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Research article

Relations between ordinary energy and energy of a self-loop graph

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ABSTRACT

Let G be a graph on n vertices with vertex set $V(G)$ and let $S \subseteq V(G)$ with $|S| = \alpha$. Denote by G_S, the graph obtained from G by adding a self-loop at each of the vertices in S. In this note, we first give an upper bound and a lower bound for the energy of G_S ($\mathcal{E}(G_S)$) in terms of ordinary energy ($\mathcal{E}(G)$), order (n) and number of self-loops (α). Recently, it is proved that for a bipartite graph G_S, $\mathcal{E}(G_S) \geq \mathcal{E}(G)$. Here we show that this inequality is strict for an unbalanced bipartite graph G_S with $0 < \alpha < n$. In other words, we show that there exits no unbalanced bipartite graph G_S with $0 < \alpha < n$ and $\mathcal{E}(G_S) = \mathcal{E}(G)$.

1. Introduction

Let G be a simple graph on n vertices with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $S \subseteq V(G)$ with $|S| = \alpha$. The graph G_S is obtained from G by adding a self-loop at each of the vertices in S. Adjacency matrix of G_S is denoted by $A(G_S)$ and is defined as $A(G_S) = A(G) + D_S(G)$, where $A(G)$ is the well-known adjacency matrix of G and $D_S(G)$ is the diagonal matrix with its i-th diagonal entry equal to 1 or 0 according as $v_i \in S$ or $v_i \notin S$, respectively.

We denote the i-th largest eigenvalue of $A(G_S)$ (resp. A(G)) by λ_i^S (resp. λ_i). The sum of absolute values of λ_i for $i = 1, 2, \dots, n$ is called the energy of a graph G. Nowadays it is called as ordinary energy. For its basic mathematical properties, including various lower and upper bounds, see [1,3–5,7] and especially the book [11]. Applications of graph energy can be found in the articles [13,14]. The sum $\sum_{i=1}^{n} \left|\lambda_i^S - \frac{\alpha}{n}\right|$ is called the energy of graph G_S and is denoted by $\mathcal{E}(G_S)$. The definition of energy of a graph with self-loops was put forward very recently by Gutman et al. in [8]. The authors in [8] showed that if G is a bipartite graph, then $\mathcal{E}(G_S) = \mathcal{E}(G_{V(G)\setminus S})$. Also, they conjectured that $\mathcal{E}(G_S) > \mathcal{E}(G)$ for $1 \leq \alpha \leq n-1$. However this conjecture was disproved in [10] by means of counterexamples. In [2], Akbari et al. showed that this conjecture was nevertheless not a complete miss by proving that energy of a bipartite graph G_S is always greater than or equal to its ordinary energy. In [12], Popat et al. obtained a family of graphs which satisfies the property $\mathcal{E}(G_S) = \mathcal{E}(G)$ and $0 < \alpha < n$. Some bounds on energy of self-loop graph are presented in [2,8]

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Motivated by this, in this paper, we give an upper bound and a lower bound for the energy of G_S ($\mathcal{E}(G_S)$) in terms of ordinary energy ($\mathcal{E}(G)$), order (n) and number of self-loops (α). Also, we show that for an unbalanced bipartite graph G_S with $0 < \alpha < n$, $\mathcal{E}(G_S) > \mathcal{E}(G)$. In other words, we show that there exits no unbalanced bipartite self-loops graph G_S with $0 < \alpha < n$ and $\mathcal{E}(G_S) = \mathcal{E}(G)$.

2. Main results

Let A be an $n \times n$ complex matrix and denote its singular values by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. Since the matrices $A(G)$ and $A(G_S)$ are symmetric, $\mathcal{E}(G) = \sum_{i=1}^{n} s_i(A(G))$ and $\mathcal{E}(G_S) = \sum_{i=1}^{n} s_i\left(A(G_S) - \frac{\alpha}{n} I_n\right)$. We need the following lemmas to prove our main result.

Lemma 2.1. [6] *Let A and B be two $n \times n$ matrices. Then*

$$\sum_{i=1}^{n} s_i(A+B) \leq \sum_{i=1}^{n} s_i(A) + \sum_{i=1}^{n} s_i(B)$$

and the equality holds if and only if there exists an unitary matrix P such that both the matrices PA and PB are positive semi-definite. Moreover, if A and B are real matrices, then the matrix P can be taken as real orthogonal.

Lemma 2.2. [9] *Let $A = [a_{ij}]_{n\times n}$ be a positive semi-definite matrix such that $a_{ii} = 0$ for some $1 \leq i \leq n$. Then $a_{ij} = a_{ji} = 0$ for all $1 \leq j \leq n$.*

The following lemma is the polar decomposition of a matrix.

Lemma 2.3. [9] *Let A be a real square matrix of order n. Then there exists an orthogonal matrix P and a positive definite matrix Q such that $A = PQ$. Moreover the matrix Q is uniquely determined by A as $Q = \sqrt{A^T A}$.*

Lemma 2.4. [9] *Let A a real matrix of order $m \times n$ and let $q = min\{m, n\}$. If $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q$ are the singular values of A, then the eigenvalues of the matrix $\begin{bmatrix} \mathbf{0} & A \\ A^T & \mathbf{0} \end{bmatrix}$ are $-\sigma_1 \leq -\sigma_2 \leq \cdots \leq -\sigma_q \leq \underbrace{0 = 0 = \cdots = 0}_{|n-m|} \leq \sigma_q \leq \sigma_{q-1} \leq \cdots \leq \sigma_1$.*

We denote the complete graph on n vertices by K_n and its complement graph by $\overline{K_n}$ or nK_1. The following theorem gives an upper bound and a lower bound for $\mathcal{E}(G_S)$ in terms of $\mathcal{E}(G)$, n and α.

Theorem 2.5. *Let G be a graph on n vertices and let $S \subseteq V(G)$ with $|S| = \alpha$. Then*

$$\mathcal{E}(G) - \frac{2\alpha(n-\alpha)}{n} \leq \mathcal{E}(G_S) \leq \mathcal{E}(G) + \frac{2\alpha(n-\alpha)}{n}.$$

Moreover the right inequality holds if and only if $\alpha = 0, n$ or $G \cong nK_1$. Whereas the left inequality holds if and only if $\alpha = 0, n$.

Proof. Let $G[S]$ and $G[V(G)\backslash S]$ be the subgraphs of G induced by the vertex sets S and $V(G)\backslash S$, respectively. Then

$$A(G) = \begin{bmatrix} A(G[S]) & X \\ X^T & A(G[V(G)\backslash S]) \end{bmatrix} \text{ and } A(G_S) = \begin{bmatrix} A(G[S]) + I_\alpha & X \\ X^T & A(G[V(G)\backslash S]) \end{bmatrix},$$

where X is a $(0,1)$-matrix.

Right inequality: We have

$$A(G_S) - \frac{\alpha}{n} I_n = A(G) + \begin{bmatrix} \left(1 - \frac{\alpha}{n}\right) I_\alpha & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{n} I_{n-\alpha} \end{bmatrix}. \tag{1}$$

Applying Lemma 2.1 to equation (1), we get

$$\sum_{i=1}^{n} s_i\left(A(G_S) - \frac{\alpha}{n} I_n\right) \leq \sum_{i=1}^{n} s_i(A(G)) + \sum_{i=1}^{n} s_i\left(\begin{bmatrix} \left(1 - \frac{\alpha}{n}\right) I_\alpha & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{n} I_{n-\alpha} \end{bmatrix}\right). \tag{2}$$

That is,

$$\mathcal{E}(G_S) \leq \mathcal{E}(G) + \frac{2\alpha(n-\alpha)}{n}. \tag{3}$$

Proving the right equality. Suppose the equality in (3) holds. Then the equality in (2) also holds and thus by Lemma 2.1 there exists an orthogonal matrix P such that the matrices $PA(G)$ and $P\begin{bmatrix} \left(1-\frac{\alpha}{n}\right)I_\alpha & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{n}I_{n-\alpha}\end{bmatrix}$ are positive semi-definite. Moreover, since $A(G)$ is symmetric, we can further assume that P is symmetric. Let $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}$ and $0<\alpha<n$. Since $P\begin{bmatrix} \left(1-\frac{\alpha}{n}\right)I_\alpha & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{n}I_{n-\alpha}\end{bmatrix}$ and P are symmetric matrices, we must have

$$\begin{bmatrix} \left(1-\frac{\alpha}{n}\right)P_{11} & -\frac{\alpha}{n}P_{12} \\ \left(1-\frac{\alpha}{n}\right)P_{21} & -\frac{\alpha}{n}P_{22}\end{bmatrix}=\begin{bmatrix} \left(1-\frac{\alpha}{n}\right)P_{11} & \left(1-\frac{\alpha}{n}\right)P_{12} \\ -\frac{\alpha}{n}P_{21} & -\frac{\alpha}{n}P_{22}\end{bmatrix}.$$

Therefore $P_{12}=P_{21}=\mathbf{0}$, otherwise $1-\frac{\alpha}{n}=-\frac{\alpha}{n}$, that is, $1=0$, a contradiction. Thus $P=\begin{bmatrix} P_{11} & \mathbf{0} \\ \mathbf{0} & P_{22}\end{bmatrix}$, where P_{11} and P_{22} are orthogonal matrices of order α and $n-\alpha$. Now, $P\begin{bmatrix} \left(1-\frac{\alpha}{n}\right)I_\alpha & \mathbf{0} \\ \mathbf{0} & -\frac{\alpha}{n}I_{n-\alpha}\end{bmatrix}$ is positive semi-definite and so P_{11} is positive semi-definite and P_{22} is negative semi-definite. As, $P_{11}^2=I_\alpha$, $P_{22}^2=I_{n-\alpha}$, P_{11} is a positive semi-definite matrix and P_{22} is negative semi-definite matrix, the minimal polynomial equation of P_{11} and P_{22} must be $x-1=0$ and $x+1=0$, respectively. Therefore, $P_{11}=I_\alpha$ and $P_{22}=-I_{n-\alpha}$. Hence, $P=\begin{bmatrix} I_\alpha & \mathbf{0} \\ \mathbf{0} & -I_{n-\alpha}\end{bmatrix}$. Now, $PA(G)=\begin{bmatrix} A(G[S]) & X \\ -X^T & -A(G[V(G)\backslash S])\end{bmatrix}$ and since $PA(G)$ is positive semi-definite with all its diagonal entries equal to 0, by Lemma 2.2, we must have $A(G)=\mathbf{0}$. So, $G\cong nK_1$. Conversely, if $\alpha=0,n$ or $G\cong nK_1$, then one can easily see that equality holds.

Left inequality: From (1) and by Lemma 2.1, we get

$$\sum_{i=1}^{n}s_i(A(G))\le\sum_{i=1}^{n}s_i\left(A(G_S)-\frac{\alpha}{n}I_n\right)+\sum_{i=1}^{n}s_i\left(\begin{bmatrix} \left(\frac{\alpha}{n}-1\right)I_\alpha & \mathbf{0} \\ \mathbf{0} & \frac{\alpha}{n}I_{n-\alpha}\end{bmatrix}\right). \tag{4}$$

Therefore,

$$\mathcal{E}(G)\le\mathcal{E}(G_S)+\frac{2\alpha(n-\alpha)}{n} \tag{5}$$

Proving the left equality. Suppose the equality in (5) holds. Then the equality in (4) also holds and thus by Lemma 2.1 there exists an orthogonal matrix P such that the matrices $P\left(A(G_S)-\frac{\alpha}{n}I_n\right)$ and $P\begin{bmatrix} \left(\frac{\alpha}{n}-1\right)I_\alpha & \mathbf{0} \\ \mathbf{0} & \frac{\alpha}{n}I_{n-\alpha}\end{bmatrix}$ are positive semi-definite. Assume that $0<\alpha<n$. Let $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}$. By the similar argument as above, we get $P_{12}=P_{21}=\mathbf{0}$, $P_{11}=-I_\alpha$ and $P_{22}=I_{n-\alpha}$. Therefore,

$$P\left(A(G_S)-\frac{\alpha}{n}I_n\right)=\begin{bmatrix} -A(G[S])+\left(\frac{\alpha}{n}-1\right)I_\alpha & -X \\ X^T & A(G[V(G)\backslash S])-\frac{\alpha}{n}I_{n-\alpha}\end{bmatrix},$$

and its diagonal entries are negative for $0<\alpha<n$. This implies that $P(A(G_S)-\frac{\alpha}{n}I_n)$ is not positive semi-definite, a contradiction. Thus $\alpha=0$ or n. Conversely, if $\alpha=0,n$, then one can easily see that equality holds. This completes the proof. □

The following corollary follows immediately from the above theorem.

Corollary 2.6. *Let G be a graph on n vertices. Then for any vertex v of G, we have*

$$\mathcal{E}(G)-2<\mathcal{E}(G_{\{v\}})<\mathcal{E}(G)+2.$$

Theorem 2.7. *Let G be an unbalanced bipartite graph on n vertices and let $S\subseteq V(G)$ with $|S|=\alpha$. Then $\mathcal{E}(G)<\mathcal{E}(G_S)$ for $0<\alpha<n$.*

Proof. Let U and W be the vertex partition sets of the bipartite graph G such that $|U|=n_1$ and $|W|=n_2$, where $n=n_1+n_2$. Suppose $S_1=S\cap U$ and $S_2=S\cap W$. Then

$$A(G) = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix} \text{ and } A(G_S) = \begin{bmatrix} A([n_1K_1]_{S_1}) & X \\ X^T & A([n_2K_1]_{S_2}) \end{bmatrix},$$

where X is a (0, 1)-matrix of order $n_1 \times n_2$.

Let $A'(G_S) = \begin{bmatrix} -A([n_1K_1]_{S_1}) & X \\ X^T & -A([n_2K_1]_{S_2}) \end{bmatrix}$. Then

$$\left(A(G_S) - \frac{\alpha}{n} I_n\right) + \left(A'(G_S) + \frac{\alpha}{n} I_n\right) = 2A(G). \tag{6}$$

Applying Lemma 2.1 to the equation (6), we get

$$\sum_{i=1}^{n} s_i\left(A(G_S) - \frac{\alpha}{n} I_n\right) + \sum_{i=1}^{n} s_i\left(A'\left(G_S\right) + \frac{\alpha}{n} I_n\right) \geq 2\sum_{i=1}^{n} s_i(A(G)). \tag{7}$$

Since $A(G_S) - \frac{\alpha}{n} I_n = \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & -I_{n_2} \end{bmatrix} \left[-A'(G_S) - \frac{\alpha}{n} I_n\right] \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & -I_{n_2} \end{bmatrix}$, the matrices $A(G_S) - \frac{\alpha}{n} I_n$ and $-\left[A'(G_S) + \frac{\alpha}{n} I_n\right]$ are similar. Therefore these matrices share the same singular eigenvalues. Thus from equation (7), we have

$$\mathcal{E}(G_s) \geq \mathcal{E}(G) \tag{8}$$

Suppose that the equality in relation (8) holds. Then the equality in relation (7) holds and thus by Lemma 2.1 there exists a real orthogonal matrix P such that the matrices $P\left(A(G_S) - \frac{\alpha}{n} I_n\right)$ and $P\left(A'(G_S) + \frac{\alpha}{n} I_n\right)$ are positive semi-definite. Now by Lemma 2.3,

$$P\left(A(G_S) - \frac{\alpha}{n} I_n\right) = \sqrt{\left(A(G_S) - \frac{\alpha}{n} I_n\right)^T \left(A(G_S) - \frac{\alpha}{n} I_n\right)} \tag{9}$$

and

$$P\left(A'(G_S) + \frac{\alpha}{n} I_n\right) = \sqrt{\left(A'(G_S) + \frac{\alpha}{n} I_n\right)^T \left(A'(G_S) + \frac{\alpha}{n} I_n\right)}. \tag{10}$$

Since the matrices $A(G_S) - \frac{\alpha}{n} I_n$ and $-\left[A'(G_S) + \frac{\alpha}{n} I_n\right]$ are similar, from equations (9) and (10) it follows that the matrices $P\left(A(G_S) - \frac{\alpha}{n} I_n\right)$ and $P\left(A'(G_S) + \frac{\alpha}{n} I_n\right)$ are similar. In particular,

$$P\left(A(G_S) - \frac{\alpha}{n} I_n\right) = \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & -I_{n_2} \end{bmatrix} \left[P\left(A'(G_S) + \frac{\alpha}{n} I_n\right)\right] \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & -I_{n_2} \end{bmatrix} \tag{11}$$

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, where P_{11} and P_{22} are square matrices of order n_1 and n_2. Then from equation (11), we get

$$\begin{bmatrix} P_{11}\left[A([n_1K_1]_{S_1}) - \frac{\alpha}{n} I_{n_1}\right] + P_{12}X^T & P_{11}X + P_{12}\left[A([n_2K_1]_{S_2}) - \frac{\alpha}{n} I_{n_2}\right] \\ P_{21}\left[A([n_1K_1]_{S_1}) - \frac{\alpha}{n} I_{n_1}\right] + P_{22}X^T & P_{21}X + P_{22}\left[A([n_2K_1]_{S_2}) - \frac{\alpha}{n} I_{n_2}\right] \end{bmatrix}$$
$$= \begin{bmatrix} -P_{11}\left[A([n_1K_1]_{S_1}) - \frac{\alpha}{n} I_{n_1}\right] + P_{12}X^T & -P_{11}X + P_{12}\left[A([n_2K_1]_{S_2}) - \frac{\alpha}{n} I_{n_2}\right] \\ P_{21}\left[A([n_1K_1]_{S_1}) - \frac{\alpha}{n} I_{n_1}\right] - P_{22}X^T & P_{21}X - P_{22}\left[A([n_2K_1]_{S_2}) - \frac{\alpha}{n} I_{n_2}\right] \end{bmatrix}.$$

Therefore, $P_{11}\left[A([n_1K_1]_{S_1}) - \frac{\alpha}{n} I_{n_1}\right] = \mathbf{0}$ and $P_{22}\left[A([n_2K_1]_{S_2}) - \frac{\alpha}{n} I_{n_2}\right] = \mathbf{0}$. Since $0 < \alpha < n$, we must have $P_{11} = P_{22} = 0$. Thus, $P = \begin{bmatrix} \mathbf{0} & P_{12} \\ P_{21} & \mathbf{0} \end{bmatrix}$. Since $n_1 \neq n_2$, by Lemma 2.4, 0 is an eigenvalue of P (an orthogonal matrix), a contradiction. Thus $\mathcal{E}(G_S) > \mathcal{E}(G)$. □

CRediT authorship contribution statement

B.R. Rakshith: Writing – review & editing, Writing – original draft, Investigation, Formal analysis. **Kinkar Chandra Das:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis. **B.J. Manjunatha:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis. **Yilun Shang:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yilun Shang is a Section Editor for Heliyon.

Data availability

No data was used for the research described in the article.

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