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On the skew characteristics polynomial/eigenvalues of operations on bipartite oriented graphs and applications

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ABSTRACT
Let \( \tilde{G} \) be an oriented graph with \( n \) vertices and \( m \) arcs having underlying graph \( G \). The skew matrix of \( \tilde{G} \), denoted by \( S(\tilde{G}) \), is a \((-1, 0, 1)\)-skew symmetric matrix. The skew eigenvalues of \( \tilde{G} \) are the eigenvalues of \( S(\tilde{G}) \) and its characteristic polynomial is the skew characteristic polynomial of \( \tilde{G} \). The sum of the absolute values of the skew eigenvalues is the skew energy of \( \tilde{G} \). In this paper, we study the skew characteristic polynomial and skew eigenvalues of joined union of oriented bipartite graphs and some of its variations. We show that the skew eigenvalues of the joined union of oriented bipartite graphs and some variations of oriented bipartite graphs is the union of the skew eigenvalues of the component oriented graphs except some eigenvalues, which are given by an auxiliary matrix associated with the joined union. As a special case we obtain the skew eigenvalues of join of two oriented bipartite graphs and the lexicographic product of an oriented graph and an oriented bipartite graph. Some examples of orientations of well-known graphs are presented to highlight the importance of the results. As applications to our result we obtain some new infinite families of skew equienergetic oriented graphs. Our results extend and generalize some of the results obtained in [C. Adiga and B.R. Rakshith, More skew-equienergetic digraphs, Commun. Comb. Optim., 1(1) (2016) 55–71].

1. Introduction
Let \( G \) be a simple graph having \( n \) vertices and \( m \) edges. The vertex set is \( \{v_1, v_2, \ldots, v_n\} \). Let \( \tilde{G} \) be a digraph, where edge is assigned arbitrarily a direction. The digraph \( \tilde{G} \) is said to be an orientation of \( G \) or oriented graph associated with \( G \). The graph \( \tilde{G} \) is viewed as the underlying graph of \( \tilde{G} \). Let \( d^+_i = d^+(v_i) \) be the out-degree, \( d^-_i = d^-(v_i) \) the in-degree and \( d_i = d^+_i + d^-_i \) the degree of the vertex \( v_i \in V(\tilde{G}) \). Let \( N^+_G(v_i) \) be the set of out-neighbours, \( N^-_G(v_i) \) the set of in-neighbours and \( N_G(v_i) = N^+_G(v_i) \cup N^-_G(v_i) \) be the set of neighbours of the vertex \( v_i \) in \( G \). The adjacency matrix \( A(G) = (a_{ij}) \) of a graph \( G \) is an \( n \times n \) square matrix with \( a_{ij} = 1 \), if there is an edge between the vertices \( v_i \) and \( v_j \), and \( a_{ij} = 0 \), otherwise. All the eigenvalues of \( A(G) \) are real numbers as it is a real symmetric matrix. The eigenvalues of the matrix \( A(G) \) are called eigenvalues (or adjacency eigenvalues) of \( G \) and are denoted by \( \lambda_1, \lambda_2, \ldots, \lambda_n \). The sum of the absolute values of the eigenvalues of \( G \) is called energy of \( G \) and is denoted by \( E(G) \). That is, 
\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]
This spectral graph invariant is one among the most studied spectral graph invariants in spectral graph theory because of its applications in mathematical and other sciences. For some recent works on energy of graphs, we refer to Akbari et al. (2022) and the book Li et al. (2012).

The skew adjacency matrix \( S = S(\tilde{G}) = (s_{ij}) \) of an oriented graph \( \tilde{G} \) is an \( n \times n \) matrix with \( s_{ij} = 1 \) when there is an arc from \( v_i \) to \( v_j \), and \( s_{ij} = -1 \) when there is an arc from \( v_j \) to \( v_i \), and \( s_{ij} = 0 \) otherwise. It is clear that the matrix \( S(\tilde{G}) \) is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The characteristic polynomial of \( S(\tilde{G}) \) is the skew characteristic polynomial of \( \tilde{G} \) and is denoted by \( P_{\tilde{G}}(\tilde{G}, x) \). The zeros of the polynomial \( P_{\tilde{G}}(\tilde{G}, x) \) are the eigenvalues of the matrix \( S(\tilde{G}) \) and are called skew eigenvalues of \( \tilde{G} \). The skew spectrum of \( \tilde{G} \) is denoted by \( \text{Sp}_{\tilde{G}}(\tilde{G}) \), which describes the eigenvalues of \( S(\tilde{G}) \) as well as their multiplicities.

The skew energy of the oriented graph \( \tilde{G} \) is called the energy of the matrix \( S(\tilde{G}) \). It is defined by the following equation.
\[
E(\tilde{G}) = \sum_{i=1}^{n} |\xi_i|,
\]
where $\xi_1, \xi_2, \ldots, \xi_n$ are the skew eigenvalues of $\overline{G}$. This type of spectral invariant appears in the literature with numerous results regarding their bounds and it has abundant connections with the different graph parameters like matching number, vertex covering number and independence number, its connections with the skew rank (the rank of the matrix $S(\overline{G})$ is called skew rank of $\overline{G}$). One of the most studied problems in the theory of skew energy is the determination of extremal oriented graphs for $E(\overline{G})$ in a given class of oriented graphs. In fact, due to the hardness of this problem, many researchers have started with a graph $G$ and tried to find the orientations of $G$ which attain the extremal value for $E(\overline{G})$. This problem is the topic of many papers in literature. Some recent examples can be found in (Deng et al., 2018; Taghvae & Fath-Tabar, 2020). We refer to (Alhevaz et al., 2020; Bhat, 2017; Ganie et al., 2019; Ganie et al., 2019; Ganie et al., 2021; Li & Lian, 2015; Pirzada et al., 2020; Qiu et al., 2021; Rather et al., 2023; Shang, 2018) for more development of skew energy theory.

Given a cycle $C_k = u_1u_2 \ldots u_ku_1$, its sign is $\text{sgn}(C_k) = s_{12}s_{23} \ldots s_{k1}$. Here, $s_{ij}$ means the entry of the skew matrix $\overline{S(G)}$ in the intersection of $u_i$ row and $u_j$ column. If the sign of an even oriented cycle $C_k$ is positive or negative, it is referred to as evenly-oriented or oddly-oriented, respectively. We say $\overline{G}$ is evenly-oriented if every even cycle in $\overline{G}$ is evenly-oriented. When $\text{sgn}(C_{2k}) = (-1)^k$, the even oriented cycle $C_{2k}$ becomes uniformly oriented.

The rest of the papers is organized as follows. In Section 2, we study the joined union of oriented bipartite graphs and some of its variations. We obtain the skew spectrum of joined union of oriented bipartite graphs and its some of its variations, in terms of the component oriented graphs and an auxiliary matrix determined by the operation. In Section 3, we use the results obtained in Section 2 to obtain the skew spectrum of various families of oriented graphs. As applications to results obtained in Section 2 and 3, we construct various new families of skew equienergetic oriented graphs in Section 4.

2. The skew spectrum of joined union of oriented graphs

Consider an $n \times n$ complex matrix

$$M = \begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1s} \\
X_{21} & X_{22} & \cdots & X_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
X_{s1} & X_{s2} & \cdots & X_{ss}
\end{pmatrix},$$

where $X_s$ is an $n_i \times n_j$ block matrix for $1 \leq i, j \leq s$ and $n = \sum_{i=1}^{s} n_i$. The element $b_{ij}$ is the average row sum of $X_{ij}$. We define an $s \times s$ matrix with elements being the average row sums of $X_{ij}$ and we call it the quotient matrix $B = (b_{ij})$. The matrix $B$ becomes an \textit{equitable quotient matrix} when each block $X_{ij}$ has constant row sum. A complex matrix has a connection with \textit{equitable quotient matrix} in terms of its spectrum as below (You et al., 2019).

**Lemma 2.1.** The equitable quotient matrix $B$ and the matrix $M$ defined in (1) share the same eigenvalues.

The generalized join (also called joined union) of graphs has different versions of definition. The spectrum of generalized join of graphs in terms of different matrices has been investigated in (Ganie, 2022; Rather et al., 2021, 2023). The joined union was extended to digraphs in (Ganie, 2022). In (Ganie, 2022), the authors have discussed the $A_s$-spectrum of the joined union of diagonalizable digraphs and as applications the $A_s$-spectrum of various families of digraphs are found. Recently, in Ganie, Ingle et al., (2022), the authors defined generalized join of oriented graphs as follows.

Let $\overline{G}(V, E)$ be an oriented graph of order $n$ and let $G_i(V, E_i)$ be oriented graphs of order $n_i$, where $i = 1, \ldots, n$. The \textit{joined union} of the oriented graphs $G_1, G_2, \ldots, G_n$ with respect to oriented graph $\overline{G}$ is denoted by $G[\overline{G}]$, and is defined as the oriented graph $\overline{H}(W, F)$ with vertex set $W = \bigcup_{i=1}^{n} V_i$ and arc set

$$F = \bigcup_{i=1}^{n} E_i \cup \left\{ (u, v) \in E(H), \text{ whenever } u \in G_i, v \in G_j \text{ and } v \in N_\overline{G}(v_j) \right\}.$$ 

In other words, the joined union is the union of oriented graphs $\overline{G}_1, \ldots, \overline{G}_n$, together with the arcs $(v_{ik}, v_{kj})$, where $v_{ik} \in \overline{G}_i$ and $v_{kj} \in \overline{G}_j$, whenever $(v_i, v_j)$ is an arc in $\overline{G}$. Clearly, the usual join of two oriented graphs $\overline{G}_1$ and $\overline{G}_2$ defined in Ramane et al., (2016) is a special case of the joined union of oriented graphs $\overline{G}_1, \ldots, \overline{G}_n$. We extend the concept of the \textit{left vertex equitable quotient} of a graph $G$ to the \textit{left vertex equitable quotient} of a graph $G$. That is, $\overline{H}_1 = \overline{G}(G_1, G_2)$, where $\overline{H}_1$ is the oriented graph corresponding to the complete graph of order 2. By taking each of the component in joined union as bipartite oriented graphs, we can define the following variations of the \textit{joined union} of the oriented graphs.

Let $\overline{G}(V, E)$ be an oriented graph of order $n$ and let $\overline{G}_1 = \overline{G}(V_i, U_i)$, be a bipartite oriented graph with partite sets $V_i$ and $U_i$, for all $i = 1, 2, \ldots, n$. Let $\overline{H}_1$ be the \textit{joined union} of the oriented graphs $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_n$ with respect to oriented graph $\overline{G}$. That is, $\overline{H}_1 = \overline{G}(\overline{G}_1, \overline{G}_2)$. Note that if there is an arc between the vertices $v_i, v_j$ in $\overline{G}$, then there are arcs between all the vertices of $V_i$ and $V_j$ between all the vertices of $V_i$ and $U_j$ between all the vertices of $U_i$ and $V_j$. Let $\overline{H}_2$ be the oriented graph obtained from $\overline{H}_1$ by deleting all
the arcs between $U_i$ and $V_j$ and all the arcs between $U_i$ and $U_j$. Let $\overline{H}_i$ be the oriented graph obtained from $H_i$ by deleting all the arcs between $V_i$ and $V_j$ and all the arcs between $V_i$ and $U_j$.

A digraph $D$ is said to be Eulerian if the out-degree of any vertex in $D$ is same as its in-degree, that is, $d^+_i = d^-_i$, for all $v_i \in V(D)$. The following theorem was obtained in Ganie, Ingle, et al., (0000) and gives the skew spectrum of the joined union of Eulerian oriented graphs $\overline{G}_1, \overline{G}_2, ..., \overline{G}_n$, in terms of the skew spectrum of the component oriented graphs $G_1, G_2, ..., G_n$ and the eigenvalues of an auxiliary matrix determined by the joined union.

**Theorem 2.2.** Let $\overline{G}$ be an oriented graph of order $n \geq 2$ having $m$ arcs. Let $\overline{G}_i$ be Eulerian oriented graph of order $n_i$, having skew characteristic polynomial $P_i(\overline{G}_i, x)$, where $i = 1, 2, ..., n$. Then the skew characteristic polynomial of the oriented graph $\overline{G}[\overline{G}_1, ..., \overline{G}_n]$ of order $N = \sum_{i=1}^{n} n_i$ is

$$P_i(\overline{G}[\overline{G}_1, ..., \overline{G}_n], x) = \phi(M, x) \prod_{i=1}^{n} P_i(\overline{G}_i, x),$$

(2)

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{bmatrix}
0 & \psi_{12} & \cdots & \psi_{1n}\\
\psi_{21} & 0 & \cdots & \psi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{n1} & \psi_{n2} & \cdots & 0
\end{bmatrix},$$

where $\psi_{ij} = n_j$, if there is an arc from $v_i$ to $v_j$; $\psi_{ij} = -n_j$, if there is an arc from $v_j$ to $v_i$ and $\psi_{ij} = 0$, if there is no arc between $v_i$ and $v_j$.

It is clear that Theorem 2.2 is applicable to Eulerian oriented graphs only. However, in the next theorem we will show that for the bipartite oriented digraphs, the condition of being Eulerian can be relaxed.

For $i = 1, 2, ..., n$, let $\overline{B}_i = \overline{B}_i(V_i, U_i)$, be a bipartite oriented graph with partite sets $V_i$ and $U_i$ of same cardinality $n_i$, having the skew adjacency matrix $S(\overline{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where $X_i$ is a $(0, 1)$-matrix satisfying $X_i e_n = r_i e_n$ and $e_n$ is the all one column vector.

In the next theorem we determine the skew characteristic polynomial of the joined union of oriented bipartite graphs $\overline{B}_1, \overline{B}_2, ..., \overline{B}_n$, in terms of the skew characteristic polynomial of the component oriented graphs and the eigenvalues of an auxiliary matrix determined by the joined union.

**Theorem 2.3.** Let $\overline{G}$ be an oriented graph of order $n \geq 2$ having $m$ arcs. For $i = 1, 2, ..., n$, let $\overline{B}_i = \overline{B}_i(V_i, U_i)$, be a bipartite oriented graph with $|V_i| = |U_i| = n_i$, having the skew adjacency matrix $S(\overline{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}$, where $X_i$ is a $(0, 1)$-matrix satisfying $X_i e_n = r_i e_n$. Let $P_i(\overline{B}_i, x)$, where $i = 1, 2, ..., n$ be the skew characteristic polynomial of $\overline{B}_i$. Then the skew characteristic polynomial of the oriented graph $H_1 = \overline{G}[\overline{B}_1, ..., \overline{B}_n]$ of order $N = 2 \sum_{i=1}^{n} n_i$ is

$$P_i(\overline{H}_1, x) = \phi(M, x) \prod_{i=1}^{n} P_i(\overline{B}_i, x),$$

(3)

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{bmatrix}
\phi_1 & \phi_{12} & \cdots & \phi_{1n} \\
\phi_{21} & \phi_2 & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_n
\end{bmatrix},$$

where $\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}$, if there is an arc from $v_i$ to $v_i$; $\phi_{ij} = \begin{pmatrix} -n_j & -n_j \\ n_j & n_j \end{pmatrix}$, if there is an arc from $v_j$ to $v_i$ and $\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, if there is no arc between $v_i$ and $v_j$.

**Proof.** Let $V(\overline{G}) = \{v_1, ..., v_n\}$ be the vertex set of $\overline{G}$ and let $V(\overline{B}_i) = \{x_{i1}, ..., x_{in}, y_{i1}, ..., y_{in}\}$ be the vertex set of $\overline{B}_i$, for $i = 1, 2, ..., n$. Let $H_1 = \overline{G}[\overline{B}_1, ..., \overline{B}_n]$. Let us label the vertices in $H_1$ in such a way that the vertices in $\overline{B}_1$ are labelled first, the vertices of $\overline{B}_2$ are labelled after the vertices in $\overline{B}_1$, and so on. With this labelling, the skew matrix of $H_1$ takes the form

$$S(\overline{H}_1) = \begin{bmatrix}
\Gamma_1 & \Gamma_{12} & \cdots & \Gamma_{1n} \\
\Gamma_{21} & \Gamma_2 & \cdots & \Gamma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{n1} & \Gamma_{n2} & \cdots & \Gamma_n
\end{bmatrix},$$

where

$$\Gamma_i = S(\overline{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix},$$

for $i = 1, 2, ..., n$,

and $\Gamma_{ij} = \begin{pmatrix} A_{n_i \times n_i} & B_{n_i \times n_j} \\ C_{n_i \times n_j} & D_{n_i \times n_j} \end{pmatrix}$, with $A_{n_i \times n_i} = B_{n_i \times n_j} = I_{n_i \times n_j}$ and $C_{n_i \times n_j} = D_{n_i \times n_j} = I_{n_i \times n_j}$, if $(v_i, v_j) \in E(\overline{G})$; $A_{n_i \times n_j} = \overline{B}_{n_i \times n_j} = I_{n_i \times n_j} = \overline{D}_{n_i \times n_j} = \overline{I}_{n_i \times n_j}$, if $(v_i, v_j) \not\in E(\overline{G})$ and $A_{n_i \times n_j} = B_{n_i \times n_j} = 0_{n_i \times n_j}$ and $C_{n_i \times n_j} = D_{n_i \times n_j} = 0_{n_i \times n_j}$, if $(v_i, v_j), (v_j, v_i) \not\in E(\overline{G})$. Note that $I_{n_i \times n_j}$ is the all
one matrix of order \(n_1 \times n_j\) and \(0_{n, n_0}\) is the zero matrix of order \(n_1 \times n_j\).

By assumption \(\tilde{B}_i = \tilde{B}_i(V_i, U_i)\) is a bipartite oriented graph for all \(i\) with partite sets \(V_i\) and \(U_i\) of same cardinality \(n_i\) and skew matrix, \(S(\tilde{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}\), where \(X_i\) is a \((0,1)\)-matrix satisfying \(X_i e_{n_i} = r_i e_{n_i}\). It is easy to verify that \(r_i\) is an eigenvalue of \(S(\tilde{B}_i)\) with corresponding eigenvector \(e_{2n_i} = (-r_i e_{n_i})\). Similarly, we can verify that 

\[-r_i\] is an eigenvalue of \(S(\tilde{B}_i)\) with corresponding eigenvector \((e_{n_i})\). Since \(S(\tilde{B}_i)\) is a skew symmetric matrix, so it is a diagonalisable matrix with its \(2n_i\) eigenvectors forming an orthogonal set. Let \(\lambda_{ik}\) be an eigenvalue of \(S(\tilde{B}_i)\) other than \(\pm r_i\), with the corresponding eigenvector \(X = (s_{i1}, s_{i2},\ldots, s_{in}, t_{i1}, t_{i2},\ldots, t_{in})^T\) satisfying \(e_{2n_i}^T X = 0\).

That is, 

\[-r_i \sum_{k=1}^{n_i} s_{ik} + \sum_{k=1}^{n_i} t_{ik} = 0.\]

Now, consider the vector 

\[Y = (y_1, y_2,\ldots, y_{2n_i})^T,\]

where

\[y_j = \begin{cases} 
 s_{ij} & \text{if } v_j \in V(\tilde{B}_i) \cap V_i \\
 t_{ij} & \text{if } v_j \in V(\tilde{B}_i) \cap U_i \\
 0 & \text{otherwise.} 
\end{cases}\]

As \(e_{2n_i}^T X = 0\) gives that \(\Gamma_y X = 0\) and coordinates of the vector \(Y\) corresponding to vertices of \(\overline{H}_1\) which are not in \(\tilde{B}_i\) are zeros, we have

\[S(\overline{H}_1) Y = \lambda_{ik} X = \lambda_{ik} Y.\]

This shows that \(Y\) is an eigenvector of \(S(\overline{H}_1)\) corresponding to the eigenvalue \(\lambda_{ik}\) and so every eigenvalue \(\lambda_{ik}\) (other than \(\pm r_i\)) of \(S(\tilde{B}_i)\) is an eigenvalue of \(S(\overline{H}_1)\). So, using this process we will obtain \(\sum_{i=1}^{n} 2n_i - 2n = N - 2n\) eigenvalues of \(S(\overline{H}_1)\). To determine the remaining \(2n\) eigenvalues of \(S(\overline{H}_1)\), we use the equitable quotient matrix. The equitable quotient matrix of \(S(\overline{H}_1)\) is

\[M = \begin{pmatrix} \phi_1 & \phi_{12} & \cdots & \phi_{1n} \\
\phi_{21} & \phi_2 & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_n \end{pmatrix},\]

where \(\phi_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}\); \(\phi_{ij} = \begin{pmatrix} n_j & n_j \\ -n_j & -n_j \end{pmatrix}\), if \((v_i, v_j) \in E(\tilde{B}_i);\)

\(\phi_{ij} = \begin{pmatrix} -n_j & -n_j \\ n_j & n_j \end{pmatrix}\), if \((v_i, v_j) \in E(\tilde{B}_i)\) and \(\phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\), if \((v_i, v_j), (v_j, v_i) \notin E(\tilde{B}_i)\). Since by Lemma 2.1, the eigenvalues of \(M\) are the eigenvalues of \(S(\overline{H}_1)\), the result follows.

The lexicographic product \(G[H]\) of graphs \(G\) and \(H\) is the graph whose vertex set \(V(G) \times V(H)\) and edge \((a, x)(b, y) \in E(G[H])\) whenever \(ab \in E(G)\), or \(a = b; x, y \in E(H)\). It is interesting to see that the lexicographic product \(G[H]\) can be constructed by joined union \(G_1, G_2,\ldots, G_n\) where \(G_i = H\) for \(1 \leq i \leq n\). Note that in the case \(G_i = K_1\) we get \(G[K_1, K_1,\ldots, K_1] = G\).

If in particular the oriented bipartite graphs \(\tilde{B}_1, \tilde{B}_2,\ldots, \tilde{B}_n\) in Theorem 2.2 are same, say \(\tilde{B}_i = \tilde{B}_1\), for \(2 \leq i \leq n\), then we obtain the following Theorem, which gives the skew spectrum of the joined union \(\overline{G}_1[\tilde{B}_1,\ldots, \tilde{B}_1]\), which represents an orientation of the lexicographic product \(G[B_1]\).

Theorem 2.4. Let \(G\) be an oriented graph of order \(n \geq 2\) having \(m\) arcs. Let \(\tilde{B}_i = \tilde{B}_i(V_i, U_i)\), be a bipartite oriented graph with \(|V_i| = |U_i| = n_i\), having the skew adjacency matrix \(S(\tilde{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix}\), where \(X_i\) is a \((0,1)\)-matrix satisfying \(X_i e_{n_i} = r_i e_{n_i}\). Let \(P_1(\tilde{B}_i, x)\) be the skew characteristic polynomial of \(\tilde{B}_1\). Then the skew characteristic polynomial of the oriented graph \(\overline{G}[\tilde{B}_1,\ldots, \tilde{B}_1]\) of order \(N = 2nn_1\) is

\[P_1(\overline{G}[\tilde{B}_1,\ldots, \tilde{B}_1], x) = \phi(M, x) \left[\frac{P_1(\tilde{B}_1, x)}{(x^2 + r_1^2)}\right]^n,\]

(4)

where \(\phi(M, x)\) is the characteristic polynomial of the matrix

\[M = \begin{pmatrix} \phi_1 & \phi_{12} & \cdots & \phi_{1n} \\
\phi_{21} & \phi_2 & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_n \end{pmatrix},\]

where \(\phi_1 = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}\); \(\phi_{ij} = n_1J_2,\) if there is an arc from \(v_i\) to \(v_j\); \(\phi_{ij} = -n_1J_2,\) if there is an arc from \(v_i\) to \(v_j\), and \(\phi_{ij} = 0_2,\) if there is no arc between \(v_i\) and \(v_j\). where \(J_2\) is the all one matrix of order \(2 \times 2\) and \(0_2\) is the zero matrix of order \(2 \times 2\).

Proof. If \(\tilde{B}_1 = \tilde{B}_2 = \cdots = \tilde{B}_n\), then from Theorem 2.3 the \(2n\) eigenvalues of \(\overline{G}_1[\tilde{B}_1,\ldots, \tilde{B}_1]\) are given by the matrix.
\[ M = \begin{pmatrix}
\phi_1 & \phi_{12} & \ldots & \phi_{1n} \\
\phi_{21} & \phi_1 & \ldots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \ldots & \phi_1
\end{pmatrix},
\]

where \( \phi_1 = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \); \( \phi_{ij} = \begin{pmatrix} n_i & n_j \\ -n_j & n_i \end{pmatrix} \), if there is an arc from \( v_i \) to \( v_j \); \( \phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), if there is no arc between \( v_i \) and \( v_j \).

In the next theorem we determine the skew characteristic polynomial of the oriented graph \( \overrightarrow{H}_2 \), when the component oriented graphs are bipartite \( \overrightarrow{B}_i(V_i, U_i) \) with partite sets of same cardinality \( n_i \), having the skew adjacency matrix \( S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a \((0, 1)\)-matrix satisfying \( X_i e_{n_i} = r_i e_{n_i} \).

**Theorem 2.6.** Let \( \overrightarrow{G} \) be an oriented graph of order \( n \geq 2 \) having \( m \) arcs. For \( i = 1, 2, \ldots, n \), let \( \overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i) \), be a bipartite oriented graph with \( |V_i| = |U_i| = n_i \), having the skew adjacency matrix \( S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a \((0, 1)\)-matrix satisfying \( X_i e_{n_i} = r_i e_{n_i} \). Let \( P_i(\overrightarrow{B}_i, x) \), where \( i = 1, 2, \ldots, n \) be the skew characteristic polynomial of \( \overrightarrow{B}_i \). Then the skew characteristic polynomial of the oriented graph \( \overrightarrow{H}_3 \) of order \( N = \sum_{i=1}^{n} n_i \) is

\[
P_i(\overrightarrow{H}_3, x) = \phi(M, x) \prod_{i=1}^{n} P_i(\overrightarrow{B}_i, x) \frac{1}{(x^2 + r_i^2)},
\]

where \( \phi(M, x) \) is the characteristic polynomial of the matrix

\[
M = \begin{pmatrix}
\phi_1 & \phi_{12} & \ldots & \phi_{1n} \\
\phi_{21} & \phi_1 & \ldots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \ldots & \phi_1
\end{pmatrix},
\]

where \( \phi_1 = \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \); \( \phi_{ij} = \begin{pmatrix} n_i & n_j \\ -n_j & n_i \end{pmatrix} \), if there is an arc from \( v_i \) to \( v_j \); \( \phi_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), if there is no arc between \( v_i \) and \( v_j \).

**Proof.** The proof follows on similar lines as in Theorem 2.3 and is therefore omitted.

Let \( \overrightarrow{B}_1 = \overrightarrow{B}_1(V_1, U_1) \) and \( \overrightarrow{B}_2 = \overrightarrow{B}_2(V_2, U_2) \) be two oriented bipartite graphs of order \( 2n_1 \) and \( 2n_2 \), respectively. Let \( \overrightarrow{B} = \overrightarrow{B}_1 \oplus \overrightarrow{B}_2 \) be the join of \( \overrightarrow{B}_1 \) and \( \overrightarrow{B}_2 \). Clearly, \( \overrightarrow{B} = \overrightarrow{K}_2[\overrightarrow{B}_1, \overrightarrow{B}_2] \). The following consequence of Theorem 2.3, gives the skew spectrum of the join of two oriented bipartite graphs. We note that Theorem 2.7 is Theorem 5 obtained in (Adiga & Rakshith, 2016).

**Theorem 2.7.** For \( i = 1, 2 \), let \( \overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i) \), be a bipartite oriented graph with \( |V_i| = |U_i| = n_i \), having the skew adjacency matrix \( S(\overrightarrow{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a \((0, 1)\)-matrix satisfying \( X_i e_{n_i} = r_i e_{n_i} \). Let \( P_i(\overrightarrow{B}_i, x) \), where
Let \( \bar{B}_1 = \bar{B}_1(V_1, U_1) \) and \( \bar{B}_2 = \bar{B}_2(V_2, U_2) \) be oriented bipartite graphs with partite sets \( U_1, V_1, U_2 \) and \( V_2 \), respectively. The join-1 of \( \bar{B}_1 \) and \( \bar{B}_2 \), denoted by \( \bar{B}_1j_1\bar{B}_2 \), is defined in (Adiga & Rakshith, 2016) as the oriented graph obtained from \( \bar{B}_1 \) and \( \bar{B}_2 \) by joining arcs from all the vertices of \( U_1 \) to each the vertex of \( U_2 \) and \( V_2 \). The next Theorem was obtained as part first of Theorem 8 in (Adiga & Rakshith, 2016) and gives the skew characteristic polynomial of join-1 \( \bar{B}_1j_1\bar{B}_2 \) of the oriented graphs \( \bar{B}_1 \) and \( \bar{B}_2 \).

**Theorem 2.9.** For \( i = 1, 2 \), let \( \bar{B}_i = \bar{B}_i(V_i, U_i) \), be a bipartite oriented graph with \( |V_i| = |U_i| = n_i \), having the skew adjacency matrix \( S(\bar{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a \((0,1)\)-matrix satisfying \( X_i e_{n_i} = r_i e_{n_i} \). Let \( P_i(\bar{B}_i, x) \), where \( i = 1, 2 \), be the skew characteristic polynomial of \( \bar{B}_i \). Then the skew characteristic polynomial of join-1 \( \bar{B}_1j_1\bar{B}_2 \) is

\[
P_j(\bar{B}_1, \bar{B}_2, x) = \frac{P_1(\bar{B}_1, x)P_2(\bar{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)} \times \frac{P_j(\bar{B}_1, \bar{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)}.
\]

This shows that Theorem 2.9 is a generalization of the part first of Theorem 8 in (Adiga & Rakshith, 2016). In fact, the operation defined to obtain the oriented graph \( \bar{H}_2 \) is actually the generalization of the join-1 operation defined in (Adiga & Rakshith, 2016).

Let \( \bar{B}_1 = \bar{B}_1(V_1, U_1) \) and \( \bar{B}_2 = \bar{B}_2(V_2, U_2) \) be oriented bipartite graphs with partite sets \( U_1, V_1, U_2 \) and \( V_2 \), respectively. We define the join-1’ of \( \bar{B}_1 \) and \( \bar{B}_2 \), denoted by \( \bar{B}_1j_1'\bar{B}_2 \), as the oriented graph obtained from \( \bar{B}_1 \) and \( \bar{B}_2 \) by joining arcs from all the vertices of \( V_1 \) to each vertex of \( U_2 \) and \( V_2 \). In the next Theorem we obtain the skew characteristic polynomial of join-1’, \( \bar{B}_1j_1'\bar{B}_2 \) of oriented graphs \( \bar{B}_1 \) and \( \bar{B}_2 \).

**Theorem 2.10.** For \( i = 1, 2 \), let \( \bar{B}_i = \bar{B}_i(V_i, U_i) \), be a bipartite oriented graph with \( |V_i| = |U_i| = n_i \), having the skew adjacency matrix \( S(\bar{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a...
(0, 1)-matrix satisfying $X_i e_n = r_i e_n$. Let $P_i(\tilde{B}_i, x)$, where $i = 1, 2$ be the skew characteristic polynomial of $\tilde{B}_i$. Then the skew characteristic polynomial of join-1, $\tilde{B}_1 \tilde{f}_1 \tilde{B}_2$ is

$$P_i(\tilde{B}_1 \tilde{f}_1 \tilde{B}_2, x) = \left[ x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2 \right] \times \frac{P_i(\tilde{B}_1, x)P_i(\tilde{B}_2, x)}{(x^2 + r_1^2)(x^2 + r_2^2)}.$$

**Proof.** The proof follows from Theorem 2.6 by taking $\tilde{G} = \tilde{K}_2$ and using the fact that the characteristic polynomial of matrix $M$ is given by

$$M = \begin{pmatrix}
0 & r_1 & 0 & 0 \\
-r_1 & 0 & n_2 & n_3 \\
0 & -n_1 & 0 & r_3 \\
0 & -n_1 & -r_2 & 0
\end{pmatrix}$$

is $x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2r_2^2$.

### 3. Skew spectrum of some oriented graphs

As applications to the resulted obtained in Section 2, we obtain the skew spectrum of some special classes of oriented graphs.

Let $K_n$ be a complete graph on $n$ vertices. Any orientation of $K_n$ is said to be a tournament. Consider the complete $t$-partite graph $K_{2n_1, 2n_2, \ldots, 2n_t}$, it is easy to verify that $K_{2n_1, 2n_2, \ldots, 2n_t} = K_t[\overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_t}}]$. Let us orient the edges in $K_t$ arbitrarily to obtain the oriented graph $\overline{K}_t$, then oriented graph $\overline{K}_t[\overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_t}}]$ gives an orientation of the complete $t$-partite graph $K_{2n_1, 2n_2, \ldots, 2n_t}$, which we denote by $CT(2n_1, 2n_2, \ldots, 2n_t)$. In the following result we obtain the skew characteristic polynomial of $CT(2n_1, 2n_2, \ldots, 2n_t)$.

**Corollary 3.1.** The skew characteristic polynomial of $CT(2n_1, 2n_2, \ldots, 2n_t)$ is $\phi(M, x) = x^{N-\nu}$, where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix}
\phi_1 & \phi_{12} & \ldots & \phi_{1n} \\
\phi_{21} & \phi_2 & \ldots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \ldots & \phi_n
\end{pmatrix}, \quad (8)$$

with $\phi_{ij} = n_j J_2$ or $-n_j J_2$, according to as there is an arc from $v_i$ to $v_j$ or from $v_j$ to $v_i$ in $\overline{K}_n$ and $\phi_i = 0_2$, for all $i$.

**Proof.** Taking $\overline{G} = \overline{K}_n$, a tournament on $n$ vertices and $\overline{G}_i = \overline{K}_{2n_i}$, an empty graph, for all $i = 1, 2, \ldots, n$ in Theorem 2.3 and using the fact that the skew characteristic polynomial of $\overline{K}_{2n_i}$ is $P_i(\overline{K}_{2n_i}, x) = x^{2n_i}$, for all $i$, the result follows. Note that the empty graph $\overline{K}_{2n_i}$ can be considered as the bipartite graph with partite sets $V_i$ and $U_i$ of same cardinality $n_i$ and the skew adjacency matrix $S(\overline{K}_{2n_i})$ of $\overline{K}_{2n_i}$ given by $S(\overline{K}_{2n_i}) = \begin{pmatrix} 0_{n_i,n_i} & X_i \\ -X_i & 0_{n_i,n_i} \end{pmatrix}$, which satisfies $X_i e_n = 0 e_n$.

If $n_1 = n_2 = \ldots = n_t = a$, then by Corollary 3.1, it follows that the skew eigenvalues of $CT(2a, 2a, \ldots, 2a)$ is $\phi_i[\overline{K}_{2a_1}, \overline{K}_{2a_2}, \ldots, \overline{K}_{2a_t}]$ consists of the eigenvalue 0 with multiplicity $(2a - 1)t$ and the $t$ eigenvalues $2a\lambda_1, 2a\lambda_2, \ldots, 2a\lambda_t$, where $\lambda_1, \lambda_2, \ldots, \lambda_t$ are the skew eigenvalues of tournament $\overline{K}_t$. This follows from Corollary 3.1 and the fact that the quotient matrix $M$ for the oriented graph $\overline{K}_t[\overline{K}_{2a_1}, \overline{K}_{2a_2}, \ldots, \overline{K}_{2a_t}]$ is $M = S(\overline{K}_n) \otimes a I_t$.

Now using the Theorem 4.2.12 from (Horn & Johnson, 1985) the result follows. In fact, if $\overline{G}$ is any oriented graph of order $t$, then the skew eigenvalues of oriented graph $\overline{G}_i[\overline{K}_{2a_1}, \overline{K}_{2a_2}, \ldots, \overline{K}_{2a_t}]$ consists of the eigenvalue 0 with multiplicity $(2a - 1)t$ and the $t$ eigenvalues $2a\lambda_1, 2a\lambda_2, \ldots, 2a\lambda_t$, where $\lambda_1, \lambda_2, \ldots, \lambda_t$ are the skew eigenvalues of $\overline{G}$.

Taking $\overline{G} = \overline{K}_n$, a tournament on $n$ vertices and $\overline{G}_i = \overline{K}_{n_i}$, an orientation of the complete bipartite graph $K_{n_i}$ with all edges oriented from one partite set to another, for all $i = 1, 2, \ldots, n$, in Theorem 2.3 and using the fact that the skew characteristic polynomial of $\overline{K}_{n_i}$ is $P_i(\overline{K}_{n_i}, x) = x^{n_i - 2}(x^2 + n_i^2)$, we obtain the following consequence of Theorem 2.3.

**Corollary 3.2.** The skew characteristic polynomial of $\overline{K}_n[\overline{K}_{n_1}, \overline{K}_{n_2}, \ldots, \overline{K}_{n_m}]$, where $2n_1 + 2n_2 + \ldots + 2n_m = N$ with each $n_i \geq 1$, is given by

$$P_i(\overline{K}_n[\overline{K}_{n_1}, \overline{K}_{n_2}, \ldots, \overline{K}_{n_m}], x) = x^{N-\nu} \phi(M, x),$$

where $\phi(M, x)$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix}
\phi_1 & \phi_{12} & \ldots & \phi_{1n} \\
\phi_{21} & \phi_2 & \ldots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \ldots & \phi_n
\end{pmatrix}, \quad (9)$$

with $\phi_{ij} = n_j J_2$ or $-n_j J_2$, according to as there is an arc from $v_i$ to $v_j$ or from $v_j$ to $v_i$ in $\overline{K}_n$ and $\phi_i = 0_n$, for all $i$. 

$$\phi_i = \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix}.$$
Proof. Note that the skew adjacency matrix $S(\overline{K}_{n,n})$ of $\overline{K}_{n,n}$, given by $S(\overline{K}_{n,n}) = \begin{pmatrix} 0 & X_n \\ -X_n & 0 \end{pmatrix}$, satisfies $X_n e_n = n e_n$.

In particular, if $n_1 = n_2 = \cdots = n_r = a$ then by Corollary 3.2, it follows that the skew eigenvalues of $K_{n_1}[\overline{K}_{n_1,a}, \overline{K}_{n_1,a}, \cdots, \overline{K}_{n_1,a}]$ consists of the eigenvalue 0 with multiplicity $(2a - 2)n$ and the 2 $n$ eigenvalues of the matrix $S(\overline{K}_{n_i}) \otimes a_j + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \otimes I$, where I is the identity matrix of order $n$ and $S(\overline{K}_{n_i})$ is the skew matrix of $\overline{K}_{n_i}$. In fact, if $\overline{G}$ is any oriented graph of order $n$, then the skew eigenvalues of oriented graph $\overline{G}[\overline{K}_{n_i,a}, \overline{K}_{n_i,a}, \cdots, \overline{K}_{n_i,a}]$ consists of the eigenvalue 0 with multiplicity $(2a - 2)n$ and the 2 $n$ eigenvalues of the matrix $S(\overline{G}) \otimes a_j + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \otimes I$.

Let $B_i = B_i(V_i, U_i)$ be a $n_i$-regular bipartite graph with partite sets $|V_i| = |U_i| = n_i$. Let us orient the edges of $B_i$ from $V_i$ to $U_i$, then it is clear that the resulting oriented graph $\overline{B}_i$ is evenly-oriented. So, using the fact (see Theorem 5.4 in [Li & Lian, 2015]) that the skew spectrum of $\overline{B}_i$ is $i$ times the adjacency spectrum of $B_i$. It follows that the skew eigenvalues of $\overline{B}_i$ are $i \lambda_1, i \lambda_2, \ldots, i \lambda_n$, where $n_i = \lambda_1, \lambda_2, \ldots, -\lambda_i = \lambda_n$ are the adjacency eigenvalues (of $B_i$). Also, it is clear that the skew adjacency matrix $S(\overline{B}_i)$ is $\begin{pmatrix} 0 & X_i \\ -X_i & 0 \end{pmatrix} \otimes n_i$ satisfies $X_i e_n = n_i e_n$. Therefore, it follows from Theorem 2.3 that the skew spectrum of the oriented graph $\overline{G}[\overline{B}_1, \ldots, \overline{B}_n]$ consists of the eigenvalues $i \lambda_1, i \lambda_2, \ldots, i \lambda_{2n-1}$, for $i = 1, 2, \ldots, n$, the remaining 2 $n$ eigenvalues are given by the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_2 & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_n \end{pmatrix}$$

with $\phi_{ij} = n_j J_{2} - n_{-j} J_{2}$ or $0 J_{2}$, according to as there is an arc from $v_i$ to $v_j$ or from $v_j$ to $v_i$ or there is no arc between $v_i$ and $v_j$ in $\overline{G}$ and $\phi_i = \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix}$.

Consider the complete bipartite graphs $K_{a,a}$ and $K_{b,b}$. Let $V_1, U_1$ and $V_2, U_2$ be the partite sets of $K_{a,a}$ and $K_{b,b}$. Let us orient the edges of $K_{a,a}$ and $K_{b,b}$ in such a way that all the edges are directed from one partite set to another. Let $\overline{K}_{a,a}$ and $\overline{K}_{b,b}$ be the resulting oriented graphs. Since the oriented graphs $\overline{K}_{a,a}$ and $\overline{K}_{b,b}$ are evenly-oriented, it follows that their skew spectrum is $i$ times their adjacency spectrum. Therefore, the skew spectrum of $\overline{K}_{a,a}$ and $\overline{K}_{b,b}$ are $\{\pm a, 0, 0^{2a-2}\}$ and $\{\pm b, 0, 0^{2b-2}\}$, respectively. Moreover, their skew adjacency matrices $S(\overline{K}_{a,a}) = \begin{pmatrix} 0 & I_{a,a} \\ -I_{a,a} & 0 \end{pmatrix}$ and $S(\overline{K}_{b,b}) = \begin{pmatrix} 0 & J_{b,b} \\ -J_{b,b} & 0 \end{pmatrix}$ satisfies $I_{a,a} e_a = a e_a$ and $J_{b,b} e_b = b e_b$, respectively. We have the following consequence of Theorem 2.7, Theorem 2.9 and Theorem 2.10.

Corollary 3.3. Let $\overline{K}_{a,a}$ and $\overline{K}_{b,b}$ be the orientations of the complete bipartite graphs $K_{a,a}$ and $K_{b,b}$ defined above.

(i) The skew spectrum of oriented graph $\overline{K}_{a,a} \rightarrow \overline{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a + 2b - 4$, the remaining four eigenvalues are the zeros of the polynomial $x^4 + (a^2 + b^2 + 4ab)x^2 + a^2 b^2$.

(ii) The skew spectrum of oriented graph $\overline{K}_{a,a} \overline{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a + 2b - 4$, the remaining four eigenvalues are the zeros of the polynomial $x^4 + (a^2 + b^2 + 4ab)x^2 + a^2 b^2$.

(iii) The skew spectrum of oriented graph $\overline{K}_{a,a} \overline{K}_{b,b}$ consists of the eigenvalues 0 with multiplicity $2a + 2b - 4$, the remaining four eigenvalues are the zeros of the polynomial $x^4 + (a + b)^2 x^2 + a^2 b^2$.

Let $C_n$ be the cycle of order $n \geq 3$, where $n$ is even. Since $C_n$ is a bipartite graph, let us orient all the edges with direction from one partite set to another and let $\overline{C}_n$ be the resulting oriented graph. It is shown in [Adiga et al., 2010] that the skew spectrum of the oriented graph $C_n$ is $\{\pm i, 2 \sin n \frac{2 \pi}{n}, \ldots, 2 \sin n \frac{2 \pi}{n} \}$, for $j = 1, 2, \ldots, n - 2$. Let us orient the edges of the cycles $C_{2n_1}, C_{2n_2}$ and $C_{2n_3}$ in such a way that the orientations $\overline{C}_{2n_1}, \overline{C}_{2n_2}$ and $\overline{C}_{2n_3}$ are even-oriented, then using Theorem 2.8, we have the following observation, which gives the skew spectrum of the oriented graph $\overline{C}_{2n_1} \rightarrow (\overline{C}_{2n_2} \cup \overline{C}_{2n_3})$.

Corollary 3.4. Let $\overline{C}_{2n_i}$ be an evenly oriented cycle of order $2n_i$, for $i = 1, 2, 3$. Then the skew spectrum of the oriented graph $\overline{C}_{2n_1} \rightarrow (\overline{C}_{2n_2} \cup \overline{C}_{2n_3})$ consists of the eigenvalues $\pm 2 \sin \frac{2 \pi}{2n_1} \sin \frac{2 \pi}{2n_2}$, for $j = 1, 2, \ldots, 2n_k - 1$, where $k = 1, 2, 3$, the remaining six eigenvalues are given by the matrix $M$ in Theorem 2.8.

In particular, if $n_1 = n_2 = n_3 = a$, then the skew spectrum of the oriented graph $\overline{C}_{2a} \rightarrow (\overline{C}_{2a} \cup \overline{C}_{2a})$ consists of the eigenvalues $\pm 2 \sin \frac{2 \pi}{2a} \sin \frac{2 \pi}{2a}$, for $j = 1, 2, \ldots, 2a - 1$, each with multiplicity three and the six zeros of the polynomial $(x^2 + 4)(x^4 + (8 + 8a^2)x^2 + 16)$. In fact, if $B$ is a r-regular bipartite graph with partite sets $U$ and $V$ of same cardinality $n$ and $\overline{B}$ an orientation of $B$ with all edges directed from $U$ to $V$, then it is clear that the skew adjacency matrix $S(\overline{B})$ is $\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$ of $\overline{B}$ satisfies $X e_n = r e_n$. Therefore, using Theorem 2.8 the skew eigenvalues of the oriented graph $B \rightarrow (\overline{B} \cup \overline{B})$ consists of the eigenvalues $i \lambda_1, \ldots, i \lambda_{2n-1}$ each with multiplicity three, where $r = \lambda_1, \lambda_2, \ldots, \lambda_{2n-1}$ are the eigenvalues of $B$ and the remaining six eigenvalues are given by the zeros of the polynomial $(x^2 + r_1^2)(x^4 + (2r_1^2 + 8n_1^2)x^2 + r_1^4)$. 

Let \( \overline{E}_1 = \overline{K}_{2n_1}, \overline{E}_2 = \overline{K}_{n_2,n_3} \) and \( \overline{E}_3 = \overline{C}_{2n_1}, \) where \( \overline{K}_{2n_1} \) is the empty oriented graph of order \( 2n_1 \), \( \overline{K}_{n_2,n_3} \) is the even-oriented complete bipartite oriented graph of order \( 2n_2 \) and \( \overline{C}_{2n_1} \) is the directed cycle. Then from Theorem 2.7 and Theorem 2.8, we obtain the skew spectrum of the oriented graphs \( \overline{K}_{2n_1} \to \overline{K}_{n_2,n_3} \to \overline{C}_{2n_1} \to \overline{K}_{n_2,n_3} \to \overline{C}_{2n_1} \), \( \overline{K}_{2n_1} \to (\overline{K}_{n_2,n_3} \cup \overline{C}_{2n_1}) \to (\overline{K}_{2n_1} \cup \overline{C}_{2n_1}) \) and \( \overline{C}_{2n_1} \to (\overline{K}_{n_2,n_3} \cup \overline{C}_{2n_1}) \).

**Corollary 3.5.**

(i) The skew spectrum of \( \overline{K}_{2n_1} \to \overline{K}_{n_2,n_3} \) consists of the eigenvalue 0 with multiplicity \( 2n_1 + 2n_2 - 2 \) and the remaining two eigenvalues are \( \pm \sqrt{n_2^2 + 4n_1n_2} \).

(ii) The skew spectrum of \( \overline{K}_{2n_1} \to \overline{C}_{2n_1} \) consists of the eigenvalue 0 with multiplicity \( 2n_1 - 2 \), the eigenvalues \( 2i \sin \frac{2\pi(j - 1)}{2n_1} \), for \( j = 1, 2, \ldots, 2n_1 - 1 \) and the remaining two eigenvalues are \( \pm 2 \sqrt{n_1n_3 + 1} \).

(iii) The skew spectrum of \( \overline{K}_{n_2,n_3} \to \overline{C}_{2n_1} \) consists of the eigenvalue 0 with multiplicity \( 2n_2 - 4 \), the eigenvalues \( 2i \sin \frac{2\pi(j - 1)}{2n_1} \), for \( j = 1, 2, \ldots, 2n_1 - 1 \) and the remaining six eigenvalues are given by the matrix \( M \) given by Theorem 2.8 with \( r_1 = 0, r_2 = n_2 \) and \( r_3 = 2 \).

(iv) The skew spectrum of oriented graph \( \overline{K}_{2n_1} \to (\overline{K}_{n_2,n_3} \cup \overline{C}_{2n_1}) \) consists of the eigenvalue 0 with multiplicity \( 2n_1 + 2n_2 - 4 \), the eigenvalues \( 2i \sin \frac{2\pi(j - 1)}{2n_1} \), for \( j = 1, 2, \ldots, 2n_1 - 1 \) and the remaining six eigenvalues are given by the matrix \( M \) given by Theorem 2.8 with \( r_1 = n_2, r_2 = 0, r_3 = 2, n_1 = n_2 \) and \( n_2 = n_3 \).

(v) The skew spectrum of oriented graph \( \overline{K}_{n_2,n_3} \to (\overline{K}_{2n_1} \cup \overline{K}_{n_2,n_3}) \) consists of the eigenvalue 0 with multiplicity \( 2n_1 + 2n_2 - 4 \), the eigenvalues \( 2i \sin \frac{2\pi(j - 1)}{2n_1} \), for \( j = 1, 2, \ldots, 2n_1 - 1 \) and the remaining six eigenvalues are given by the matrix \( M \) in Theorem 2.8 with \( r_1 = 2, r_2 = 0, r_3 = n_3, n_1 = n_3, n_2 = n_1 \) and \( n_3 = n_2 \).

Similarly, we can obtain the skew spectrum of the oriented graphs \( \overline{C}_{n_1} \to (\overline{C}_{n_1} \cup \overline{K}_{n_1}), \overline{K}_{n_1} \to (\overline{C}_{n_1} \cup \overline{C}_{n_1}), \overline{C}_{n_1} \to (\overline{K}_{n_1} \cup \overline{K}_{n_1}), \overline{K}_{n_1} \to (\overline{C}_{n_1} \cup \overline{K}_{n_1}), \) etc.

### 4. Skew equienergetic oriented graphs

In this section, by using the results obtained in Section 2, we construct some new infinite families of non-cospectral skew equienergetic digraphs.

Two oriented graphs \( D_1 \) and \( D_2 \) are said to be skew equienergetic if they have same skew energy, that is, \( E_s(D_1) = E_s(D_2) \). If two oriented graphs are cospectral, then they are trivially skew equienergetic. Therefore, in what follows, we will be interested in finding skew equienergetic non-cospectral oriented graphs. The following problem was proposed in (Li & Lian, 2015) by Li and Lian. How to construct families of oriented graphs such that they have equal skew energy, but they do not have the same skew spectra?

The above problem was addressed by Ramane et al. (Ramane et al., 2016), Adiga et al. (Adiga & Rakshit, 2016) and Liu et al. (Liu et al., 2019). In (Ramane et al., 2016) the authors have extended the definition of join of graphs to oriented graphs. They obtained the skew spectrum of the join of two oriented graphs \( \overline{G}_1 \) and \( \overline{G}_2 \) with the property that the out-degree and in-degree of each vertex in \( \overline{G}_1 \) and \( \overline{G}_2 \) is same (that is the oriented graphs \( \overline{G}_1 \) and \( \overline{G}_2 \) are Eulerian digraphs). Using their results they have constructed some infinite families of non-cospectral skew equienergetic oriented graphs. In (Adiga & Rakshit, 2016) the authors have introduced some variations of the join of two oriented graphs for bipartite oriented graphs. They have defined four types of join operations for the bipartite oriented graphs. Using their results they were able to obtain some more infinite families of non-cospectral skew equienergetic oriented graphs. Recently, in (Liu et al., 2019) the authors have introduced the concept of corona and neighborhood corona of oriented graphs. Using these operations together with join operation they have constructed some new infinite families of non-cospectral skew equienergetic oriented graphs. Recently, the authors (Ganie, Ingle, et al., 2020) have extended the definition of join of two oriented graph by defining the joined union of oriented graphs. They have discussed the skew spectrum of the joined union of oriented Eulerian graphs and as applications they have added some new infinite families of non-cospectral skew equienergetic oriented graphs. Moreover, the results obtained in (Ramane et al., 2016) were obtained as particular cases. In the rest of this section, we aim to construct some new infinite families of non-cospectral skew equienergetic oriented graphs.

The following result gives the skew energy of the joined union \( \overline{G}[\overline{B}_1, \ldots, \overline{B}_n] \) of oriented bipartite graphs \( \overline{B}_1, \overline{B}_2, \ldots, \overline{B}_n \).

**Theorem 4.1.** Let \( \overline{G} \) be an oriented graph of order \( n \geq 2 \). For \( i = 1, 2, \ldots, n \), let \( \overline{B}_i \) be an oriented bipartite graph with partite sets of same cardinality \( n_i \) having the skew adjacency matrix \( S(\overline{B}_i) = \begin{pmatrix} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{pmatrix} \), where \( X_i \) is a


\[(0, 1)\)-matrix satisfying \( X_i e_n = r_i e_n. \) Then

\[
E_S(\overline{G}[\overline{B}_1, \ldots, \overline{B}_n]) = \sum_{i=1}^{n} E_S(\overline{B}_i) - 2 \sum_{i=1}^{n} r_i + 2 \sum_{i=1}^{n} |X_i|,
\]

where \( \pm \iota x_1, \pm \iota x_2, \ldots, \pm \iota x_n \) are the eigenvalues of the matrix \( M \) given in Theorem 2.3.

Proof. Since the skew adjacency matrix \( S(\overline{B}_i) = \begin{pmatrix} 0_{n \times n} & X_i \\ -X_i & 0_{n \times n} \end{pmatrix} \) of \( \overline{B}_i \) has the property that \( X_i e_n = r_i e_n. \)

It is easy to verify that \( \pm \iota r_i \) is eigenvalue of \( S(\overline{B}_i) \). Let \( \iota r_i, \psi_{i1}, \psi_{i1}, \ldots, \psi_{i2n-1} \) be the skew eigenvalues of \( \overline{B}_i \), for \( i = 1, 2, \ldots, n \). Then, it is clear from Theorem 2.3 that the skew eigenvalues of the oriented graph \( \overline{G}[\overline{B}_1, \ldots, \overline{B}_n] \)

\[= \psi_{i2}, \psi_{i3}, \ldots, \psi_{i2n-1} \] for \( i = 1, 2, \ldots, n \) and the remaining \( 2n \) eigenvalues are the eigenvalues of the matrix \( M \). Now, using the definition of skew energy the result follows.

Taking in particular \( \overline{B}_1 = \overline{B}_2 = \ldots = \overline{B}_n \) in Theorem 4.1, we obtain the skew energy of the oriented graph \( \overline{G}[\overline{B}_1, \ldots, \overline{B}_1] \).

Corollary 4.2. Let \( \overline{G} \) be an oriented graph of order \( n \geq 2 \). Let \( \overline{B}_1 \) be an oriented bipartite graph with partite sets of same cardinality \( n \) having the skew adjacency matrix \( S(\overline{B}_1) = \begin{pmatrix} 0_{n \times n} & X_1 \\ -X_1 & 0_{n \times n} \end{pmatrix} \), where \( X_1 \) is a \((0, 1)\)-matrix satisfying \( X_1 e_n = r_1 e_n. \) Then

\[
E_S(\overline{G}[\overline{B}_1, \ldots, \overline{B}_1]) = nE_S(\overline{B}_1) - 2nr_1 + \sum_{i=1}^{n} |X_i|,
\]

where \( \pm \iota x_1, \pm \iota x_2, \ldots, \pm \iota x_n \) are the eigenvalues of the matrix \( S(\overline{G}) \otimes n_1 J_2 + \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \otimes I. \)

Since the matrix \( M \) is determined by the structure of \( \overline{G} \) and the orders \( n_1 \) of the oriented bipartite graphs \( \overline{B}_i \), for \( i = 1, 2, \ldots, n \). We have the following observation from Theorem 4.1.

Corollary 4.3. Let \( \overline{G} \) be an oriented graph of order \( n \geq 2 \). Let \( \overline{B}_i \) and \( \overline{G}_i \) be oriented bipartite graphs of order \( n \), for \( i = 1, 2, \ldots, n \), and let \( S(\overline{B}_i) = \begin{pmatrix} 0_{n \times n} & X_i \\ -X_i & 0_{n \times n} \end{pmatrix} \) and \( S(\overline{G}_i) = \begin{pmatrix} 0_{n \times n} & Y_i \\ -Y_i & 0_{n \times n} \end{pmatrix} \) be their skew adjacency matrices with \( X_i e_n = r_i e_n = Y_i e_n. \) If the oriented bipartite graphs \( \overline{B}_i \) and \( \overline{G}_i \) are non-cospectral with \( E_S(\overline{B}_i) = E_S(\overline{G}_i) \), for all \( i = 1, 2, \ldots, n \), then the oriented graphs \( \overline{G}[\overline{B}_1, \ldots, \overline{B}_n] \) and \( \overline{G}[\overline{G}_1, \ldots, \overline{G}_n] \) are non-cospectral with

\[
E_S(\overline{G}[\overline{B}_1, \ldots, \overline{B}_n]) = E_S(\overline{G}[\overline{G}_1, \ldots, \overline{G}_n]).
\]

For \( i = 1, 2, \ldots, n \), let \( \overline{B}_i \) be a bipartite graph with partite sets \( V_i \) and \( U_i \) of same cardinality \( n_i \). Let us orient the edges of \( \overline{B}_i \) in such a way that the resulting orientation \( \overline{B}_i \) is uniformly oriented. Then, using the fact (see Theorem 5.4 in (Li & Lian, 2015)) that the skew spectrum of \( \overline{B}_i \) is \( \iota \) times the adjacency spectrum of \( B_i \). It follows that the skew energy of \( \overline{B}_i \) is same as the energy of \( B_i \), that is, \( E_S(\overline{B}_i) = E(B_i) \), for all \( i = 1, 2, \ldots, n \). Therefore, we have following observation which gives the construction of skew equienergetic oriented graphs from the equienergetic graphs.

**Corollary 4.4.** Let \( \overline{G} \) be an oriented graph of order \( n \geq 2 \). Let \( \overline{B}_i \) and \( \overline{G}_i \) be \( r_i \)-regular bipartite equienergetic graphs with partite sets of same cardinality \( n_i \), for \( i = 1, 2, \ldots, n \). If the orientations \( \overline{B}_i \) and \( \overline{G}_i \) are uniformly oriented, then the oriented graphs \( \overline{G}[\overline{B}_1, \ldots, \overline{B}_n] \) and \( \overline{G}[\overline{G}_1, \ldots, \overline{G}_n] \) are non-cospectral with

\[
E_S(\overline{G}[\overline{B}_1, \ldots, \overline{B}_n]) = E_S(\overline{G}[\overline{G}_1, \ldots, \overline{G}_n]).
\]

A lot of papers can be found in the literature regarding the construction of equienergetic graphs, see the book (Li et al., 2012) and the references therein. Let \( D(\overline{G}) \) be the duplication digraph of a digraph \( G \) defined in (Adiga & Rakshith, 2016). Since, the graph \( D(G) \) is always a bipartite graph with \( E(D(G)) = 2E(G) \), giving that if \( G_i \) and \( H_i \) are equienergetic graphs then the bipartite graphs \( D(G_i) \) and \( D(H_i) \) are also equienergetic. Thus, from any given pair of equienergetic regular graphs we can construct a pair of bipartite equienergetic regular graphs which in turn can be used to construct a pair of skew equienergetic oriented graphs by Corollary 4.4.

Taking in particular \( \overline{G} = \overline{K}_2 \) in Theorem 4.1 and using Theorem 2.7, we obtain the following result which is Theorem 6 in (Adiga & Rakshith, 2016), and gives the skew energy of the join of oriented bipartite graphs \( \overline{B}_1 \) and \( \overline{B}_2 \).

**Corollary 4.5.** For \( i = 1, 2 \), let \( \overline{B}_i = \overline{B}_i(V_i, U_i) \) be an oriented bipartite graph with partite sets \( V_i \) and \( U_i \) of same cardinality \( n_i \) and skew adjacency matrix \( S(\overline{B}_i) = \begin{pmatrix} 0_{n \times n} & X_i \\ -X_i & 0_{n \times n} \end{pmatrix} \), where \( X_i \) is a \((0, 1)\)-matrix satisfying \( X_i e_n = r_i e_n. \) Then

\[
E_S(\overline{B}_1 \rightarrow \overline{B}_2) = E_S(\overline{B}_1) + E_S(\overline{B}_2) + 2(|x_1| + |x_2| - r_1 - r_2),
\]
where \( \pm ix_1, \pm ix_2 \) are the zeros of the polynomial 
\[ x^4 + (r_1^2 + r_2^2 + 4n_1n_2)x^2 + r_1^2 + r_2^2. \]

If \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are two oriented graphs which are non-
cospectral with respect to skew matrix, then we have the following consequence of 
Corollary 4.2, which gives a new infinite family of non-cospectral skew equienergetic 
oriented graphs.

**Corollary 4.6.** Let \( \overrightarrow{G} \) and \( \overrightarrow{H} \) be two non-cospectral oriented 
graphs of order \( n \geq 2 \). Let \( \overrightarrow{B}_1 = \overrightarrow{B}_1(V_i, U_i) \) be an oriented 
bipartite graph with partite sets \( V_i \) and \( U_i \) of same cardinality \( n_1 \) and skew adjacency matrix 
\( S(\overrightarrow{B}_1) = \left( \begin{array}{cc} 0_{n_1 \times n_1} & X_i \\ -X_i & 0_{n_1 \times n_1} \end{array} \right) \), where \( X_i \) is a \((0,1)\)-matrix satisfying 
\( X_i e_n = r_i e_n \). If

\[
\sum_{i=1}^{n} \lambda \left( \overrightarrow{S}(\overrightarrow{G}) \otimes n_1j_2 + A \right) = \sum_{i=1}^{n} \lambda \left( \overrightarrow{S}(\overrightarrow{H}) \otimes n_1j_2 + A \right),
\]

where \( A = \left( \begin{array}{cc} 0 & r_i \\ -r_i & 0 \end{array} \right) \). Then

\[
E_S(\overrightarrow{G}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]) = E_S(\overrightarrow{H}[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n]).
\]

Let \( \overrightarrow{H}_2 \) be the variation of the joined union of oriented 
bipartite graphs defined in Section 2. Proceeding similar 
to Theorem 4.1, we have the following result which gives the 
skew energy of \( \overrightarrow{H}_2 \).

**Theorem 4.7.** Let \( \overrightarrow{G} \) be an oriented graph of order \( n \geq 2 \). For \( i = 1, 2, \ldots, n \), let \( \overrightarrow{B}_i \) be a bipartite oriented graph 
with partite sets of same cardinality \( n_i \) having the skew 
adjacency matrix \( S(\overrightarrow{B}_i) = \left( \begin{array}{cc} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{array} \right) \), where \( X_i \) is a 
\((0,1)\)-matrix satisfying \( X_i e_n = r_i e_n \). Let \( \overrightarrow{H}_2 \) be the oriented 
graph defined in Section 2, then

\[
E_S(\overrightarrow{H}_2) = \sum_{i=1}^{n} E_S(\overrightarrow{B}_i) - 2 \sum_{i=1}^{n} r_i + 2 \sum_{i=1}^{n} |x_i|,
\]

where \( \pm ix_1, \pm ix_2, \ldots, \pm ix_n \) are the eigenvalues of the 
matrix \( M \) given in Theorem 2.5.

Since the matrix \( M \) is determined by the structure of 
\( \overrightarrow{G} \) and the orders \( n_i \) of the oriented bipartite graphs \( \overrightarrow{B}_i \), for 
\( i = 1, 2, \ldots, n \). We have the following observation from 
Theorem 4.7.

**Corollary 4.8.** Let \( \overrightarrow{G} \) be an oriented graph of order \( n \geq 2 \). Let 
\( \overrightarrow{B}_i = \overrightarrow{B}_i(V_i, U_i) \) and \( \overrightarrow{G}_i = \overrightarrow{G}_i(P_i, Q_i) \) be oriented 
bipartite graphs with partite sets \( V_i, U_i \) and \( P_i, Q_i \) of same 
cardinality \( n_i \), for \( i = 1, 2, \ldots, n \). Let \( S(\overrightarrow{B}_i) = \left( \begin{array}{cc} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{array} \right) \)

and \( S(\overrightarrow{G}_i) = \left( \begin{array}{cc} 0_{n_i \times n_i} & Y_i \\ -Y_i & 0_{n_i \times n_i} \end{array} \right) \) be their skew adjacency 
matrices with \( X_i e_n = r_i e_n = Y_i e_n \). Let \( \overrightarrow{H}_1 \) be the oriented 
graph obtained from \( \overrightarrow{G}_i[\overrightarrow{B}_1, \ldots, \overrightarrow{B}_n] \) by deleting all the arcs 
between \( U_i \) and \( V_j \), \( U_j \), \( i \neq j \) and let \( \overrightarrow{H}_1' \) be the oriented 
graph obtained from \( \overrightarrow{G}_i[\overrightarrow{B}_1, \ldots, \overrightarrow{G}_i] \) by deleting all the arcs 
between \( Q_i, P_i, Q_i, \) \( i \neq j \). If the oriented bipartite graphs \( \overrightarrow{B}_i \) and \( \overrightarrow{G}_i \) are non-cospectral with \( E_G(\overrightarrow{B}_i) = E_G(\overrightarrow{G}_i), \) for all 
\( i = 1, 2, \ldots, n \), then the oriented graphs \( \overrightarrow{H}_1 \) and \( \overrightarrow{H}_1' \) are non-cospectral with

\[
E_S(\overrightarrow{H}_1) = E_S(\overrightarrow{H}_1').
\]

Taking \( \overrightarrow{G} = \overrightarrow{K}_2 \) in Theorem 4.7, we obtain the following result obtained in (Adiga & Rakshith, 2016) as part first of 
Theorem 9.

**Corollary 4.9.** For \( i = 1, 2 \), let \( \overrightarrow{B}_i \) be an oriented bipartite 
graph with partite sets of same cardinality \( n_i \) having the skew 
adjacency matrix \( S(\overrightarrow{B}_i) = \left( \begin{array}{cc} 0_{n_i \times n_i} & X_i \\ -X_i & 0_{n_i \times n_i} \end{array} \right) \), where \( X_i \) is a 
\((0,1)\)-matrix satisfying \( X_i e_n = r_i e_n \). Then

\[
E_S(\overrightarrow{B}_1, \overrightarrow{B}_2) = E_S(\overrightarrow{B}_1) + E_S(\overrightarrow{B}_2) + 2(|x_1| + |x_2| - r_1 - r_2),
\]

where \( \pm ix_1, \pm ix_2 \) are the zeros of the polynomial 
\[ x^4 + (r_1^2 + r_2^2 + 2n_1n_2)x^2 + r_1^2 + r_2^2. \]

5. Conclusion

In this paper we have discussed the skew characteristic 
polynomial and the skew eigenvalues of the joined 
union and some of its variations for the oriented bipartite graphs. As applications, we have given a general method 
to construct infinite families of oriented graphs with same 
skew energy but different skew spectrum.. Our ideas and 
results obtained generalize some of the ideas and results 
in Adiga & Rakshith, (2016).

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