Palatini formulation of non-local gravity

F. Briscese[1, 2, *] and M. L. Pucheu[3, †]

[1] *Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne, UK*

[2] *Istituto Nazionale di Alta Matematica Francesco Severi, Gruppo Nazionale di Fisica Matematica, Città Universitaria, P. le A. Moro 5, 00185 Rome, EU*

[3] *Departamento de Física, CCEN, Universidade Federal da Paraíba, Cidade Universitária, 58051-970 João Pessoa, PB, Brazil.*

We derive the dynamical equations for a non-local gravity model in the Palatini formalism and we discuss some of the properties of this model. We have show that, in some specific case, the vacuum solutions of general relativity are also vacuum solutions of the non-local model, so we conclude that, at least in this case, the singularities of Einstein's gravity are not removed.

I. INTRODUCTION

Recently, the possibility of using higher-derivative theories to construct a viable non-local theory of quantum gravity has been considered [1, 2]. These models have been constructed in order to fulfill the following set of hypotheses: (1) Einstein-Hilbert action must be a good approximation of the theory below the Planck energy scale; (2) The theory must be perturbatively quantum-renormalizable on a flat background; (3) The theory must be unitary; (4) Lorentz invariance must be preserved; (5) Possibly, the presence of singularities is avoided.

Non-local models have been shown to be both renormalizable and ghost-free [1, 2] and are currently believed to be a viable alternative to other quantum gravity scenarios, i.e. Loop Quantum Gravity, Strings and Noncommutative Geometries (see [3] for a review).

The typical Lagrangian density for non-local gravity is an extension of the Stelle theory[4] (which is renormalizable but plagued by ghosts) and has the following form:

$$\mathcal{L} = R - \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) \gamma(\Box/\Lambda^2) R^{\mu\nu} \,, \qquad (1)$$

where the form factor $\gamma(z)$ is a non-polynomial analytic function, $\Box$ is the covariant D'Alembertian operator and Λ is an invariant mass scale close to the Planck mass. The propagator of the theory is

$$G(k^2) = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{2}\right) \,, \qquad (2)$$

where $P^{(0)}$ and $P^{(2)}$ are the spin zero and spin two projectors [1] and where $V(\Box/\Lambda^2)$ is defined by

$$\gamma(\Box/\Lambda^2) \equiv \frac{V(\Box/\Lambda^2)^{-1} - 1}{\Box} \,. \qquad (3)$$

We stress that (1) is not the most general realization of non-local gravity and more general examples will be considered in the following. For instance, one may consider a model with two form factors as in [5] (which has an extra scalar degree of freedom which is not a ghost), add quadratic non-local combinations of the Ricci tensor, etcetera. However, (1) is very useful to describe the key properties of non-local gravity, so we focus on this model first.

In order to recover Einstein's gravity for small momenta, one requires that $V(z) \simeq 1$ for $|z| \ll 1$. Moreover, if one imposes that $V(z)$ has no poles, the theory contains only the two massless gravitons of Einstein's theory, with no extra propagating particles, and thus one avoids the occurrence of ghosts. Thus, if one wants to avoid ghosts, the form factor $\gamma(z)$ cannot be polynomial, so that the theory must contain an infinite number of derivatives and it has to be non-local. We stress that $1/\Lambda$ represents the length scale above which the theory is fully nonlocal and that the local behavior of the theory is recovered at energies below Λ.

We remember that the study of non-local quantum field theory has been introduced by Efimov in a series of seminal papers [8–10], where he discussed the quantization scheme [8], the unitarity of the theory [9] and the causality [10] in the case of a non-local scalar field (see also [11] for a recent discussion). We also mention that a nonlocal version of QED has been studied in [12], nonlocal vector field theory has been introduced in [13] and nonlocal gauge theories have been studied in [14]. More recently, in [15] it has been considered the case of a nonlocal scalar field with specific self–interactions

---

*Electronic address: briscese.phys@gmail.com,fabio.briscese@northumbria.ac.uk

†Electronic address: mlaurapucheu@fisica.ufpb.br

[1] In four dimensions the projectors are defined as [1] $P^{(2)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}$, $P^{(1)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right)$, $P^{(0)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}$, $\bar{P}^{(0)}_{\mu\nu,\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}$, $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$, $\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$.

which have been chosen to present the same symmetries of the non-local quantum gravity (NLQG); in [16] the non-locality-induced one loop corrections to the scalar field potential have been calculated and the implications for cosmology have been discussed.

The second key property of non-local gravity, i.e. its renormalizability, can be understood from the observation that, if $V(z)$ goes to zero for $|z| \gg 1$ sufficiently fast, the convergence of the propagator in the ultraviolet is improved in such a way that the theory results to be renormalizable, since this improves the convergence of the integrations over loops (see for instance [17] for details).

At cosmological level, non-local gravity has nice properties: in [5–7] it has been shown that on a FRW background it reduces to the $R + \epsilon R^2$ Starobinsky model [18] with the identification $\epsilon \equiv 1/\Lambda^2$ up to corrections of order $1/\Lambda^4$. Therefore, non-local gravity gives a viable inflation in agreement with Planck data [19] for $\Lambda \sim 10^{-5}\, M_P$ [5–7].

It is notable that non-local gravity models can be constructed, which are free from the singularities which affect Einstein's gravity. In fact, in some specific models, the linearized equations for gravitational perturbations of Minkowski background typically reads $\exp(\Box/\Lambda^2)\Box h_{\mu\nu} = M_P^{-2}\tau_{\mu\nu}$ [2], where $\tau_{\mu\nu}$ is the stress energy tensor of matter. For a point-like source of mass m this gives a Newtonian potential $h_{00} \sim erf(r\Lambda/2)\, m/rM_P^2$, where $erf(z)$ is the error function of argument z, which is finite for all $r \geq 0$. This shows how black holes singularities of general relativity can be removed in non-local gravity (see also [20]). With similar arguments one can show that also the big bang singularity can be removed and a non singular bouncing cosmology can be obtained [21]. We stress that the disappearance of singularities is very model dependent and for instance the specific model (1) still has all the singular vacuum solutions of Einstein's gravity [22], as the Schwarzschild and Kerr metrics among others (see also [23] for further discussions on non-singular spacetimes).

In this paper we are interested in deriving the Palatini formulation of non-local gravity. The Palatini formalism [25] assumes that the metric tensor and the affine connection are independent variables, so that the field equations are obtained by varying the action with respect to both variables. If applied to the Einstein-Hilbert action, the Palatini method gives the same equations of motion of general relativity but, if one consider modifications of Einstein's gravity (see [26] for a review of modified gravity models), it gives a completely different theory of gravitation [27].

For instance, in the case of modified $f(R)$ gravity, the action is

$$\mathcal{S} = -\frac{1}{2\kappa^2}\int d^4x\, \sqrt{-g}\, f(R) + S_m\,, \tag{4}$$

where S_m represents the action of all the matter fields, $\kappa^2 = 8\pi G$ in natural units, G is the gravitational constant and R is the Ricci scalar constructed with the affine connection $\Gamma^{\alpha}_{\ \beta\gamma}$ [2]. Furthermore, henceforth we limit our analysis to a symmetric connection $\Gamma^{\alpha}_{\ \beta\gamma} = \Gamma^{\alpha}_{\ \gamma\beta}$.

The variation of the action (4) with respect to the metric tensor gives the set of equations

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa^2 T_{\mu\nu}\,, \tag{5}$$

where $f'(R) \equiv \partial_R f(R)$ and $T_{\mu\nu}$ is the energy-momentum tensor of matter, which is defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,, \tag{6}$$

while the variation of the action (4) with respect to the affine connection gives the equations [3]

$$\frac{1}{2\kappa^2}\left[\nabla_\lambda\left(\sqrt{-g} f'(R) g^{\mu\nu}\right) - \nabla_\rho\left(\sqrt{-g} f'(R) g^{\rho(\mu}\right)\delta^{\nu)}_\lambda\right] = \\ = -\frac{\delta S_m}{\delta \Gamma^\lambda_{\mu\nu}}\,, \tag{7}$$

where the second term in the l.h.s. of the last equation is symmetrized in the indices μ and ν.

In most physical cases, the matter action S_m is independent from the affine connection, thus $\delta S_m/\delta\Gamma^\lambda_{\mu\nu} \equiv 0$ [4]. In this case, taking the trace on λ and μ in Eq.(7), it is evident that if $f(R) = R$, i.e. in the case of the Einstein-Hilbert action, the affine connection is exactly the metric connection associated to the metric tensor $g_{\mu\nu}$ and general relativity is recovered. However, when $f(R)' \neq 1$, equation (7) gives the compatibility condition

$$\nabla_\lambda\left(f'(R)\, g^{\mu\nu}\right) = 0, \tag{8}$$

and the theory is genuinely different from the corresponding $f(R)$ model in the metric formalism. In fact, equation (8) implies that the connection $\Gamma^{\alpha}_{\beta\gamma}$ is the metric connection of the tensor $h^{\mu\nu} \equiv f'(R)\, g^{\mu\nu}$, which is conformal to the metric tensor $g^{\mu\nu}$.

In what follows we derive the Palatini formulation of non-local gravity in the case of an action containing non-local quadratic terms constructed with the Ricci scalar

---

[2] Hereafter we use the following convention: $R^\rho_{\ \sigma\mu\nu} = -\partial_\nu\Gamma^\rho_{\sigma\mu} + \partial_\mu\Gamma^\rho_{\sigma\nu} - \Gamma^\eta_{\sigma\mu}\Gamma^\rho_{\nu\eta} + \Gamma^\eta_{\sigma\nu}\Gamma^\rho_{\mu\eta}$, $R_{\mu\nu} = R^\alpha_{\mu\alpha\beta}$ and $R = g^{\mu\nu}R_{\mu\nu}$. This definition differs for a minus sign from the one used in [24].

[3] We remember that a tensor density of weight w can be written as $t^{\mu\nu\dots}_{\alpha\beta\dots} \equiv (-g)^{w/2}\, T^{\mu\nu\dots}_{\alpha\beta\dots}$, where $T^{\mu\nu\dots}_{\alpha\beta\dots}$ is a tensor, and the covariant derivative of a tensor density is defined as $\nabla_\rho t^{\mu\nu\dots}_{\alpha\beta\dots} \equiv (-g)^{w/2}\,\nabla_\rho T^{\mu\nu\dots}_{\alpha\beta\dots}$, see [28].

[4] This is not the most general case which can be considered. For instance, in appendix B it is shown that in the case of a nonlocal scalar field one has $\delta S_m/\delta\Gamma^\lambda_{\mu\nu} \neq 0$.

and with the Ricci and Riemann tensors. Specifically, our lagrangian density will be of the type

$$\mathcal{L}_{grav} = \sqrt{-g}\left[-\tfrac{1}{2\kappa^2}\left(R+\Lambda_c\right) + R\, h_1\left(-\Box_\Lambda\right) R + \right. \\ \left. + R_{\mu\nu}\, h_2\left(-\Box_\Lambda\right) R^{\mu\nu} + R_{\mu\nu\alpha\beta}\, h_3\left(-\Box_\Lambda\right) R^{\mu\nu\alpha\beta}\right] , \tag{9}$$

where we have included an optional cosmological constant Λ_c and where $\Box_\Lambda \equiv \Box/\Lambda^2$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator constructed with the covariant derivatives associated to the non-metric connection $\Gamma^\alpha{}_{\beta\gamma}$ and the three form factors $h_i(z)$ are analytic functions of their arguments and their action extend to the objects on their right hand side, see [17] for review.

Therefore the complete action for the gravitational field $\mathcal{S}_{grav} = \int d^4x \mathcal{L}_{grav}$ contains the usual Einstein-Hilbert term

$$\mathcal{S}_{EH} = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R+\Lambda_c\right) , \tag{10}$$

plus the three contributions

$$\mathcal{S}_{Scalar} = \int d^4x\sqrt{-g}\, R\, h_1\left(-\Box_\Lambda\right) R , \tag{11}$$

$$\mathcal{S}_{Ricci} = \int d^4x\sqrt{-g}\, R_{\mu\nu}\, h_2\left(-\Box_\Lambda\right) R^{\mu\nu} , \tag{12}$$

$$\mathcal{S}_{Riemann} = \int d^4x\sqrt{-g}\, R_{\mu\nu\alpha\beta}\, h_3\left(-\Box_\Lambda\right) R^{\mu\nu\alpha\beta} , \tag{13}$$

The lagrangian (9) includes for instance, the case of the model presented in [17], corresponding to the choice $h_1 = -h_2/2$ and $h_3 = 0$, which is of particular interest since it is quantum-renormalizable and ghost free on a flat background. We stress that the condition $h_3 = 0$ does not spoil renormalizability and unitarity of the theory around flat spacetime.

In section II, following the same line outlined in [22, 24], we derive the field equations for the lagrangian (9). The variation of the terms (11-12-13) are derived in sections II A-II B-II C respectively. We stress that, since our connection is non-metric, the operators h_i do not commute with the metric tensor $g_{\mu\nu}$ and therefore the following action

$$\mathcal{S}^*_{Ricci} = \int d^4x\sqrt{-g}\, R^{\mu\nu}\, h^*_2\left(-\Box_\Lambda\right) R_{\mu\nu} , \tag{14}$$

is physically different from $\mathcal{S}_{Ricci}$ defined in (12). Since we are interested in giving a method to derive the equations of motion for a generic nonlocal theory assuming the Palatini variation, in what follows we will limit our analysis to the terms (11-13). However, with an illustrative scope, we will calculate the variation of the action (14) in Appendix A.

Then, in section (III) we will consider the full set of equations of motion and we will show that, when $h_3 = 0$, the vacuum solutions of general relativity are also vacuum solutions of the theory (9), see [22] for an analog result for the non-local metric theory and [29] for the local $f(R)$ models. This fact shows that, at least in this case, the singularities of Einstein's gravity are not removed. Finally, in section IV we will resume the main results of this paper and we will conclude.

II. EQUATIONS OF MOTION

As it was mentioned, the Palatini's method does not assume *a priori* a standard form, given by the Christoffel symbols, for the components of the affine connection. Instead, it is based on the hypothesis that the metric tensor and the connection are independent variables [28] (for a review of the history of the method see [30]).

If the connection is symmetric, the variation of the Riemann tensor can be expressed as

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu\left(\delta\Gamma^\rho{}_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\mu\sigma}\right) . \tag{15}$$

Accordingly, for the variations of the Ricci tensor and the Ricci scalar we have

$$\delta R_{\sigma\nu} = \nabla_\rho\left(\delta\Gamma^\rho{}_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\rho\sigma}\right) , \tag{16}$$

and

$$\delta R = g^{\sigma\nu}\delta R_{\sigma\nu} + \delta g^{\sigma\nu} R_{\sigma\nu} . \tag{17}$$

In the following, we will make extensive use of the formula of integration by parts, which allows to show the following formula

$$\int d^4x\,\sqrt{-g}\, A^{\beta_1\beta_2\dots}_{\alpha_1\alpha_2\dots}\Box\, B^{\alpha_1\alpha_2\dots}_{\beta_1\beta_2\dots} = \\ \int d^4x\,\sqrt{-g}\left(\Box^\dagger A^{\alpha_1\alpha_2\dots}_{\beta_1\beta_2\dots}\right) B^{\beta_1\beta_2\dots}_{\alpha_1\alpha_2\dots} , \tag{18}$$

which is valid for any couple of tensors $A^{\beta_1\beta_2\dots}_{\alpha_1\alpha_2\dots}$ and $B^{\alpha_1\alpha_2\dots}_{\beta_1\beta_2\dots}$ with the condition that one of them is null on the boundary of integration, and where the operator $\Box^\dagger$ is defined as

$$\Box^\dagger T^{\beta_1\beta_2\dots}_{\alpha_1\alpha_2\dots} \equiv \frac{1}{\sqrt{-g}}\nabla_\mu\nabla_\nu\left(\sqrt{-g}g^{\mu\nu}T^{\beta_1\beta_2\dots}_{\alpha_1\alpha_2\dots}\right) . \tag{19}$$

Moreover, from Eq.(16), one also has

$$\int d^4x\,\sqrt{-g}\, T^{\mu\nu}\delta R_{\mu\nu} = \\ = \int d^4x\left[\nabla_\beta\left(\sqrt{-g}\,\delta^{(\nu}_\alpha\, T^{\mu)\,\beta}\right) - \nabla_\alpha\left(\sqrt{-g}T^{(\mu\nu)}\right)\right]\delta\Gamma^\alpha_{\mu\nu} . \tag{20}$$

To derive the field equations we have to calculate the functional derivatives of the action of the gravitational field with respect to the metric and the affine connection, which are obtained by variation of the action with respect to such variables. The variation of the Einstein-Hilbert term (10) is straightforward (see [28, 30]), and one has

$$\frac{\delta \mathcal{S}_{EH}}{\delta g^{\mu\nu}} = -\frac{1}{2\kappa^2}\sqrt{-g}\left(G_{\mu\nu} - \frac{\Lambda_c}{2} g_{\mu\nu}\right), \qquad (21)$$

where $G_{\mu\nu} = R_{\mu\nu} - R\, g_{\mu\nu}/2$ is the Einstein tensor, and

$$\frac{\delta \mathcal{S}_{EH}}{\delta \Gamma^{\alpha}_{\mu\nu}} = \frac{1}{2\kappa^2}\left[\nabla_\alpha\left(\sqrt{-g}g^{\mu\nu}\right) - \nabla_\beta\left(\sqrt{-g}g^{\beta(\mu}\right)\delta^{\nu)}_{\alpha}\right], \qquad (22)$$

where in the last term the indices μ and ν are symmetrized. Therefore, all we need is to calculate the functional derivatives of the terms (11-13), which is the subject of the following sections.

A. Scalar action

To begin with, let us regard the action (11), which depends on the Ricci scalar. The variation of this action is

$$\delta \mathcal{S}_{Scalar} = \int d^4x\sqrt{-g}\Big\{ R\,\delta h_1\left(-\Box_\Lambda\right) R + \\ + R\, h_1\left(-\Box_\Lambda\right)\delta R + \delta R\, h_1\left(-\Box_\Lambda\right) R + \\ -\tfrac{1}{2} g_{\mu\nu} R\, h_1\left(-\Box_\Lambda\right) R\,\delta g^{\mu\nu}\Big\}, \qquad (23)$$

where we have used $\delta\sqrt{-g} = -1/2 g_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}$.

To handle the first term in the r.h.s. of Eq.(23) we follow the method outlined in [22, 24], and we expand the analytic function $h_1\left(-\Box_\Lambda\right)$ in power series as

$$h_1(-\Box_\Lambda) = \sum_{m=0}^{\infty} h_1^{(m)}\left(-\frac{\Box}{\Lambda^2}\right)^m, \qquad (24)$$

which allows to express $\delta h_1\left(-\Box_\Lambda\right)$ as

$$\delta h_1\left(-\Box_\Lambda\right) = \sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{h_1^{(m)}}{\left(-\Lambda^2\right)^m}\Box^r\delta\Box\,\Box^{m-r-1}. \qquad (25)$$

The action of $\delta\Box$ on a tensor X is schematically defined by

$$(\delta\Box)\, X = \delta\left(\Box X\right) - \Box\delta X\,, \qquad (26)$$

and in the case of a scalar field ψ one has

$$(\delta\Box)\,\psi = \left[\delta g^{\mu\nu}\nabla_\mu\nabla_\nu\psi - g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}\nabla_\alpha\psi\right]. \qquad (27)$$

Using Eqs.(25,27) one has

$$\int d^4x\sqrt{-g}R\delta h_1\left(-\Box_\Lambda\right) R = -\tfrac{1}{\Lambda^2}\textstyle\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_1^{(m)}\times \\ \times\int d^4x\sqrt{-g}\Big[\Big(R^{[r]^\dagger}\nabla_\mu\nabla_\nu R^{[m-r-1]}\Big)\delta g^{\mu\nu} + \\ -\Big(R^{[r]^\dagger} g^{\mu\nu}\nabla_\alpha R^{[m-r-1]}\Big)\delta\Gamma^{\alpha}_{\mu\nu}\Big], \qquad (28)$$

where we have defined

$$R^{[k]} \equiv \left(-\frac{\Box}{\Lambda^2}\right)^k R\,, \qquad (29)$$

and

$$R^{[k]^\dagger} \equiv \left(-\frac{\Box^\dagger}{\Lambda^2}\right)^k R\,. \qquad (30)$$

Let us consider the second and third terms in Eq.(23). They can be recast as

$$\int d^4x\sqrt{-g}\Big\{R\, h_1\left(-\Box/\Lambda^2\right)\delta R + \delta R\, h_1\left(-\Box/\Lambda^2\right) R\Big\} = \\ \int d^4x\sqrt{-g}\Big\{h_1\left(-\Box/\Lambda^2\right) R + h_1\left(-\Box^\dagger/\Lambda^2\right) R\Big\}\delta R \equiv \\ \equiv \int d^4x\sqrt{-g}F\delta R\,, \qquad (31)$$

with the definition

$$F \equiv h_1\left(-\Box/\Lambda^2\right) R + h_1\left(-\Box^\dagger/\Lambda^2\right) R\,. \qquad (32)$$

Using Eq.(17) one also has

$$\int d^4x\sqrt{-g}F\delta R = \int d^4x\sqrt{-g}F\left[\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right] = \\ = \int d^4x\Big\{\sqrt{-g}\,F\,R_{\mu\nu}\delta g^{\mu\nu} + \\ + \left[\nabla_\alpha\left(\sqrt{-g}\,F\,g^{\mu\nu}\right) - \nabla_\beta\left(\sqrt{-g}\,F\,g^{\beta(\mu}\right)\delta^{\nu)}_{\alpha}\right]\delta\Gamma^{\alpha}_{\mu\nu}\Big\}, \qquad (33)$$

where we have used Eq.(20) with the identification $T^{\mu\nu} = F\,g^{\mu\nu}$.

Therefore, from Eqs.(23,28,33) one has

$$\frac{\delta \mathcal{S}_{Scalar}}{\delta g^{\mu\nu}} = \sqrt{-g}\Big\{-\tfrac{1}{2}g_{\mu\nu}R\,h_1\left(-\Box_\Lambda\right)R + F\,R_{(\mu\nu)} + \\ -\tfrac{1}{\Lambda^2}\textstyle\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_1^{(m)}\,R^{[r]^\dagger}\nabla_{(\mu}\nabla_{\nu)}R^{[m-r-1]}\Big\}, \qquad (34)$$

and

$$\frac{\delta\mathcal{S}_{Scalar}}{\delta\Gamma^{\alpha}_{\mu\nu}} = \nabla_{\alpha}\left(\sqrt{-g}\,F\,g^{\mu\nu}\right) - \nabla_{\beta}\left(\sqrt{-g}\,F\,g^{\beta(\mu}\right)\delta^{\nu)}_{\alpha} + $$
$$+\frac{\sqrt{-g}}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_1^{(m)}\, R^{[r]^\dagger} g^{\mu\nu}\nabla_{\alpha}R^{[m-r-1]}\,. \tag{35}$$

B. Ricci action

Let us consider in the present section, the variation of the action (12), which can be written as

$$\delta\mathcal{S}_{Ricci} = \int d^4x\sqrt{-g}\Big\{-\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}h_2\left(-\Box_{\Lambda}\right)R^{\alpha\beta}\delta g^{\mu\nu} + $$
$$+\delta R_{\mu\nu}h_2\left(-\Box_{\Lambda}\right)R^{\mu\nu} + R_{\mu\nu}\delta h_2\left(-\Box_{\Lambda}\right)R^{\mu\nu} + $$
$$+R_{\mu\nu}h_2\left(-\Box_{\Lambda}\right)\delta R^{\mu\nu}\Big\}\,. \tag{36}$$

It is not difficult to see that the second and the last terms on the r.h.s of (36) can be handled to give

$$\int d^4x\sqrt{-g}\Big\{\delta R_{\mu\nu}h_2\left(-\Box_{\Lambda}\right)R^{\mu\nu} + R_{\mu\nu}h_2\left(-\Box_{\Lambda}\right)\delta R^{\mu\nu}\Big\} = $$
$$\int d^4x\Big\{\sqrt{-g}\Big[R_{\nu}{}^{\alpha}h_2\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\mu\alpha} + R^{\alpha}{}_{\mu}h_2\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\alpha\nu}\Big]\delta g^{\mu\nu} + $$
$$+\Big[\nabla_{\beta}\left(\sqrt{-g}\delta^{(\mu}_{\alpha}F^{\nu)\beta}\right) - \nabla_{\alpha}\left(\sqrt{-g}F^{(\mu\nu)}\right)\Big]\delta\Gamma^{\alpha}_{\mu\nu}\Big\}\,, \tag{37}$$

with $F^{\alpha\beta} \equiv h_2\left(-\frac{\Box}{\Lambda^2}\right)R^{\alpha\beta} + g^{\alpha\sigma}g^{\beta\rho}h_2\left(-\frac{\Box^\dagger}{\Lambda^2}\right)R_{\sigma\rho}$ and where Eq.(16) was taken into account.

Now, let us consider the following expression for the function h_2, analogous to the one given in (24):

$$h_2(-\Box_{\Lambda}) = \sum_{m=0}^{\infty} h_2^{(m)}\left(-\frac{\Box}{\Lambda^2}\right)^{m}\,. \tag{38}$$

Therefore, its consequent variation leads to

$$\delta h_2\left(-\Box_{\Lambda}\right) = \sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{h_2^{(m)}}{\left(-\Lambda^2\right)^m}\Box^r\delta\Box\;\Box^{m-r-1}\,. \tag{39}$$

Defining for this case

$$R^{[k]\alpha\beta} \equiv \left(-\frac{\Box}{\Lambda^2}\right)^{k} R^{\alpha\beta}\,, \tag{40}$$

and

$$R^{[k]^\dagger}_{\alpha\beta} \equiv \left(-\frac{\Box^\dagger}{\Lambda^2}\right)^{k} R_{\alpha\beta}\,, \tag{41}$$

where $\Box^\dagger$ is given by Eq.(19), the third term on the r.h.s. of (36) can be expressed as

$$\int d^4x\sqrt{-g}R_{\mu\nu}\delta h_2\left(-\Box_{\Lambda}\right)R^{\mu\nu} = $$
$$-\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_2^{(m)}\int d^4x\sqrt{-g}R^{[r]^\dagger}_{\mu\nu}\,\delta\left(\Box\right)R^{[m-r-1]\mu\nu}\,. \tag{42}$$

The calculation of Eq.(42) requires the following identity, which can be obtained with the aid of Eq.(26), for an arbitrary symmetric contravariant tensor of second order:

$$\left(\delta\Box\right)T^{\alpha\beta} = \delta g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}T^{\alpha\beta} + 2\delta\Gamma^{\beta}_{\nu\rho}\nabla^{\nu}T^{\alpha\rho} + 2\delta\Gamma^{\alpha}_{\nu\rho}\nabla^{\nu}T^{\rho\beta} + $$
$$+T^{\rho\beta}\nabla^{\nu}\delta\Gamma^{\alpha}_{\nu\rho} + T^{\alpha\rho}\nabla^{\nu}\delta\Gamma^{\beta}_{\nu\rho} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\nabla_{\lambda}T^{\alpha\beta}\,. \tag{43}$$

Applying this last result in (42), one finally gets

$$\int d^4x\,\sqrt{-g}\,R_{\mu\nu}\delta h_2\left(-\Box_{\Lambda}\right)R^{\mu\nu} = -\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_2^{(m)}\times$$
$$\times\int d^4x\Big\{\Big(\sqrt{-g}R^{[r]^\dagger}_{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R^{[m-r-1]\alpha\beta}\Big)\delta g^{\mu\nu} + $$
$$+\Big[-\nabla_{\lambda}\Big(\sqrt{-g}g^{\mu\lambda}\Big(R^{[r]^\dagger}_{\alpha\beta}R^{[m-r-1]\nu\beta} + R^{[r]^\dagger}_{\beta\alpha}R^{[m-r-1]\beta\nu}\Big)\Big) + $$
$$+\sqrt{-g}\Big(2R^{[r]^\dagger}_{\beta\alpha}\nabla^{\mu}R^{[m-r-1]\beta\nu} + 2R^{[r]^\dagger}_{\alpha\beta}\nabla^{\mu}R^{[m-r-1]\nu\beta} + $$
$$-g^{\mu\nu}R^{[r]^\dagger}_{\lambda\beta}\nabla_{\alpha}R^{[m-r-1]\lambda\beta}\Big)\Big]\delta\Gamma^{\alpha}_{\mu\nu}\Big\}\,. \tag{44}$$

In conclusion, the variation of the Ricci action leads to the following equations

$$\frac{\delta\mathcal{S}_{Ricci}}{\delta g^{\mu\nu}} = \sqrt{-g}\Big\{-\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}h_2\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\alpha\beta} + $$
$$+R^{\alpha}{}_{(\nu}h_2\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\alpha\mu)} + R_{(\nu}{}^{\alpha}h_2\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\mu)\alpha} + $$
$$-\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_2^{(m)}R^{[r]^\dagger}_{\alpha\beta}\nabla_{(\mu}\nabla_{\nu)}R^{[m-r-1]\alpha\beta}\Big\}\,, \tag{45}$$

and

$$\frac{\delta\mathcal{S}_{Ricci}}{\delta\Gamma^{\alpha}_{\mu\nu}} = \nabla_\beta\left(\sqrt{-g}\delta^{(\mu}_{\alpha}F^{\nu)\beta}\right) - \nabla_\alpha\left(\sqrt{-g}F^{(\mu\nu)}\right) +$$

$$+\tfrac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_2^{(m)}\times$$

$$\left\{\nabla_\lambda\left[\sqrt{-g}g^{\lambda(\mu}\left(R^{[r]^\dagger}_{\alpha\beta}R^{[m-r-1]\nu)\beta} + R^{[r]^\dagger}_{\beta\alpha}R^{[m-r-1]\beta\nu)}\right)\right]+\right.$$

$$-\sqrt{-g}\left[2R^{[r]^\dagger}_{\beta\alpha}\nabla^{(\mu}R^{[m-r-1]\beta\nu)} + 2R^{[r]^\dagger}_{\alpha\beta}\nabla^{(\mu}R^{[m-r-1]\nu)\beta}+\right.$$

$$\left.\left.-g^{\mu\nu}R^{[r]^\dagger}_{\lambda\beta}\nabla_\alpha R^{[m-r-1]\lambda\beta}\right]\right\}. \tag{46}$$

where we have symmetrized these expressions with respect to the indices μ and ν.

C. Riemann action

In this last section, let us expand the calculation of the variation of the Riemann action (13). To begin with, let us consider the expansion in power series of the function $h_3(-\Box_\Lambda)$:

$$h_3(-\Box_\Lambda) = \sum_{m=0}^{\infty} h_3^{(m)}\left(-\frac{\Box}{\Lambda^2}\right)^m . \tag{47}$$

After some simple manipulation, considering (47) and (18), the variation of (13) can be written as

$$\delta\mathcal{S}_{Riemann} = \int d^4x\sqrt{-g}\left\{\left[3R^{\pi}{}_{\mu}{}^{\alpha\beta}\left(h_3\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\pi\nu\alpha\beta}\right)+\right.\right.$$

$$-g_{\sigma\mu}R_{\nu\omega\alpha\beta}\left(h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\sigma\omega\alpha\beta}\right) - \tfrac{1}{2}g_{\mu\nu}R_{\pi\sigma\alpha\beta}\times$$

$$\left.\times\left(h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\pi\sigma\alpha\beta}\right)\right]\delta g^{\mu\nu} + \left[g_{\mu\rho}\left(h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\mu\nu\alpha\beta}\right)+\right.$$

$$\left.+g^{\mu\nu}g^{\sigma\alpha}g^{\beta\pi}\left(h_3\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\rho\mu\sigma\pi}\right)\right]\delta R^{\rho}{}_{\nu\alpha\beta}+$$

$$\left.+R_{\mu\nu\alpha\beta}\delta h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\mu\nu\alpha\beta}\right\}. \tag{48}$$

The terms in (48) involving the variation of the Riemann tensor give

$$\int d^4x\sqrt{-g}\delta R^{\rho}{}_{\nu\alpha\beta}\left[g_{\mu\rho}h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\mu\nu\alpha\beta} + g^{\mu\nu}g^{\sigma\alpha}g^{\beta\pi}\times\right.$$

$$\left.\times h_3\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\rho\mu\sigma\pi}\right] = \int d^4x\left[\nabla_\beta\left(\sqrt{-g}g^{\rho\nu}g^{\sigma\mu}g^{\beta\pi}\times\right.\right.$$

$$\left.\left.\times h_3\left(-\tfrac{\Box^\dagger}{\Lambda^2}\right)R_{\alpha\rho\sigma\pi}\right) - \nabla_\rho\left(\sqrt{-g}g_{\alpha\beta}h_3\left(-\tfrac{\Box}{\Lambda^2}\right)R^{\beta\nu\rho\mu}\right)\right]\delta\Gamma^{\alpha}_{\mu\nu}, \tag{49}$$

where we have used Eq.(15).

Analogously to the previous sections, the calculation of the last term in (48) makes use of the expression of the variation of $h_3(-\Box_\Lambda)$

$$\delta h_3\left(-\Box_\Lambda\right) = \sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{h_3^{(m)}}{\left(-\Lambda^2\right)^m}\Box^r\delta\Box\,\Box^{m-r-1}, \tag{50}$$

and of the following property for an arbitrary fourth rank tensor

$$\left(\delta\Box\right)T^{\mu\nu\alpha\beta} = \delta g^{\sigma\lambda}\nabla_\sigma\nabla_\lambda T^{\mu\nu\alpha\beta} + g^{\sigma\lambda}\left[T^{\rho\nu\alpha\beta}\nabla_\sigma\delta\Gamma^{\mu}_{\lambda\rho}+\right.$$

$$+T^{\mu\rho\alpha\beta}\nabla\delta\Gamma^{\nu}_{\lambda\rho} + T^{\mu\nu\rho\beta}\nabla_\sigma\delta\Gamma^{\alpha}_{\lambda\rho} + T^{\mu\nu\alpha\rho}\nabla_\sigma\delta\Gamma^{\beta}_{\lambda\rho}+$$

$$+2\delta\Gamma^{\mu}_{\lambda\rho}\nabla_\sigma T^{\rho\nu\alpha\beta} + 2\delta\Gamma^{\nu}_{\lambda\rho}\nabla_\sigma T^{\mu\rho\alpha\beta} + 2\delta\Gamma^{\alpha}_{\lambda\rho}\nabla_\sigma T^{\mu\nu\rho\beta}+$$

$$\left.+2\delta\Gamma^{\beta}_{\lambda\rho}\nabla_\sigma T^{\mu\nu\alpha\rho} - \delta\Gamma^{\rho}_{\sigma\lambda}\nabla_\rho T^{\mu\nu\alpha\beta}\right]. \tag{51}$$

Thus, considering Eqs.(50) and (51) after applying the symmetry properties of the Riemann tensor, we finally have

$$\int d^4x\sqrt{-g}R_{\mu\nu\alpha\beta}\delta h_3(-\Box_\Lambda)R^{\mu\nu\alpha\beta} =$$

$$-\tfrac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}h_3^{(m)}\int d^4x\sqrt{-g}\left\{R^{[r]^\dagger}_{\pi\sigma\alpha\beta}\times\right.$$

$$\times\left(\nabla_\mu\nabla_\nu R^{[m-r-1]\pi\sigma\alpha\beta}\right)\delta g^{\mu\nu} + \left[8g^{\pi\mu}R^{[r]^\dagger}_{\alpha\omega\sigma\beta}\times\right.$$

$$\times\left(\nabla_\pi R^{[m-r-1]\nu\omega\sigma\beta}\right) - \tfrac{4}{\sqrt{-g}}\nabla_\pi\left(\sqrt{-g}g^{\pi\mu}R^{[r]^\dagger}_{\alpha\sigma\omega\beta}\times\right.$$

$$\left.\left.\left.\times R^{[m-r-1]\nu\sigma\omega\beta}\right) - g^{\mu\nu}R^{[r]^\dagger}_{\pi\sigma\rho\beta}\left(\nabla_\alpha R^{[m-r-1]\pi\sigma\rho\beta}\right)\right]\delta\Gamma^{\alpha}_{\mu\nu}\right\}, \tag{52}$$

where, as usual, we have denoted

$$R^{[k]\alpha\beta\sigma\omega} \equiv \left(-\frac{\Box}{\Lambda^2}\right)^k R^{\alpha\beta\sigma\omega}, \tag{53}$$

and

$$R^{[k]^\dagger}_{\alpha\beta\sigma\omega} \equiv \left(-\frac{\Box^\dagger}{\Lambda^2}\right)^k R_{\alpha\beta\sigma\omega}\,. \tag{54}$$

Finally, we have that the variation of the Riemann action leads to

$$\frac{\delta\mathcal{S}_{Riemann}}{\delta g^{\mu\nu}} = \sqrt{-g}\Bigg\{-3R^{\pi}{}_{(\mu}{}^{\alpha\beta}\, h_3\left(-\frac{\Box^\dagger}{\Lambda^2}\right) R_{\nu)\pi\alpha\beta} +$$
$$-g_{\sigma(\mu}R_{\nu)\omega\alpha\beta}\, h_3\left(-\frac{\Box}{\Lambda^2}\right) R^{\sigma\omega\alpha\beta} - \frac{1}{2} g_{\mu\nu} R_{\pi\sigma\alpha\beta}\times$$
$$\times h_3\left(-\frac{\Box}{\Lambda^2}\right) R^{\pi\sigma\alpha\beta} - \frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_3^{(m)} R^{[r]^\dagger}_{\pi\sigma\alpha\beta}\times$$
$$\times\nabla_{(\mu}\nabla_{\nu)} R^{[m-r-1]\pi\sigma\alpha\beta}\Bigg\}\,, \tag{55}$$

and

$$\frac{\delta\mathcal{S}_{Riemann}}{\delta\Gamma^{\alpha}_{\mu\nu}} = \nabla_\beta\left[\sqrt{-g}g^{\rho(\nu}g^{\mu)\sigma}g^{\beta\pi} h_3\left(-\frac{\Box^\dagger}{\Lambda^2}\right) R_{\alpha\rho\sigma\pi}\right] +$$
$$+\nabla_\rho\left[\sqrt{-g}g_{\alpha\beta} h_3\left(-\frac{\Box}{\Lambda^2}\right) R^{\beta(\nu\mu)\rho}\right] +$$
$$-\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} h_3^{(m)}\Bigg\{8g^{\pi(\mu}R^{[r]^\dagger}_{\alpha\omega\sigma\beta}\left[\nabla_\pi R^{[m-r-1]\nu)\omega\sigma\beta}\right] +$$
$$-\frac{4}{\sqrt{-g}}\nabla_\pi\left[\sqrt{-g}\, g^{\pi(\mu}\, R^{[r]^\dagger}_{\alpha\sigma\omega\beta} R^{[m-r-1]\nu)\sigma\omega\beta}\right] +$$
$$-g^{\mu\nu}R^{[r]^\dagger}_{\pi\sigma\rho\beta}\left[\nabla_\alpha R^{[m-r-1]\pi\sigma\rho\beta}\right]\Bigg\}\,. \tag{56}$$

III. DISCUSSION

In the last sections we have derived the functional derivatives of the action of the non-local gravitational field with respect to the metric tensor and the connection, see Eqs.(34,35,45,46,55,56). The dynamical equations for the metric tensor and the connection read

$$\frac{\delta\mathcal{S}_{grav}}{\delta g^{\mu\nu}} = \frac{\delta\mathcal{S}_{EH}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_{Scalar}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_{Ricci}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_{Riemann}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{S}_{m}}{\delta g^{\mu\nu}} = 0\,, \tag{57}$$

and

$$\frac{\delta\mathcal{S}_{grav}}{\delta\Gamma^{\alpha}_{\mu\nu}} = \frac{\delta\mathcal{S}_{EH}}{\delta\Gamma^{\alpha}_{\mu\nu}} + \frac{\delta\mathcal{S}_{Scalar}}{\delta\Gamma^{\alpha}_{\mu\nu}} + \frac{\delta\mathcal{S}_{Ricci}}{\delta\Gamma^{\alpha}_{\mu\nu}} + \frac{\delta\mathcal{S}_{Riemann}}{\delta\Gamma^{\alpha}_{\mu\nu}} + \frac{\delta\mathcal{S}_{m}}{\delta\Gamma^{\alpha}_{\mu\nu}} = 0\,. \tag{58}$$

At a first look Eqs.(57,58) seem extremely complicated. An initial observation is that it is impossible to recast (58) in the form (8), i.e. it is not immediate to find a metric tensor $h_{\mu\nu}$ for which $\Gamma^{\alpha}_{\mu\nu}$ is the metric connection. Therefore we lose one of the main simplifications of the Palatini formalism in $f(R)$ theories.

However, with certain assumptions, one can show that some of the well known solutions of the Einsteinian gravity verify (57,58). Let us set $\mathcal{S}_m = 0$ and $h_3(\Box_\Lambda) = 0$, which guarantees that $\delta\mathcal{S}_{Riemann}/\delta g_{\mu\nu} = 0$ and $\delta\mathcal{S}_{Riemann}/\delta\Gamma^{\alpha}_{\mu\nu} = 0$.

As first example, let us assume that $\Lambda_c = 0$ and let us consider the vacuum solutions of the Einstein's equations, which are such that $R_{\mu\nu} = 0$, where the Ricci tensor is constructed with the metric connection $\Gamma^{(met)\alpha}_{\mu\nu}$ of $g_{\mu\nu}$, that verifies $\nabla^{(met)}_{\alpha} g_{\mu\nu} = 0$. It is easy to show that the couple $(\Gamma^{(met)\alpha}_{\mu\nu},\, g_{\mu\nu})$ is solution of the system (57,58) in vacuum and with zero cosmological constant. In fact, since $R = 0$ and $R_{\mu\nu} = 0$, one has that $\delta\mathcal{S}_{EH}/\delta g^{\mu\nu} = 0$, $\delta\mathcal{S}_{Scalar}/\delta g^{\mu\nu} = 0$, $\delta\mathcal{S}_{Ricci}/\delta g^{\mu\nu} = 0$ independently. For the same reason, using the fact that the connection is metric, one also has $\delta\mathcal{S}_{EH}/\delta\Gamma^{\alpha}_{\mu\nu} = 0$, $\delta\mathcal{S}_{Scalar}/\delta\Gamma^{\alpha}_{\mu\nu} = 0$, $\delta\mathcal{S}_{Ricci}/\delta\Gamma^{\alpha}_{\mu\nu}$, so that the system (57,58) is fully verified.

A second class of exact solutions of (57,58) is obtained considering a nonzero cosmological constant Λ_c. One can thus show that the couple $(\Gamma^{(met)\alpha}_{\mu\nu},\, g_{\mu\nu})$ of solutions of the Einstein's equations in vacuum with cosmological constant, is a solution of (57,58). In fact, in this case one has $R_{\mu\nu} = -\Lambda_c\, g_{\mu\nu}/2$ and therefore $\nabla^{(met)}_{\alpha} R_{\mu\nu} = 0$. One can easily verify that $\delta\mathcal{S}_{grav}/\delta g_{\mu\nu} = 0$ and $\delta\mathcal{S}_{grav}/\delta\Gamma^{\alpha}_{\mu\nu} = 0$, so that the equations of motion are satisfied.

These two cases exhaust the known examples of exact solutions of our dynamical equations. In more general cases one can include a contribution (13) to the action of the gravitational field, or extend the non-locality of the gravitational field to the matter fields, as in the case of a non-local scalar field briefly studied in appendix B, which implies $\delta\mathcal{S}_m/\delta\Gamma^{\alpha}_{\mu\nu} \neq 0$.

These simple examples show that, at least in the case $h_3(\Box_\Lambda) = 0$, the singularities of Einstein's gravity are not removed, since for instance black hole solutions are still there. However, one hopes that in the more general case $h_3(\Box_\Lambda) \neq 0$, models can be constructed which are singularity free.

IV. CONCLUSIONS

In this paper we have found the dynamical equations of the non-local gravity model (9) in the Palatini formalism. We have discussed some of the properties of the model and we have shown that, in the case $h_3 = 0$, the vacuum solutions of general relativity are also vacuum solutions of the model (9). We have concluded that, at least in this case, the singularities of Einstein's gravity are not removed.

Acknowledgements.

MPL thanks CAPES for financial support.

APPENDIX A: VARIATION OF THE ACTION (14)

In this section, the goal will be to display the calculus of the variation of the action given by Eq.(14). Thus, after some straightforward manipulation considering Eq.(19) and expanding h_2^* in power series similarly to Eq.(38), it is not difficult to show that the variation of the Ricci action can be written in the following manner

$$\delta\mathcal{S}^*_{Ricci} = \int d^4x\sqrt{-g}\Bigg\{\Bigg[-\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}h_2^*(-\tfrac{\Box}{\Lambda^2})R_{\alpha\beta}+ +R_{(\nu}{}^{\alpha}h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\mu)\alpha} + R^{\alpha}{}_{(\nu}h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\alpha\mu)}\Bigg]\delta g^{\mu\nu}+ +\Bigg[g^{\alpha\mu}g^{\beta\nu}\left(h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\alpha\beta}\right)+\left(h_2^*(-\tfrac{\Box^\dagger}{\Lambda^2})R^{\mu\nu}\right)\Bigg]\delta R_{\mu\nu}+ +R^{\mu\nu}\delta h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\mu\nu}\Bigg\}. \quad \text{(A1)}$$

By considering (16), the terms in the second bracket on the r.h.s. of (A1) can be handled to give

$$\int d^4x\sqrt{-g}\delta R_{\mu\nu}\left[g^{\alpha\mu}g^{\beta\nu}h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\alpha\beta}+h_2^*(-\tfrac{\Box^\dagger}{\Lambda^2})R^{\mu\nu}\right] = \int d^4x\left[\nabla_\rho\left(\sqrt{-g}\delta^{(\nu}_{\alpha}F^{*\mu)\rho}\right)-\nabla_\alpha\left(\sqrt{-g}F^{*(\nu\mu)}\right)\right]\delta\Gamma^{\alpha}_{\mu\nu}\,, \quad \text{(A2)}$$

where $F^{*\alpha\beta} \equiv g^{\omega\alpha}g^{\pi\beta}h_2^*\left(-\frac{\Box}{\Lambda^2}\right)R_{\omega\pi} + h_2^*\left(-\frac{\Box^\dagger}{\Lambda^2}\right)R^{\alpha\beta}$.

With the aim of treating the last term on the r.h.s. of (A1), by using (26) an expression analogous to (43) can be found. In this case, it is given by

$$(\delta\Box)\,T_{\alpha\beta} = \delta g^{\mu\nu}\nabla_\mu\nabla_\nu T_{\alpha\beta} - T_{\sigma\beta}\nabla^\mu\delta\Gamma^\sigma_{\mu\alpha} - T_{\alpha\sigma}\nabla^\mu\delta\Gamma^\sigma_{\mu\beta}+ -g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda T_{\alpha\beta} - 2\delta\Gamma^\sigma_{\mu\alpha}\nabla^\mu T_{\sigma\beta} - 2\delta\Gamma^\sigma_{\mu\beta}\nabla^\mu T_{\alpha\sigma}\,, \quad \text{(A3)}$$

with $T_{\alpha\beta}$ denoting an arbitrary covariant tensor of rank 2.

Thereby, taking into account Eq.(A3) and that here we have

$$\delta h_2^*\left(-\Box_\Lambda\right) = \sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{h_2^{*(m)}}{(-\Lambda^2)^m}\Box^r\delta\Box\;\Box^{m-r-1}\,, \quad \text{(A4)}$$

we write

$$\int d^4x\sqrt{-g}R^{\mu\nu}\delta h_2^*\left(-\Box_\Lambda\right)R_{\mu\nu} = -\tfrac{1}{\Lambda^2}\textstyle\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}h_2^{*(m)}\times \int d^4x\Bigg\{\left(\sqrt{-g}R^{[r]^\dagger\alpha\beta}\nabla_{(\mu}\nabla_{\nu)}R^{[m-r-1]}_{\alpha\beta}\right)\delta g^{\mu\nu}+ +\Bigg[\nabla_\rho\left(\sqrt{-g}g^{\mu\rho}\left(R^{[r]^\dagger\nu\beta}R^{[m-r-1]}_{\alpha\beta}+R^{[r]^\dagger\beta\nu}R^{[m-r-1]}_{\beta\alpha}\right)\right)+ -\sqrt{-g}\left(g^{\mu\nu}R^{[r]^\dagger\rho\beta}\nabla_\alpha R^{[m-r-1]}_{\rho\beta}+2R^{[r]^\dagger\nu\beta}\nabla^\mu R^{[m-r-1]}_{\alpha\beta}+ +2R^{[r]^\dagger\beta\nu}\nabla^\mu R^{[m-r-1]}_{\beta\alpha}\right)\Bigg]\delta\Gamma^\alpha_{\mu\nu}\Bigg\}\,, \quad \text{(A5)}$$

where we have defined

$$R^{[k]}_{\alpha\beta} \equiv \left(-\frac{\Box}{\Lambda^2}\right)^k R_{\alpha\beta}\,, \quad \text{(A6)}$$

and

$$R^{[k]^\dagger\alpha\beta} \equiv \left(-\frac{\Box^\dagger}{\Lambda^2}\right)^k R^{\alpha\beta}\,. \quad \text{(A7)}$$

Therefore, the complete variation of the Ricci action (14) reads

$$\frac{\delta\mathcal{S}^*_{Ricci}}{\delta g^{\mu\nu}} = \sqrt{-g}\Bigg[-\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}h_2^*(-\tfrac{\Box}{\Lambda^2})R_{\alpha\beta}+ +R_{(\nu}{}^{\alpha}h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\mu)\alpha}+R^{\alpha}{}_{(\nu}h_2^*\left(-\tfrac{\Box}{\Lambda^2}\right)R_{\alpha\mu)}+ -\tfrac{1}{\Lambda^2}\textstyle\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}h_2^{*(m)}R^{[r]^\dagger\alpha\beta}\nabla_{(\mu}\nabla_{\nu)}R^{[m-r-1]}_{\alpha\beta}\Bigg]\,, \quad \text{(A8)}$$

and

$$\frac{\delta\mathcal{S}^*_{Ricci}}{\delta\Gamma^\alpha_{\mu\nu}} = \nabla_\rho\left(\sqrt{-g}\delta^{(\nu}_{\alpha}F^{*\mu)\rho}\right)-\nabla_\alpha\left(\sqrt{-g}F^{*(\nu\mu)}\right)+ -\tfrac{1}{\Lambda^2}\textstyle\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}h_2^{*(m)}\times \Bigg\{\nabla_\rho\left[\sqrt{-g}g^{\rho(\mu}\left(R^{[r]^\dagger\nu)\beta}R^{[m-r-1]}_{\alpha\beta}+R^{[r]^\dagger\beta\nu)}R^{[m-r-1]}_{\beta\alpha}\right)\right]+ -\sqrt{-g}\left[g^{\mu\nu}R^{[r]^\dagger\rho\beta}\nabla_\alpha R^{[m-r-1]}_{\rho\beta}+2R^{[r]^\dagger(\nu\beta}\nabla^{\mu)}R^{[m-r-1]}_{\alpha\beta}+ +2R^{[r]^\dagger\beta(\nu}\nabla^{\mu)}R^{[m-r-1]}_{\beta\alpha}\right]\Bigg\} \quad \text{(A9)}$$

where the indices μ and ν have been symmetrized.

APPENDIX B: NON-LOCAL SCALAR FIELD

Let us consider a non-local scalar field whose action is

$$\delta\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left\{\phi\, Q\left(-\Box_\Lambda\right)\phi - V(\phi)\right\}, \qquad \text{(B1)}$$

where $Q(z)$ is analytic, and let us calculate the functional derivatives of this action with respect to the metric and the connection. The variation of (B1) gives

$$\delta\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left\{\phi\,\delta Q\left(-\Box_\Lambda\right)\phi + \right.$$
$$\left. + \frac{V(\phi)-\phi\, Q(-\Box_\Lambda)\phi}{2} g_{\mu\nu}\delta g^{\mu\nu}\right\}, \qquad \text{(B2)}$$

therefore all wee need to calculate is the first term in the r.h.s. of Eq.(B2). To do that, we follow the same procedure of section II A, and we expand $Q\left(-\Box_\Lambda\right)$ in power series as

$$Q(-\Box_\Lambda) = \sum_{m=0}^{\infty} Q^{(m)}\left(-\frac{\Box}{\Lambda^2}\right)^m, \qquad \text{(B3)}$$

which allows to express $\delta Q\left(-\Box_\Lambda\right)$ as

$$\delta Q\left(-\Box_\Lambda\right) = \sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{Q^{(m)}}{\left(-\Lambda^2\right)^m}\Box^r\delta\Box\;\Box^{m-r-1}, \qquad \text{(B4)}$$

so that by use of (27) one has

$$\int d^4x\sqrt{-g}\phi\delta Q\left(-\Box_\Lambda\right)\phi = -\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} Q^{(m)}\times$$
$$\times\int d^4x\sqrt{-g}\left[\left(\phi^{[r]^\dagger}\nabla_\mu\nabla_\nu\phi^{[m-r-1]}\right)\delta g^{\mu\nu} + \right.$$
$$\left. - \left(\phi^{[r]^\dagger} g^{\mu\nu}\nabla_\alpha\phi^{[m-r-1]}\right)\delta\Gamma^\alpha_{\mu\nu}\right], \qquad \text{(B5)}$$

where we have defined

$$\phi^{[k]} \equiv \left(-\frac{\Box}{\Lambda^2}\right)^k\phi, \qquad \text{(B6)}$$

and

$$\phi^{[k]^\dagger} \equiv \left(-\frac{\Box^\dagger}{\Lambda^2}\right)^k\phi. \qquad \text{(B7)}$$

Therefore, from Eqs.(B2,B5) one has

$$\frac{\delta\mathcal{S}_\phi}{\delta g^{\mu\nu}} = \sqrt{-g}\left\{\frac{V(\phi)-\phi\, Q(-\Box_\Lambda)\phi}{2} g_{\mu\nu} + \right.$$
$$\left. -\frac{1}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} Q^{(m)}\,\phi^{[r]^\dagger}\nabla_{(\mu}\nabla_{\nu)}\phi^{[m-r-1]}\right\}, \qquad \text{(B8)}$$

and

$$\frac{\delta\mathcal{S}_\phi}{\delta\Gamma^\alpha_{\mu\nu}} = \frac{\sqrt{-g}}{\Lambda^2}\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} Q^{(m)}\,\phi^{[r]^\dagger} g^{\mu\nu}\nabla_\alpha\phi^{[m-r-1]}. \qquad \text{(B9)}$$

[1] L. Modesto, Astron. Rev. **8.2**, 4 (2013); Phys. Rev. D86 (2012) 044005, arXiv: 1107.2403
[2] T. Biswas *et al.*, Phys. Rev. Lett. **108**, 031101 (2012), arXiv: 1110.5249.
[3] L. Smolin, *Three roads to quantum gravity*, London, UK: Weidenfeld & Nicolson (2000).
[4] K. S. Stelle, Phys. Rev. D **16**, 953 (1977); E. T. Tomboulis, hep-th/9702146.
[5] F. Briscese *et al.* Phys. Rev. D **89**, 024029 (2014), arXiv: 1308.1413.
[6] F. Briscese *et al.*, Phys. Rev. D **87**, 083507 (2013), arXiv: 1212.3611.
[7] A. S. Koshelev, L. Modesto, L. Rachwal, A. A. Starobinsky, arXiv:1604.03127 [hep-th].
[8] G. V. Efimov, Int. J. Theor. Phys. **10**, 19 (1974) G. V. Efimov, Commun. Math. Phys. **7**, 138 (1968)
[9] V. A. Alebastrov, G. V. Efimov, Commun. Math. Phys. **31**, 1-24 (1973).
[10] V. A. Alebastrov, G. V. Efimov, Commun. Math. Phys. **38**, 11 (1974).
[11] E. T. Tomboulis, Phys.Rev. D **92**, 125037 (2015).
[12] G. V. Efimov, M. A. Ivanov, O. A. Mogilevsky, Ann. Phys. **103**, 169-l E4 (1977); G. V. Efimov and K. Namsrai, Teor. Mat. Fiz. **22**, 186 (1975); G. V. Efimov, Ann. Phys. **71**, 466 (1972)
[13] E. T. Tomboulis, "Superrenormalizable gauge and gravitational theories", hep-th/9702146.
[14] L. Modesto, M. Piva, L. Rachwal, arXiv:1506.06227; L.

Modesto, L. Rachwal, Nucl.Phys. B **900** (2015) 147-169.

[15] S. Talaganis, T. Biswas, A. Mazumdar, "Towards understanding the ultraviolet behavior of quantum loops in infinite-derivative theories of gravity", arXiv:1412.3467 [hep-th].

[16] F. Briscese, E. R. Bezerra de Mello, A. Yu. Petrov, V.B. Bezerra, arXiv:1508.02001 [gr-qc].

[17] L. Modesto, L. Rachwal, Nucl.Phys. B **889** (2014) 228-248, arXiv:1407.8036 [hep-th]; P. Dona, S. Giaccari, L. Modesto, L. Rachwal, Y. Zhu, JHEP **1508** (2015) 038, arXiv:1506.04589 [hep-th]; S. Giaccari, L. Modesto, arXiv:1605.03906 [hep-th].

[18] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980); S. V. Ketov and A. A. Starobinsky, JCAP **1208**, 022 (2012), arXiv: 1203.0805.

[19] Planck 2015 results, XIII, Cosmological parameters, Planck Collaboration: P. A. R. Ade et al., arXiv:1502.01589; Planck 2015, XX, Constraints on inflation, Planck Collaboration: P. A. R. Ade et al., arXiv:1502.02114.

[20] C. Bambi *et al.*, Eur. Phys. J. C **74**: 2767 (2014), arXiv: 1306.1668.

[21] G. Calcagni, L. Modesto, P. Nicolini, Eur. Phys. J. C **74**, 2999 (2014), arXiv:1306.5332; G.Calcagni, L. Modesto, Phys. Rev. D **91**, 124059 (2015), arXiv:1404.2137 [hep-th].

[22] Y.-D. Li, L. Modesto, L. Rachwal, JHEP **12**, 173 (2015), arXiv:1506.08619 [hep-th].

[23] L. Modesto, L. Rachwal, arXiv:1605.04173 [hep-th].

[24] Eran Marcus, *Higher Derivative Gauge and Gravitational Theories*, Ph.D. Thesis (1998).

[25] A. Palatini, Rend. Circ. Mat. Palermo 43: 203172 (1919). English translation by R. Hojman and C. Mukku in P. G. Bergmann and V. De Sabbata (eds.) Cosmology and Gravitation, Plenum Press, New York (1980).

[26] S. Capozziello, M. De Laurentis, Physics Reports Volume **509**, 417(2011), 167171; T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010); S. Nojiri, S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4** (2007), 115-146; S. Nojiri, S. D. Odintsov, Phys.Rept. **505** (2011).

[27] G. J. Olmo, Int. J. of Mod. Phy. D **20**, 4 (2011), 413172; T, P. Sotiriou, S. Liberati, Ann. of Phys. **322** (2007), 935176.

[28] C.W. Misner, K.S. Thorne, J.A. Wheeler - Gravitation, Freeman & Co. (1973).

[29] G, Allemandi, A. Borowiec, M. Francaviglia, S. D. Odintsov, Phys.Rev. D **72** (2005) 063505.

[30] M. Ferraris, M. Francaviglia, Gen. Rel. Grav. **14**, 3 (1982).