightarrow{H}_i)$, for all $i = 1, 2, \ldots, n$, then the oriented graphs $\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]$ and $\overrightarrow{G}[\overrightarrow{G}_1, \ldots, \overrightarrow{G}_n]$ are non-cospectral with*

$$E_S(\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]) = E_S(\overrightarrow{G}[\overrightarrow{G}_1, \ldots, \overrightarrow{G}_n]).$$

For $i = 1, 2, \ldots, n$, let B_i be a bipartite graph with partite sets V_i and U_i of same cardinality n_i. Let us orient the edges of B_i in such a way that the resulting orientation $\overrightarrow{B}_i$ is uniformly oriented. Then, using the fact (see Theorem 5.4 in [15]) that the skew spectrum of $\overrightarrow{B}_i$ is ι times the adjacency spectrum of B_i. It follows that the skew energy of $\overrightarrow{B}_i$ is same as the energy of B_i, that is, $E_S(\overrightarrow{B}_i) = \mathcal{E}(B_i)$, for all $i = 1, 2, \ldots, n$. Therefore, we have following observation which gives the construction of skew equienergetic oriented graphs from the equienergetic graphs.

Corollary 4.4 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$. Let B_i and G_i be r_i-regular bipartite equienergetic graphs with partite sets of same cardinality n_i, for $i = 1, 2, \ldots, n$. If the orientations $\overrightarrow{B}_i$ and $\overrightarrow{G}_i$ are uniformly oriented, then the oriented graphs $\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]$ and $\overrightarrow{G}[\overrightarrow{G}_1, \ldots, \overrightarrow{G}_n]$ are non-cospectral with*

$$E_S(\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]) = E_S(\overrightarrow{G}[\overrightarrow{G}_1, \ldots, \overrightarrow{G}_n]).$$

A lot of papers can be found in the literature regarding the construction of equienergetic graphs, see the book [16] and the references therein. Let $D(\overrightarrow{G})$ be the duplication digraph of a digraph $\overrightarrow{G}$ defined in [2]. Since, the graph $D(G)$ is always a bipartite graph with $\mathcal{E}(D(G)) = 2\mathcal{E}(G)$, giving that if G_i and H_i are equienergetic graphs then the bipartite graphs $D(G_i)$ and $D(H_i)$ are also equienergetic. Thus, from any given pair of equienergetic regular graphs we can construct a pair of bipartite equienergetic regular graphs which in turn can be used to construct a pair of skew equienergetic oriented graphs by Corollary 4.4.

Taking in particular $\overrightarrow{G} = \overrightarrow{K}_2$ in Theorem 4.1 and using Theorem 2.7, we obtain the following result which is Theorem 6 in [2], and gives the skew energy of the join of oriented bipartite graphs $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$.

Corollary 4.5 *For $i = 1, 2$, let $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$ be an oriented bipartite graph with partite sets V_i and U_i of same cardinality n_i and skew adjacency matrix $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where X_i is a $(0, 1)$-matrix satisfying $X_i \boldsymbol{e}_{n_i} = r_i \boldsymbol{e}_{n_i}$. Then*

$$E_S(\overrightarrow{B}_1 \to \overrightarrow{B}_2) = E_S(\overrightarrow{B}_1) + E_S(\overrightarrow{B}_2) + 2(|x_1| + |x_2| - r_1 - r_2),$$

where $\pm\iota x_1, \pm\iota x_2$ are the zeroes of the polynomial $x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2 + r_2^2$.

If $\overrightarrow{G}$ and $\overrightarrow{H}$ are two oriented graphs which are non-cospectral with respect to skew matrix, then we have the following consequence of Corollary 4.2, which gives a new infinite family of non-cospectral skew equienergetic oriented graphs.

Corollary 4.6 *Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two non-cospectral oriented graphs of order $n \geq 2$. Let $\overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1)$ be an oriented bipartite graph with partite sets V_1 and U_1 of same cardinality n_1 and skew adjacency matrix $S(\overrightarrow{B}_1) = \begin{pmatrix} 0_{n_1 \times n_1} & X_1 \\ -X_1 & 0_{n_1 \times n_1} \end{pmatrix}$, where X_1 is a $(0, 1)$-matrix satisfying $X_1 \boldsymbol{e}_{n_1} = r_1 \boldsymbol{e}_{n_1}$. If*

$$\sum_{i=1}^{n} \left| \lambda\Big(S(\overrightarrow{G}) \otimes n_1 J_2 + A \otimes I\Big) \right| = \sum_{i=1}^{n} \left| \lambda\Big(S(\overrightarrow{H}) \otimes n_1 J_2 + A \otimes I\Big) \right|,$$

where $A = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}$. Then

$$E_S(\overrightarrow{G}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_1]) = E_S(\overrightarrow{H}[\overrightarrow{B}_1, \dots, \overrightarrow{B}_1]).$$

Let $\overrightarrow{H}_2$ be the variation of the joined union of oriented bipartite graphs defined in Section 2. Proceeding similar to Theorem 4.1, we have the following result which gives the skew energy of $\overrightarrow{H}_2$.

Theorem 4.7 *Let $\overrightarrow{G}$ be an oriented graph of order $n \geq 2$. For $i = 1, 2, \dots n$, let $\overrightarrow{B}_i$ be a bipartite oriented graph with partite sets of same cardinality n_i having the skew adjacency matrix*

$S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, *where* X_i *is a* $(0,1)$*-matrix satisfying* $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. *Let* $\overrightarrow{H}_2$ *be the oriented graph defined in Section 2, then*

$$E_S(\overrightarrow{H}_2) = \sum_{i=1}^{n} E_S(\overrightarrow{B}_i) - 2\sum_{i=1}^{n} r_i + 2\sum_{i=1}^{n} |x_i|,$$

where $\pm\iota x_1, \pm\iota x_2, \ldots, \pm\iota x_n$ *are the eigenvalues of the matrix* M *given in Theorem 2.5.*

Since the matrix M is determined by the structure of $\overrightarrow{G}$ and the orders n_i of the oriented bipartite graphs $\overrightarrow{B}_i$, for $i = 1, 2, \ldots, n$. We have the following observation from Theorem 4.7.

Corollary 4.8 *Let* $\overrightarrow{G}$ *be an oriented graph of order* $n \geq 2$. *Let* $\overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i)$ *and* $\overrightarrow{G}_i = \overrightarrow{G}_i(P_i, Q_i)$ *be oriented bipartite graphs with partite sets* V_i, U_i *and* P_i, Q_i *of same cardinality* n_i, *for* $i = 1, 2, \ldots, n$. *Let* $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$ *and* $S(\overrightarrow{G}_i) = \begin{pmatrix} 0_{n_i\times n_i} & Y_i \\ -Y_i & 0_{n_i\times n_i} \end{pmatrix}$ *be their skew adjacency matrices with* $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i} = Y_i\boldsymbol{e}_{n_i}$. *Let* $\overrightarrow{H}_1$ *be the oriented graph obtained from* $\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]$ *by deleting all the arcs between* U_i *and* V_j, U_j, $i \neq j$ *and let* $\overrightarrow{H}'_1$ *be the oriented graph obtained from* $\overrightarrow{G}[\overrightarrow{G}_1, \ldots, \overrightarrow{G}_n]$ *by deleting all the arcs between* Q_i *and* P_j, Q_j, $i \neq j$. *If the oriented bipartite graphs* $\overrightarrow{B}_i$ *and* $\overrightarrow{G}_i$ *are non-cospectral with* $E_S(\overrightarrow{B}_i) = E_S(\overrightarrow{G}_i)$, *for all* $i = 1, 2, \ldots, n$, *then the oriented graphs* $\overrightarrow{H}_1$ *and* $\overrightarrow{H}'_1$ *are non-cospectral with*

$$E_S(\overrightarrow{H}_1) = E_S(\overrightarrow{H}'_1).$$

Taking $\overrightarrow{G} = \overrightarrow{K}_2$ in Theorem 4.7, we obtain the following result obtained in [2] as part first of Theorem 9.

Corollary 4.9 *For* $i = 1, 2$, *let* $\overrightarrow{B}_i$ *be an oriented bipartite graph with partite sets of same cardinality* n_i *having the skew adjacency matrix* $S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i\times n_i} & X_i \\ -X_i & 0_{n_i\times n_i} \end{pmatrix}$, *where* X_i *is a* $(0,1)$*-matrix satisfying* $X_i\boldsymbol{e}_{n_i} = r_i\boldsymbol{e}_{n_i}$. *Then*

$$E_S(\overrightarrow{B}_1 j_1 \overrightarrow{B}_2) = E_S(\overrightarrow{B}_1) + E_S(\overrightarrow{B}_2) + 2(|x_1| + |x_2| - r_1 - r_2),$$

where $\pm\iota x_1, \pm\iota x_2$ *are the zeroes of the polynomial* $x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2$.

5 Conclusion

In this paper we have discussed the skew characteristic polynomial and the skew eigenvalues of the joined union and some of its variations for the oriented bipartite graphs. As applications, we have given a general method to construct infinite families of oriented graphs with same skew energy but different skew spectrum. Our ideas and results obtained generalize some of the ideas and results in [2].

6 Conflict of interest

The authors declare that they have no conflict of interest.

7 Data availability

There is no data associated with this article.

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