On the eigenvalues and energy of the Seidel and Seidel Laplacian matrices of graphs

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Abstract

Let $S(\Gamma)$ be a Seidel matrix of a graph Γ of order n and let $D(\Gamma) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be a diagonal matrix with d_i denoting the degree of a vertex v_i in Γ. The Seidel Laplacian matrix of Γ is defined as $SL(\Gamma) = D(\Gamma) - S(\Gamma)$. In this paper, we obtain an upper bound, and a lower bound on the Seidel Laplacian Estrada index of graphs. Moreover, we find a relation between Seidel energy and Seidel Laplacian energy of graphs. We establish some lower bounds on the Seidel Laplacian energy in terms of different graph parameters. Finally, we present a relation between Seidel Laplacian Estrada index and Seidel Laplacian energy of graphs.

Keywords: Graph, Seidel matrix, Seidel Laplacian matrix, Seidel Laplacian Estrada index, Seidel Laplacian energy.

MSC: 05C05.

1 Introduction

All graphs considered in this paper are simple. Let $\Gamma = (V, E)$ be a graph with order n and size m, namely $|V(\Gamma)| = n$ and $|E(\Gamma)| = m$, where $V(\Gamma)$ and $E(\Gamma)$ are the vertex set and edge set of Γ, respectively. The adjacency matrix $A(\Gamma) = (a_{ij})$ of Γ is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. The complement of the graph Γ, denoted by $\overline{\Gamma}$, is the graph with the same vertex set $V(\Gamma)$, where two distinct vertices are adjacent if and only if they are non-adjacent in Γ. The degree of v_i, denoted by $d_i = d(v_i)$, is the number of edges incident to v_i. A graph is called *k-regular* if $d(v_i) = k$ for all vertices $v_i \in V(\Gamma)$. In 1966, Van Lint and Seidel [52] introduced the *Seidel matrix* of Γ, $S(\Gamma) = J - 2A(\Gamma) - I = A(\overline{\Gamma}) - A(\Gamma)$, where J is a square matrix whose all entries are equal to 1. It is obvious that $-S(\Gamma)$ is the

Seidel matrix of the complement of Γ, that is, $S(\overline{\Gamma}) = -S(\Gamma)$. Several results on Seidel matrix were reported in [3,4,6,8,27,52,53]. We denote the eigenvalues of $A(\Gamma)$ and $S(\Gamma)$ by $\theta_1, \theta_2, \ldots, \theta_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively ($\theta_i$'s and λ_i's are called the eigenvalues and Seidel eigenvalues of Γ, respectively). The energy of a graph Γ is defined as

$$E(\Gamma) = \sum_{k=1}^{n} \left| \theta_k - \frac{\text{trace}\,(A(\Gamma))}{n} \right| = \sum_{k=1}^{n} |\theta_k|, \tag{1}$$

where the last equality holds since $A(\Gamma)$ has zero trace. The energy of a graph Γ has been extensively studied in the literature (see, [9,22,38,40] and the references therein). Similar to the energy of a graph, the *Seidel energy* $E_S(\Gamma)$ of Γ is defined to be the sum of the absolute values of the eigenvalues of its Seidel matrix. In the last twenty years, other kinds of energies (Laplacian energy [13,15,24,54], distance energy [49,50], generalized distance energy [2] etc.) of a graph have been defined and mathematical properties studied. The Seidel Laplacian matrix of a graph was introduced in [43], where the main properties of its eigenvalues and the bounds of Seidel Laplacian energy were established. Let $D(\Gamma) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be a diagonal matrix in which d_i stands for degree of a vertex v_i in Γ. Then, in analogy with the ordinary Laplacian matrix, the Seidel Laplacian matrix of Γ is defined as $SL(\Gamma) = D(\Gamma) - S(\Gamma)$. It is obvious that $-SL(\Gamma)$ is the Seidel Laplacian matrix of the complement of Γ, that is, $SL(\overline{\Gamma}) = -SL(\Gamma)$. Let $\lambda_1^L \geq \lambda_2^L \geq \cdots \geq \lambda_n^L$ be the eigenvalues of $SL(\Gamma)$. In analogy to Eq.(1), the Seidel Laplacian energy of Γ is defined as

$$\begin{aligned} E_{SL}(\Gamma) = \sum_{k=1}^{n} \left| \lambda_k^L - \frac{\text{trace}\,(SL(\Gamma))}{n} \right| &= \sum_{k=1}^{n} \left| \lambda_k^L - \frac{\sum\limits_{k=1}^{n} (n-1-2d_k)}{n} \right| \\ &= \sum_{k=1}^{n} \left| \lambda_k^L - \frac{n(n-1)-4m}{n} \right|. \end{aligned} \tag{2}$$

We denote

$$\sigma_k = \lambda_k^L - \frac{n(n-1)-4m}{n}, \quad k = 1, 2, \ldots, n. \tag{3}$$

Then the expression for Seidel Laplacian energy becomes analogous to the formula for ordinary graph energy, Eq.(1), that is,

$$E_{SL}(\Gamma) = \sum_{k=1}^{n} |\sigma_k|, \tag{4}$$

where $\sum\limits_{k=1}^{n} \sigma_k = 0$. More results on Seidel and Seidel Laplacian energies are reported in [1, 23, 26, 30, 31, 33–35, 39, 41–43, 51]. The Estrada index of a graph Γ is defined as

$$EE = EE(\Gamma) = \sum_{k=1}^{n} e^{\theta_k}. \tag{5}$$

This graph invariant appeared for the first time in year 2000 by Ernesto Estrada dealing with the folding of protein molecules [19]. Since then a remarkable variety of other chemical and non chemical applications of EE were communicated, for details see [18, 20, 45–48]. Motivated by (5), we define the Seidel Estrada index and Seidel Laplacian Estrada index of Γ as

$$SEE = SEE(\Gamma) = \sum_{k=1}^{n} e^{\lambda_k}, \quad \textit{and} \quad SLEE = SLEE(\Gamma) = \sum_{k=1}^{n} e^{\lambda_k^L}, \tag{6}$$

respectively. For more information related to eigenvalues and Seidel eigenvalues and their properties, we refer the reader to [5, 8, 25, 27, 29, 32].

Example 1.1. *The Seidel and Seidel Laplacian matrices of K_n are $S(K_n) = I - J = -A(K_n)$ and $SL(K_n) = J - nI = -L(K_n)$, respectively. It is well-known that the adjacency spectrum of K_n is $AS(K_n) = (n-1, \underbrace{-1, -1, \ldots, -1}_{n-1})$ and the Laplacian spectrum of K_n is $LS(K_n) = (\underbrace{n, n, \ldots, n}_{n-1}, 0)$. Hence the Seidel spectrum and the Seidel Laplacian spectrum of K_n are $SS(K_n) = (\underbrace{1, 1, \ldots, 1}_{n-1}, 1-n)$ and $SLS(K_n) = (0, \underbrace{-n, -n, \ldots, -n}_{n-1})$, respectively. Therefore*

$$E_S(K_n) = E_{SL}(K_n) = 2n-2, \; SEE(K_n) = (n-1)e + e^{1-n} \quad \textit{and} \quad SLEE(K_n) = 1 + (n-1)e^{-n}.$$

Example 1.2. *Let $\Gamma \cong K_{p,q}\,(p+q=n,\ p \geq q \geq 1)$. The Seidel Laplacian matrix is*

$$SL(\Gamma) = \left(\begin{array}{cccccc|ccccc} p-q-1 & -1 & -1 & \cdots & -1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & p-q-1 & -1 & \cdots & -1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & p-q-1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & p-q-1 & 1 & 1 & \cdots & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & q-p-1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & q-p-1 & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & \cdots & q-p-1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & \cdots & -1 & q-p-1 \end{array} \right).$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue λ^L of $SL(\Gamma)$. Then we have $SL(\Gamma)\mathbf{x} = \lambda^L \mathbf{x}$. One can easily see that $p-q$ is an eigenvalue of multiplicity $p-1$ with corresponding eigenvectors $(\underbrace{1, -1, 0, \ldots, 0}_{p}; \underbrace{0, \ldots, 0}_{q})^T$, $(\underbrace{1, 0, -1, 0, \ldots, 0}_{p}; \underbrace{0, \ldots, 0}_{q})^T, \ldots,$

$(\underbrace{1, 0, \ldots, 0, -1}_{p};\ \underbrace{0, \ldots, 0}_{q})^T$. *Moreover, we have that* $q-p$ *is an eigenvalue of multiplicity* $q-1$ *with corresponding eigenvectors* $(\underbrace{0, \ldots, 0}_{p};\ \underbrace{1, -1, 0, \ldots, 0}_{q})^T$, $(\underbrace{0, \ldots, 0}_{p};\ \underbrace{1, 0, -1, 0, \ldots, 0}_{q})^T, \ldots,$ $(\underbrace{0, \ldots, 0}_{p};\ \underbrace{1, 0, \ldots, 0, -1}_{q})^T$.

The remaining two eigenvalues of $SL(\Gamma)$**:** *For this let* $\mathbf{x} = (\underbrace{x_1, \ldots, x_1}_{p};\ \underbrace{x_n, \ldots, x_n}_{q})^T$ *be an eigenvector corresponding to the eigenvalue* λ^L *of* $SL(\Gamma)$*, where* $x_1 \neq 0 \neq x_n$*. Then* $SL(\Gamma)\mathbf{x} = \lambda^L\mathbf{x}$*. The eigenvalue* λ^L *satisfy the following equations:*

$$\lambda^L x_1 = -qx_1 + qx_n, \text{ that is, } (\lambda^L + q)\, x_1 = qx_n,$$
$$\lambda^L x_n = -px_n + px_1, \text{ that is, } (\lambda^L + p)\, x_n = px_1,$$

that is, $(\lambda^L + q)(\lambda^L + p) = pq$*, that is,* $\lambda^L(\lambda^L + p + q) = 0$*, that is,* $\lambda^L = 0, -(p+q)$*. Hence*

$$SLS(\Gamma) = \Big(\underbrace{p-q, \ldots, p-q}_{p-1}, 0, \underbrace{q-p, \ldots, q-p}_{q-1}, -(p+q)\Big).$$

Therefore

$$SLEE(\Gamma) = (p-1)\, e^{p-q} + (q-1)\, e^{q-p} + 1 + e^{-(p+q)}.$$

Now,

$$\frac{n(n-1) - 4m}{n} = \frac{(p+q)(p+q-1) - 4pq}{p+q} = p + q - 1 - \frac{4pq}{p+q}.$$

Thus we have

$$\begin{aligned} E_{SL}(\Gamma) &= \sum_{k=1}^{n} \left| \lambda_k^L - \Big(p + q - 1 - \frac{4pq}{p+q}\Big) \right| \\ &= \left| p - q - (p+q-1) + \frac{4pq}{p+q} \right| (p-1) + \left| q - p - (p+q-1) + \frac{4pq}{p+q} \right| (q-1) \\ &\qquad + \left| p + q - 1 - \frac{4pq}{p+q} \right| + \left| 2(p+q) - 1 - \frac{4pq}{p+q} \right|. \end{aligned}$$

After simplifying, we obtain

$$E_{SL}(\Gamma) = \begin{cases} 4p - 2 & \text{if } p = q, \\ \frac{1}{p+q}\Big[2pq\,(2p - 2q - 1) + 2p^2 + 4q^2 - 2p - 2q\Big] & \text{if } p > q \text{ and } (p-q)^2 \geq p + q, \\ \frac{1}{p+q}\Big[2pq\,(2p - 2q + 1) + 2q^2\Big] & \text{if } p > q \text{ and } (p-q)^2 \leq p + q. \end{cases}$$

Example 1.3. *Let* $\Gamma \cong K_{p,q}\,(p + q = n,\ p \geq q \geq 1)$*. Then the Seidel matrix of* Γ *is* $S(\Gamma) = vv^T - I$*, where* I *is the identity matrix and the vector* v *has all* 1*'s in the* *first p position and all -1's in the last q positions. Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be an eigenvector corresponding to the eigenvalue λ of $S(\Gamma)$. Then $S(\Gamma)\mathbf{y} = \lambda\mathbf{y}$. One can easily see that -1 is an eigenvalue of multiplicity $p-1$ with corresponding eigenvectors* $(\underbrace{1, -1, 0, \ldots, 0}_{p}; \underbrace{0, \ldots, 0}_{q})^T$, $(\underbrace{1, 0, -1, 0, \ldots, 0}_{p}; \underbrace{0, \ldots, 0}_{q})^T$, $\ldots$, $(\underbrace{1, 0, \ldots, 0, -1}_{p}; \underbrace{0, \ldots, 0}_{q})^T$. *Moreover, we have that -1 is an eigenvalue of multiplicity $q-1$ with corresponding eigenvectors* $(\underbrace{0, \ldots, 0}_{p}; \underbrace{1, -1, 0, \ldots, 0}_{q})^T$, $(\underbrace{0, \ldots, 0}_{p}; \underbrace{1, 0, -1, 0, \ldots, 0}_{q})^T$, $\ldots$, $(\underbrace{0, \ldots, 0}_{p}; \underbrace{1, 0, \ldots, 0, -1}_{q})^T$.

The remaining two eigenvalues of $S(\Gamma)$: *For this let* $\mathbf{y} = (\underbrace{y_1, \ldots, y_1}_{p}; \underbrace{y_n, \ldots, y_n}_{q})^T$ *be an eigenvector corresponding to the eigenvalue λ of $S(\Gamma)$, where $y_1 \neq 0 \neq y_n$. Then $S(\Gamma)\mathbf{y} = \lambda\mathbf{y}$. The eigenvalue λ satisfy the following equations:*

$$\lambda\, y_1 = (p-1)y_1 - qy_n, \text{ that is, } (\lambda - p + 1)\, y_1 = -qy_n,$$

$$\lambda\, y_n = -py_1 + (q-1)y_n, \text{ that is, } (\lambda - q + 1)\, y_n = -py_1,$$

that is, $(\lambda - p + 1)(\lambda - q + 1) = pq$, *that is,* $(\lambda + 1)(\lambda + 1 - p - q) = 0$, *that is,* $\lambda = -1,\, p + q - 1$. *Hence*

$$SS(\Gamma) = \Big(p + q - 1, \underbrace{-1, \ldots, -1}_{p+q-1}\Big).$$

Therefore

$$E_S(\Gamma) = 2\,(p + q - 1).$$

and

$$SEE(\Gamma) = e^{p+q-1} + (p + q - 1)\, e^{-1}.$$

This paper is organized as follows. In Section 2, we give a list of some previously known results. Also we find some properties on the Seidel Laplacian eigenvalues of graphs. In Section 3, we give some bounds on the Seidel Laplacian Estrada index of a graph. We obtain an inequality between Seidel energy and Seidel Laplacian energy of graph Γ. We also present some lower bounds on the Seidel Laplacian energy of graphs. Finally, we give a relation between the Seidel Laplacian Estrada index and the Seidel Laplacian energy of graphs.

2 Preliminaries and basic results

It is well known that the Seidel and Seidel Laplacian eigenvalues satisfy the following relations:

$$\begin{cases} \sum\limits_{k=1}^{n} \lambda_k = 0, \quad \sum\limits_{k=1}^{n} \lambda_k^2 = n(n-1), \\ \sum\limits_{k=1}^{n} \lambda_k^L = \mathrm{trace}(SL(\Gamma)) = \sum\limits_{k=1}^{n} [n-1-2d_k] = n(n-1) - 4m, \\ \sum\limits_{k=1}^{n} (\lambda_k^L)^2 = \mathrm{trace}(SL(\Gamma)^2) = n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma), \end{cases} \tag{7}$$

where $Z_1(\Gamma) = \sum\limits_{k=1}^{n} d_k^2$ is called the first Zagreb index, whose mathematical properties have been studied in due detail [7, 10–12, 14, 16, 17].

Lemma 2.1. [43] *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Seidel eigenvalues of a k-regular graph Γ of order n, then the Seidel Laplacian eigenvalues of Γ are $n-1-2k-\lambda_i$, $i = 1, 2, \dots, n$.*

Lemma 2.2. [43] *Let Γ be a graph with n vertices, m edges and σ_k be as defined above by Eq. (3). Then*

$$\sum_{k=1}^{n} \sigma_k^2 = n(n-1) + 4Z_1(\Gamma) - \frac{16m^2}{n}.$$

Lemma 2.3. [28] *Let P and Q be two Hermitian matrices of order n and $M = P + Q$. Then*

$$\lambda_k(M) \le \lambda_\ell(P) + \lambda_{k-\ell+1}(Q) \quad (n \ge k \ge \ell \ge 1),$$

$$\lambda_k(M) \ge \lambda_\ell(P) + \lambda_{k-\ell+n}(Q) \quad (n \ge k \ge \ell \ge 1),$$

where $\lambda_k(P)$ is the k^{th} largest eigenvalue of the matrix P.

We now state the following results from matrix theory.

Lemma 2.4. [44] *Let B be a $p \times p$ symmetric matrix and let B_k be its leading $k \times k$ submatrix. Then, for $i = 1, 2, \dots, k$,*

$$\mu_{p-i+1}(B) \le \mu_{k-i+1}(B_k) \le \mu_{k-i+1}(B), \tag{8}$$

where $\mu_i(B)$ is the i-th greatest eigenvalue of B.

Lemma 2.5. [21] *Let A, B, C be three real symmetric matrices of order n such that $C = A + B$. Then*

$$E(C) \le E(A) + E(B),$$

where $E(C) = \sum\limits_{k=1}^{n} |\lambda_k(C)|$ is the energy and $\lambda_k(C)$ $(k = 1, 2, \dots, n)$ are the eigenvalues of C.

Ozeki's Inequality [36]: If b_k and c_k, $k = 1, 2, \dots, n$, are non-negative real numbers, then

$$\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2 - \left(\sum_{k=1}^{n} b_k\, c_k\right)^2 \leq \frac{n^2}{4}\,(M_1M_2 - m_1m_2)^2\,,$$

where $M_1 = \max b_k$, $M_2 = \max c_k$ and $m_1 = \min b_k$, $m_2 = \min c_k$, $1 \leq k \leq n$.

Polya-Szego Inequality [37]: Let b_k and c_k, $k = 1, \dots, n$, be positive real numbers. Then

$$\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} b_k\, c_k\right)^2,$$

where M_k and m_k are defined as before.

Lemma 2.6. *Let Γ be a graph of order n. Then there exists a positive integer i such that $\lambda_i^L(\Gamma) = 0\,(1 \leq i \leq n)$. Moreover, if $\Gamma \ncong K_n$, then $\lambda_1^L(\Gamma) \geq 1$.*

Proof. From the definition of Seidel Laplacian matrix, in each row or column, the sum of the entries is zero. Then at least one Seidel Laplacian eigenvalue is zero.

Since $\Gamma \ncong K_n$, then there exist two non-adjacent vertices v_i and v_j. Then by Lemma 2.4, we obtain $\lambda_1^L(\Gamma) \geq \mu_1$, where $\mu_1 \geq \mu_2$ and μ_1, μ_2 are the roots of the following equation:

$$\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 0, \quad \text{that is, } x = \pm 1.$$

Thus we have $\lambda_1^L(\Gamma) \geq \mu_1 = 1$. □

Lemma 2.7. *Let Γ be a graph of order n with m edges. Then $\lambda_n^L(\Gamma) < \frac{n(n-1)-4m}{n}$.*

Proof. We prove this result by contradiction. For this we assume that $\lambda_n^L \geq \frac{n(n-1)-4m}{n}$. Then

$$\lambda_1^L \geq \lambda_2^L \geq \cdots \geq \lambda_n^L \geq \frac{n(n-1)-4m}{n}.$$

Since $\sum\limits_{i=1}^{n} \lambda_i^L = n(n-1) - 4m$, from the above, we conclude that $\lambda_1^L = \lambda_2^L = \cdots = \lambda_n^L = \frac{n(n-1)-4m}{n}$. Since one of the Seidel Laplacian eigenvalue is zero, we have $\lambda_1^L = \lambda_2^L = \cdots = \lambda_n^L = 0$ and $m = \frac{n(n-1)}{4}$. By Lemma 2.6, we have $\lambda_1^L(\Gamma) \geq 1$, we get a contradiction. This completes the proof of the result. □

Finally, we state the well-known theorem in the following:

Theorem 2.8. *Let $1 \leq p \leq \infty$ be a (extended) real number. For fixed vector $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathcal{R}^n$, the mapping*

$$p \to \|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} x^p \right)^{1/p},$$

is monotone decreasing, that is, $p \geq q \geq 1 \Rightarrow \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q$.

Corollary 2.9. *Let $c_1,\, c_2, \ldots,\, c_n$ be n real numbers and $k\,(\geq 2)$ a positive integer. Then*

$$\left(\sum_{\ell=1}^{n} c_\ell^k \right)^2 \leq \left(\sum_{\ell=1}^{n} c_\ell^2 \right)^k. \tag{9}$$

Corollary 2.10. *Let $c_1,\, c_2, \ldots,\, c_n$ be n real numbers and $k\,(\geq 1)$ a positive integer. Then*

$$\sum_{\ell=1}^{n} c_\ell^k \leq \left(\sum_{\ell=1}^{n} c_\ell \right)^k.$$

3 On Seidel Laplacian Estrada index and Seidel Laplacian energy of graphs

We give some lower and upper bounds on the Seidel Laplacian Estrada index of Γ.

Theorem 3.1. *Let Γ be a graph of order n. Then the Seidel Laplacian Estrada index of Γ is bounded by*

$$SLEE(\Gamma) \geq \begin{cases} \sqrt{1 + (n-1)\, e^{2n - \dfrac{8m}{n-1}} + n(n-1)\, e^{2(n-1) - \dfrac{8m}{n}}} & \text{if } m \geq \frac{1}{4}\, n(n-1), \\ n\, e^{n - 1 - \dfrac{4m}{n}} & \text{if } m < \frac{1}{4}\, n(n-1), \end{cases}$$

and

$$SLEE(\Gamma) \leq n - 1 + e^{\sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)}}.$$

Proof. **(I)** Lower Bound: From (6), we get

$$SLEE^2(\Gamma) = \sum_{i=1}^{n} e^{2\lambda_i^L} + 2 \sum_{i<j} e^{\lambda_i^L} e^{\lambda_j^L}. \tag{10}$$

By the arithmetic-geometric-mean inequality, we obtain

$$\begin{aligned}
2\sum_{i<j} e^{\lambda_i^L} e^{\lambda_j^L} &\geq n(n-1)\left(\prod_{i<j} e^{\lambda_i^L} e^{\lambda_j^L}\right)^{\frac{2}{n(n-1)}} \\
&= n(n-1)\left[\left(\prod_{i=1}^{n} e^{\lambda_i^L}\right)^{n-1}\right]^{\frac{2}{n(n-1)}} \\
&= n(n-1)\left(e^{\sum_{i=1}^{n}\lambda_i^L}\right)^{\frac{2}{n}} \\
&= n(n-1)\left(e^{2(n-1)-\frac{8m}{n}}\right). \qquad (11)
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{i=1}^{n} e^{2\lambda_i^L} &= e^{2\lambda_1^L} + \sum_{i=2}^{n} e^{2\lambda_i^L} \\
&\geq e^{2\lambda_1^L} + (n-1)\left(e^{2\sum_{i=2}^{n}\lambda_i^L}\right)^{1/(n-1)} \\
&= e^{2\lambda_1^L} + (n-1)\left(e^{2\left(n(n-1)-4m-\lambda_1^L\right)}\right)^{1/(n-1)} \\
&= e^{2\lambda_1^L} + (n-1)\, e^{2n - \frac{8m}{n-1} - \frac{2\lambda_1^L}{n-1}}. \qquad (12)
\end{aligned}$$

Let us consider a function

$$f(x) = e^{2x} + (n-1)\, e^{2n - \frac{8m}{n-1} - \frac{2x}{n-1}}.$$

Then

$$f'(x) = 2\, e^{2x} - 2\, e^{2n - \frac{8m}{n-1} - \frac{2x}{n-1}}.$$

Thus $f(x)$ is an increasing function on $x \geq n-1-\frac{4m}{n}$ and a decreasing function on $x \leq n-1-\frac{4m}{n}$.

Case 1. $m \geq \frac{1}{4}n(n-1)$. In this case $\lambda_1^L \geq 0 \geq n-1-\frac{4m}{n}$ and hence $f(\lambda_1^L) \geq f(0) \geq f\left(n-1-\frac{4m}{n}\right)$, we obtain

$$e^{2\lambda_1^L} + (n-1)\, e^{2n - \frac{8m}{n-1} - \frac{2\lambda_1^L}{n-1}} = f(\lambda_1^L) \geq f(0) = 1 + (n-1)\, e^{2n - \frac{8m}{n-1}}.$$

Using this result with (12), we obtain

$$\sum_{i=1}^{n} e^{2\lambda_i^L} \geq 1 + (n-1)\, e^{2n - \frac{8m}{n-1}}.$$

From (10) and (11) with the above result, we obtain

$$SLEE^2(\Gamma) \geq 1 + (n-1)\, e^{2n - \frac{8m}{n-1}} + n(n-1)\, e^{2(n-1) - \frac{8m}{n}},$$

that is,

$$SLEE(\Gamma) \geq \sqrt{1 + (n-1)\, e^{2n - \frac{8m}{n-1}} + n(n-1)\, e^{2(n-1) - \frac{8m}{n}}}.$$

Case 2. $m < \frac{1}{4}\, n(n-1)$. In this case $f(0) > f\left(n-1-\frac{4m}{n}\right)$. Hence

$$e^{2\lambda_1^L} + (n-1)\, e^{2n - \frac{8m}{n-1} - \frac{2\,\lambda_1^L}{n-1}} = f(\lambda_1^L) \geq f\left(n-1-\frac{4m}{n}\right) = n\, e^{2(n-1) - \frac{8m}{n}}.$$

Using this result with (12), we obtain

$$\sum_{i=1}^{n} e^{2\lambda_i^L} \geq n\, e^{2(n-1) - \frac{8m}{n}}.$$

From (10) and (11) with the above result, we obtain

$$SLEE^2(\Gamma) \geq n^2\, e^{2(n-1) - \frac{8m}{n}}, \quad \text{that is,} \quad SLEE(\Gamma) \geq n\, e^{n-1-\frac{4m}{n}}.$$

This completes the proof of the lower bound.

(II) Upper Bound: Since $f(x) = e^x$ monotonically increases in $\mathbb{R}$, from (6) with Corollary 2.9, we obtain

$$SLEE(\Gamma) = n + n(n-1) - 4m + \sum_{k=2}^{\infty}\sum_{j=1}^{n} \frac{(\lambda_j^L)^k}{k!} \leqslant n^2 - 4m + \sum_{k=2}^{\infty}\sum_{j=1}^{n} \frac{(|\lambda_j^L|)^k}{k!}$$

$$= n^2 - 4m + \sum_{k=2}^{\infty}\sum_{j=1}^{n} \frac{\left[(\lambda_j^L)^2\right]^{k/2}}{k!}$$

$$\leqslant n^2 - 4m + \sum_{k=2}^{\infty} \frac{1}{k!}\left(\sum_{j=1}^{n}(\lambda_j^L)^2\right)^{k/2}$$

$$= n^2 - 4m - \sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)} + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{j=1}^{n}(\lambda_j^L)^2\right)^{k/2}$$

$$= n^2 - 4m - \sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)} + \sum_{k=1}^{\infty} \frac{\left(\sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)}\right)^k}{k!}$$

$$= n^2 - 4m - \sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)} + e^{\sqrt{n^2(n-1) - 8m(n-1) + 4Z_1(\Gamma)}}.$$

This completes the proof of the upper bound. □

Theorem 3.2. *Let Γ be a graph of order n. Also let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1^L \geq \cdots \geq \lambda_n^L$ be all the Seidel and Seidel Laplacian eigenvalues of Γ, respectively. Then*

$$n - 1 - 2\Delta - \lambda_{n-i+1} \leq \lambda_i^L \leq n - 1 - 2\delta - \lambda_{n-i+1}, \quad i = 1, 2, \ldots, n, \tag{13}$$

where Δ and δ are the maximum degree and the minimum degree of graph Γ, respectively. Moreover, if Γ is k-regular, then the left and right equalities hold.

Proof. We have $SL(\Gamma) = D(\Gamma) - S(\Gamma)$, that is, $D(\Gamma) = SL(\Gamma) + S(\Gamma)$. By Lemma 2.3, we have

$$(i)\ \ \lambda_i(D(\Gamma)) \ \leq \ \lambda_j(SL(\Gamma)) + \lambda_{i-j+1}(S(\Gamma)) \quad (1 \leq j \leq i \leq n), \tag{14}$$

$$(ii)\ \ \lambda_i(D(\Gamma)) \ \geq \ \lambda_j(SL(\Gamma)) + \lambda_{i-j+n}(S(\Gamma)) \quad (1 \leq i \leq j \leq n). \tag{15}$$

Setting $i = n$ in (14), we obtain

$$\lambda_n(D(\Gamma)) \leq \lambda_j(SL(\Gamma)) + \lambda_{n-j+1}(S(\Gamma)),$$

$$\text{that is, } \ \lambda_j(SL(\Gamma)) \geq n - 1 - 2\Delta - \lambda_{n-j+1}(S(\Gamma)), \quad 1 \leq i \leq n.$$

Thus we have

$$\lambda_i^L \geq n - 1 - 2\Delta - \lambda_{n-i+1}, \ \ 1 \leq i \leq n.$$

Similarly, by setting $i = 1$ in (15), we obtain,

$$\lambda_i^L \leq n - 1 - 2\delta - \lambda_{n-i+1}, \ \ 1 \leq i \leq n.$$

For k-regular graph, we have $\delta(\Gamma) = \Delta(\Gamma)$. Then by Lemma 2.1, the left and right equalities hold in (13). □

Theorem 3.3. *Let Γ be a graph of order n. Then the following inequalities hold:*

$$E_{\mathrm{S}}(\Gamma) + E(D) - E(D + S) \leq E_{\mathrm{SL}}(\Gamma) \leq E_{\mathrm{S}}(\Gamma) + E(D), \tag{16}$$

where $D = diag(n - 1 - 2d_1, n - 1 - 2d_2, \ldots, n - 1 - 2d_n)$. If Γ is a $\left(\frac{n-1}{2}\right)$-regular graph of odd order n, then the right equality holds.

Proof. **Left inequality:** We define the matrices $C = \begin{pmatrix} S & D \\ D & S \end{pmatrix}$, $C' = \begin{pmatrix} S & -D \\ -D & S \end{pmatrix}$ and $P = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$ of order $2n$. It is not hard to prove that $P^{-1} = \frac{1}{2}\begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$,

$P^{-1}CP = \begin{pmatrix} 0 & D+S \\ D-S & 0 \end{pmatrix}$ and $P^{-1}C'P = -\begin{pmatrix} 0 & D-S \\ D+S & 0 \end{pmatrix}$.

Let $X = \begin{pmatrix} 0 & D+S \\ D-S & 0 \end{pmatrix}$ and $Y = -\begin{pmatrix} 0 & D-S \\ D+S & 0 \end{pmatrix}$. Therefore, $E(C) = E(X) = E(D+S) + E_{SL}(\Gamma)$ and $E(C') = E(Y) = E_{SL}(\Gamma) + E(D+S)$. Thus we have $E(C) = E(C') = E(D+S) + E_{SL}(\Gamma)$. We observe that $C + C' = \begin{pmatrix} 2S & 0 \\ 0 & 2S \end{pmatrix}$ and $E(C+C') = 4E(S)$. By applying Lemma 2.5, we obtain $E(C+C') \leq E(C) + E(C')$. Therefore $4E(S) \leq 2E_{SL}(\Gamma) + 2E(D+S)$. So

$$2E(S) - E(D+S) \leq E_{SL}(\Gamma). \tag{17}$$

Again by Lemma 2.5 to the matrix $C - C'$, we obtain

$$2E(D) - E(D+S) \leq E_{SL}(\Gamma). \tag{18}$$

Substituting by formula (1), $E(D) = \sum_{i=1}^{n} |\lambda_i(D) - \frac{trace(D)}{n}| = \sum_{i=1}^{n} |\frac{4m}{n} - 2d(v_i)|$, we have

$$E_S(\Gamma) + E(D) - E(D+S) \leq E_{SL}(\Gamma).$$

Right inequality: Since $SL(\Gamma) = D(\Gamma) - S(\Gamma)$, by Lemma 2.5 on $SL(\Gamma)$, we obtain

$$E_{SL}(\Gamma) = E(D-S) \leq E(D) + E(-S) = E(D) + E(S) = E(D) + E_S(\Gamma).$$

Therefore, the right inequality holds. If Γ is a $\left(\frac{n-1}{2}\right)$-regular graph of odd order n, then $E(D) = 0$ and $E_{\mathrm{SL}}(\Gamma) = E_{\mathrm{S}}(\Gamma)$. So, the right equality holds. □

We now discuss on the Seidel Laplacian energy of graphs.

Theorem 3.4. *Let Γ be a connected graph of order n with m edges. Then*

$$E_{SL}(\Gamma) = 2 \max_{1 \leq i \leq n-1} \left\{ \sum_{k=1}^{i} \left(\lambda_k^L - \frac{n(n-1) - 4m}{n} \right) \right\}.$$

Proof. Let $\nu\,(1 \leq \nu \leq n-1)$ be the largest integer such that $\lambda_\nu^L \geq \frac{n(n-1)-4m}{n}$, by Lemma 2.7. From the definition, we obtain

$$\begin{aligned} E_{SL}(\Gamma) &= \sum_{k=1}^{n} \left| \lambda_k^L - \frac{n(n-1) - 4m}{n} \right| \\ &= \sum_{k=1}^{\nu} \left(\lambda_k^L - \frac{n(n-1) - 4m}{n} \right) + \sum_{k=\nu+1}^{n} \left(\frac{n(n-1) - 4m}{n} - \lambda_k^L \right) \\ &= 2 \sum_{k=1}^{\nu} \left(\lambda_k^L - \frac{n(n-1) - 4m}{n} \right) \end{aligned}$$

as $\sum_{k=1}^{n}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) = 0$, by (7).

For $i < \nu$, we obtain

$$\sum_{k=1}^{\nu}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) = \sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) + \sum_{k=i+1}^{\nu}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)$$
$$\geq \sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)$$

as $\lambda_k^L \geq \frac{n(n-1)-4m}{n}$ for $i+1 \leq k \leq \nu$.

For $i > \nu$, we obtain

$$\sum_{k=1}^{\nu}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) = \sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) - \sum_{k=\nu+1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)$$
$$> \sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)$$

as $\lambda_k^L < \frac{n(n-1)-4m}{n}$ for $\nu+1 \leq k \leq i$.

For $i = \nu$, we have

$$\sum_{k=1}^{\nu}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right) = \sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right).$$

From the above results, we obtain

$$\max_{1\leq i\leq n-1}\left\{\sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)\right\} = \sum_{k=1}^{\nu}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right).$$

Hence

$$E_{SL}(\Gamma) = 2\max_{1\leq i\leq n-1}\left\{\sum_{k=1}^{i}\left(\lambda_k^L - \frac{n(n-1)-4m}{n}\right)\right\}.$$

□

From the above theorem, we get some lower bounds on E_{SL}.

Corollary 3.5. *Let Γ be a connected graph of order n with m edges. Then*

$$E_{SL}(\Gamma) \geq 2\left(\lambda_1^L - \frac{n(n-1)-4m}{n}\right).$$

Proof. Setting $i = 1$ in Theorem 3.4, we obtain the required result. □

Corollary 3.6. *Let Γ be a connected graph of order n with m edges. Then*

$$E_{SL}(\Gamma) \geq 2\left(\frac{n(n-1)-4m}{n} - \lambda_n^L\right).$$

Proof. Setting $i = n-1$ in Theorem 3.4 with (7), we obtain the required result. $\square$

Corollary 3.7. *Let Γ be a connected graph of order n. Then*

$$E_{SL}(\Gamma) \geq \lambda_1^L - \lambda_n^L.$$

Proof. From Corollaries 3.5 and 3.6, we obtain the required result. $\square$

Theorem 3.8. *Let Γ be a graph of order n with m edges and the first Zagreb index $Z_1(\Gamma)$. Then the lower bound of Seidel Laplacian energy of Γ is as follows:*

$$E_{SL}(\Gamma) \geq n\sqrt{(n-1) + \frac{4Z_1(\Gamma)}{n} - \frac{16m^2}{n^2} - \frac{1}{4}(|\sigma_1| - |\sigma_n|)^2},$$

where $\sigma_k\,(1 \leq k \leq n)$ is defined in (3) such that $|\sigma_1| \geq \cdots \geq |\sigma_n|$.

Proof. Let $b_k = |\sigma_k|$. By assumption $b_1 \geq \cdots \geq b_n$. Setting $c_k = 1,\ 1 \leq k \leq n$, by Ozeki's Inequality and Lemma 2.2, we obtain

$$\sum_{k=1}^{n} |\sigma_k|^2 \sum_{k=1}^{n} 1^2 - (\sum_{k=1}^{n} |\sigma_k|)^2 \leq \frac{n^2}{4}\left(|\sigma_1| - |\sigma_n|\right)^2.$$

This implies that $n\left(4Z_1(\Gamma) + n(n-1) - \frac{16m^2}{n}\right) - E_{SL}^2(\Gamma) \leq \frac{n^2}{4}\left(|\sigma_1| - |\sigma_n|\right)^2$, as desired. $\square$

Theorem 3.9. *Let Γ be a graph of order n with m edges and the first Zagreb index $Z_1(\Gamma)$. Then the lower bound of Seidel Laplacian energy of Γ is as follows:*

$$E_{SL}(\Gamma) \geq \frac{2n\sqrt{\left(n - 1 + \frac{4Z_1(\Gamma)}{n} - \frac{16m^2}{n^2}\right)|\sigma_1||\sigma_n|}}{|\sigma_1| + |\sigma_n|},$$

where $\sigma_k\,(1 \leq k \leq n)$ is defined in (3) such that $|\sigma_1| \geq \cdots \geq |\sigma_n| > 0$.

Proof. Setting $b_k = |\sigma_k|$ $(1 \leq k \leq n)$ and $c_k = 1$ $(1 \leq k \leq n)$, by Pólya-Szegö Inequality, we obtain

$$\sum_{k=1}^{n} |\sigma_k|^2 \sum_{k=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\sigma_1|}{|\sigma_n|}} + \sqrt{\frac{|\sigma_n|}{|\sigma_1|}}\right)^2 \left(\sum_{k=1}^{n} |\sigma_k|\right)^2.$$

Therefore, by $\sum_{k=1}^{n} \sigma_k^2 = n(n-1) + 4Z_1(\Gamma) - \frac{16m^2}{n}$, the result follows. $\square$

We now investigate the relation between the Seidel Laplacian Estrada index and the Seidel Laplacian energy. Since $\sigma_k = \lambda_k^L - \frac{n(n-1)-4m}{n}$ $(1 \le k \le n)$ and $\lambda_1^L \ge \lambda_2^L \ge \cdots \ge \lambda_n^L$, we obtain $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$. We have

$$E_{SL} = E_{SL}(\Gamma) = \sum_{i=1}^{n} |\sigma_i|, \quad \text{where} \quad \sum_{i=1}^{n} \sigma_i = 0.$$

Let n_+ be the number of positive eigenvalues of Seidel Laplacian matrix of graph Γ. Then one can easily see that $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_{n_+} > 0 \ge \sigma_{n_++1} \ge \cdots \ge \sigma_n$. Thus we have

$$E_{SL}(\Gamma) = 2\sum_{i=1}^{n_+} \sigma_i = -2 \sum_{i=n_++1}^{n} \sigma_i. \tag{19}$$

Let $M_k = M_k(\Gamma) = \sum_{i=1}^{n} (\sigma_i)^k$. From the Taylor expansion of e^x, it is easy to see that the Seidel Laplacian Estrada index (6) and $M_K(\Gamma)$ of Γ are related by

$$\frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} = \sum_{k=0}^{\infty} \frac{M_k(\Gamma)}{k!}. \tag{20}$$

It is easy to see that any graph Γ of order n has $SLEE(\Gamma) > n$.

Theorem 3.10. *The Seidel Laplacian Estrada index $SLEE(\Gamma)$ and the Seidel Laplacian energy $E_{SL}(\Gamma)$ satisfy the following inequality:*

$$\left(\frac{e}{2}\, E_{SL}(\Gamma) + (n-n_+)\, e^{-\frac{E_{SL}(\Gamma)}{2\,(n-n_+)}}\right) \le \frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} \le \left(n - 1 + e^{E_{SL}(\Gamma)}\right). \tag{21}$$

Proof. (a) First we prove the left inequality of (21).

Since $f(x) = e^x - e\,x$ is an increasing function on $x \ge 1$ and a decreasing function on $0 \le x \le 1$, we obtain $e^x \ge e\,x$ with equality if and only if $x = 1$. By the arithmetic-geometric-mean inequality, we obtain

$$\sum_{i=n_++1}^{n} e^{\sigma_i} \ge (n-n_+) \left(\prod_{i=n_++1}^{n} e^{\sigma_i}\right)^{\frac{1}{n-n_+}} = (n-n_+) \left(e^{\sum_{i=n_++1}^{n} \sigma_i}\right)^{\frac{1}{n-n_+}} = (n-n_+)\, e^{-\frac{E_{SL}(\Gamma)}{2\,(n-n_+)}}.$$

By (6), we obtain

$$SLEE(\Gamma) = \sum_{i=1}^{n} e^{\lambda_i^L} = \sum_{i=1}^{n} e^{\sigma_i + \frac{n(n-1)-4m}{n}} = e^{\frac{n(n-1)-4m}{n}} \sum_{i=1}^{n} e^{\sigma_i}.$$

Using the above result, we have

$$\begin{aligned}
\frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} &= \sum_{i=1}^{n} e^{\sigma_i} = \sum_{\sigma_i > 0} e^{\sigma_i} + \sum_{\sigma_i \le 0} e^{\sigma_i} \\
&\ge \sum_{i=1}^{n_+} e\sigma_i + (n-n_+)\, e^{-\frac{E_{SL}(\Gamma)}{2\,(n-n_+)}} \\
&= \frac{e}{2}\, E_{SL}(\Gamma) + (n-n_+)\, e^{-\frac{E_{SL}(\Gamma)}{2\,(n-n_+)}}.
\end{aligned}$$

(b) We now prove the right inequality of (21).

From (20) with Corollary 2.10, we obtain

$$\begin{aligned}\frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} &= n+\sum_{i=1}^{n}\sum_{k\geq 1}\frac{(\sigma_i)^k}{k!} \\ &\leq n+\sum_{i=1}^{n}\sum_{k\geq 1}\frac{|\sigma_i|^k}{k!} \qquad (22)\\ &\leq n+\sum_{k\geq 1}\frac{1}{k!}\left(\sum_{i=1}^{n}|\sigma_i|\right)^k = n-1+\sum_{k=0}^{\infty}\frac{(E_{SL}(\Gamma))^k}{k!} = n-1+e^{E_{SL}(\Gamma)}\end{aligned}$$

as $E_{SL}(\Gamma)=\sum_{i=1}^{n}|\sigma_i|$. This completes the proof of the theorem. □

Remark 3.11. We now give an inequality very similar to the right inequality of Theorem 3.10. From (22) with Lemmas 2.9 and 2.2, we obtain

$$\begin{aligned}\frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} \leq n+\sum_{j=1}^{n}\sum_{k\geq 1}\frac{|\sigma_j|^k}{k!} &= n+E_{SL}(\Gamma)+\sum_{j=1}^{n}\sum_{k\geq 2}\frac{|\sigma_j|^k}{k!} \\ &= n+E_{SL}(\Gamma)+\sum_{j=1}^{n}\sum_{k\geq 2}\frac{[(\sigma_j)^2]^{\frac{k}{2}}}{k!} \\ &\leq n+E_{SL}(\Gamma)+\sum_{k\geq 2}\frac{1}{k!}\left[\sum_{j=1}^{n}(\sigma_j)^2\right]^{\frac{k}{2}} \\ &= n+E_{SL}(\Gamma)-1-\sqrt{M}+\sum_{k\geq 0}\frac{(\sqrt{M})^k}{k!},\end{aligned}$$

where $M=n(n-1)+4Z_1(\Gamma)-\frac{16m^2}{n}$. Hence

$$\frac{SLEE(\Gamma)}{e^{\frac{n(n-1)-4m}{n}}} \leq n-1+\left(E_{SL}(\Gamma)-\sqrt{M}\right)+e^{\sqrt{M}}.$$

4 Concluding Remarks and Future Work

There are many graph energy variants in the literature. In this report we discuss on the Seidel energy, Seidel Laplacian energy and Seidel Laplacian Estrada index of graphs. We give some lower and upper bounds on the Seidel Laplacian Estrada index of graphs using a nice inequality. Moreover, we obtain a relation between Seidel energy and Seidel Laplacian energy of graphs. We establish some lower bounds on the Seidel Laplacian energy in terms of different graph parameters. Finally, we present a relation between Seidel Laplacian Estrada index and Seidel Laplacian energy of graphs.

Future research directions on this topic could include deriving critical bounds and identifying corresponding extremal graphs of the Seidel energy, Seidel Laplacian energy and Seidel Laplacian Estrada index for important classes of graphs, such as c-cyclic graphs. Here we pose two related problems.

Problem 1. Characterize the extremal graph (maximal and minimal) with respect to the Seidel Laplacian energy among all connected graphs of order n with clique number ω.

Problem 2. Characterize the extremal graph (maximal and minimal) with respect to the Seidel Laplacian Estrada index among all connected graphs of order n with p pendant vertices.

Acknowledgement. The authors are much grateful to the anonymous referees for their valuable comments on our paper, which have considerably improved the presentation of this paper. K. C. Das is supported by National Research Foundation funded by the Korean government (Grant No. 2021R1F1A1050646).

Authors Contribution. Conceptualization, J. A., K.C.D. and Y. S.; investigation, J. A., K.C.D. and Y. S.; writing—original draft preparation, J. A., K.C.D. and Y. S.; writing—review and editing, J. A., K.C.D. and Y. S.; All authors have read and agreed to the submitted version of the manuscript.

Conflict of interest. The authors declare no conflict of interest.

Data availability. No underlying data was collected or produced in this study.

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