Hermite-Hadamard type inequality for non-convex functions employing the Caputo-Fabrizio Fractional Integral

Faisal Yasin[1], Zeeshan Afzal[1], Muhammad Shaoib Saleem[2], Nazia Jahangir[2], Yilun Shang[3]

[1]Department of Mathematics, The University of Lahore, Lahore, Pakistan; faisalyasin76@gmail.com, zeeshanciitlhr@gmail.com

[2]Department of Mathematics, University of Okara, Okara, Pakistan; shaby455@yahoo.com, naziajahangiruo@gmail.com

[3]Department of Computer and Information Sciences, Northumbria University, Newcastle, UK; yilun.shang@northumbria.ac.uk

Abstract

Novel Hermite-Hadamard type inequalities for p-convex functions utilizing the Caputo-Fabrizio fractional integral are established. Additionally, fresh inequalities incorporating the Caputo-Fabrizio fractional integral operator are examined. Furthermore, applications to special means of the primary outcomes are presented.
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1 Introduction

The theory of fractional calculus may not be considered a new subject as it has a long history; see e.g., [22, 23, 24, 25]. In the recent decade, this subject has attracted the attention of more and more researchers because of the fascinating fractional phenomenon as well as its wide application in physics, mechanics, engineering, and other areas [27, 28]. For example, we can employ this framework to characterize systems, media, and domains exhibiting power law non-locality and power law memory; see [24, 26].

The growing interest in the field of fractional calculus has led to the development of diverse approaches for defining fractional derivatives and fractional integrals. like Baleanu et al. [7], Caponetto [8], Caputo [9], Diethelm [10], Hilfer [11], and many more [1, 2, 3, 4, 5] in non-mathematical journals. The use of derivatives of fractional order has also spread into different fields of science besides mathematics and physics. The Hermite-Hadamard inequality is generalized by means of several fractional integral operators.

Theorem 1.1 *Let $\phi \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a_1, a_2 \in I$ with $a_1 < a_2$. Then the following inequality holds:*

$$\phi\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(x)dx \leq \frac{\phi(a_1)+\phi(a_2)}{2}. \tag{1.1}$$

The introduction of the Caputo-Fabrizio fractional derivative has opened up new avenues in the investigation of fractional differential equations. This derivative is formulated through the convolution of an ordinary derivative.

Another type of fractional derivatives, attributed to Hadamard and first presented in 1892 [12], distinguishes itself from the Riemann-Liouville and Caputo derivatives by incorporating logarithmic functions of arbitrary exponents within the integral kernel.

In this paper, we expanded the application of the fractional Caputo-Fabrizio derivative to the space $C_{\mathbb{R}}[0,1]$. Through this extension, we explore higher-order series-type fractional integro-differential equations. The investigation into the properties of Fractional Caputo-Fabrizio derivatives was conducted recently, as documented in reference [13]. The Caputo-Fabrizio fractional derivatives are discussed in the distributional setting [13]. The physical description of Caputo-Fabrizio fractional derivatives is presented in [14]. The fundamental characteristic of the Caputo-Fabrizio definition lies in its transformation of real powers into integers through Laplace transformation. This property enables the discovery of exact solutions for a variety of problems.

The present paper is summarized as follows. First, we give some preliminary material and basic definitions related to our work. Next, we develop Hermite-Hadamard type inequality via Caputo-Fabrizio fractional integral for p-convex function. Lastly, we establish some fractional integral inequalities via Caputo-Fabrizio fractional integral.

2 Mathematical Preliminaries

Here, we present some preliminary related to our work.

Definition 2.1 *Convexity in classical for a function $\phi : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is defined as*

$$\phi(\theta x_1 + (1-\theta)x_2) \leq \theta\phi(x_1) + (1-\theta)\phi(x_2). \tag{2.1}$$

The definition of p-convexity is given in [17]. Let I be a p-convex set. A function $\phi : I \to \mathbb{R}$ is said to be p-convex function if

$$\phi(\theta x^p(1-\theta)y^p)^{\frac{1}{p}} \leq \phi(x) + (1-\theta)\phi(y)), \tag{2.2}$$

where $x, y \in I$ and $\theta \in [0,1]$.

Let us recall the Caputo-Fabrizio fractional derivative.

Definition 2.2 *(See [1, 15, 16]) Let $\phi \in H^1(a_1, a), a_1 < a_2, \beta \in [0,1]$. The definition of the left fractional derivatives of Caputo-Fabrizio becomes*

$$\left({}_a^{CFC}D^{\beta}\phi\right)(k) = \frac{A(\beta)}{1-\beta}\int_a^k \phi^{'}(x)e^{\frac{-\beta(t-x)^{\beta}}{1-\beta}}dx$$

with associated fractional integral given by

$$\left({}_a^{CF}I^{\beta}\phi\right)(k) = \frac{1-\beta}{A(\beta)}\phi(k) + \frac{\beta}{A(\beta)}\int_a^k \phi(k)dk,$$

where $A(\beta) > 0$ is normalization function satisfying $A(0) = A(1) = 1$. Regarding the right fractional derivative, we have the following

$$\left({}_{a}^{CFC}D^{\beta}\phi\right)(k) = \frac{-A(\beta)}{1-\beta}\int_{k}^{b}\phi^{'}(x)e^{\frac{-\beta(x-t)^{\beta}}{1-\beta}}dx.$$

The corresponding fractional integral is expressed as follows.

$$\left({}_{a}^{CF}I_{b}^{\beta}\phi\right)(k) = \frac{1-\beta}{A(\beta)}\phi(k) + \frac{\beta}{A(\beta)}\int_{k}^{b}\phi(x)dx.$$

The generalized kernel, facilitated by Caputo-Fabrizio fractional integral operators, has been established through the work of Dragomir and Agarwal. In [18], the subsequent lemma is provided to aid in the derivation of enhancements for inequalities of the Hermite-Hadamard type.

Lemma 2.3 *Let $\phi : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o with $a_1 < a_2$. If $\phi^{'} \in L[a_1, a_2]$, then*

$$\begin{aligned}&\frac{\phi(a_1)+\phi(a_2)}{2} - \frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\\&= \frac{b^p - a_1^p}{2p}\int_0^1 M_p^{-1}(a_1,a_2;\theta)(1-2\theta)\phi^{'}(M_p(a_1,a_2;\theta))d\theta,\end{aligned}\tag{2.3}$$

where $M_p^{-1}(a_1, a_2; \theta) = [\theta a_1^p + (1-\theta)a_2^p]^{\frac{1}{p}-1}$.

3 An extension of the Hermite-Hadamard inequality through the utilization of the Caputo-Fabrizio fractional operator

In this section we aim to develop Hermite-Hadamard type inequa lities for p-convex function with respect to Caputo-Fabrizio-fractional integral.

Theorem 3.1 *Let the function $\phi : [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ be a p-convex function on $[a_1, a_2]$ and $\phi \in L_1[a_1, a_2]$, where $p > 0$. If $\beta \in [0, 1]$, then:*

$$\begin{aligned}&\phi\left[\left(\frac{a_1^p + a_2^p}{2}\right)\right]^{\frac{1}{p}}\\&\leq \frac{pA(\beta)}{\beta(a_2^p - a_1^p)}\left[\left({}_{a}^{CF}I^{\beta}\psi\right)(k) + \left({}_{b}^{CF}I^{\beta}\psi\right)(k) - \frac{2(1-\beta)}{A(\beta)}\psi(k)\right]\\&\leq \frac{\phi(a_1)+\phi(a_2)}{2},\end{aligned}\tag{3.1}$$

with $\psi(x) = \dfrac{\phi(x)}{x^{1-p}}$ holds.

Proof. Let $\phi : I \to \mathbb{R}$ be a p-convex function, where $p > 0$. We have

$$\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \le \frac{p}{a_2^p-a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}dx \le \frac{\phi(a_1)+\phi(a_2)}{2}. \tag{3.2}$$

From (3.2) we have

$$\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \le \frac{p}{a_2^p-a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}dx, \tag{3.3}$$

$$\begin{aligned}2\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} &\le \frac{2p}{a_2^p-a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}dx \\ &= \frac{2p}{a_2^p-a_1^p}\left[\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}dx+\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\right]. \end{aligned}\tag{3.4}$$

By multiplying both side of (3.4) with $\frac{\beta(a_2^p-a_1^p)}{2pA(\beta)}$ and adding $\frac{2(1-\beta)}{A(\beta)k^{1-p}}\phi(k)$ we have

$$\begin{aligned}&\frac{2(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta(a_2^p-a_1^p)}{pA(\beta)}\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \\ &\le \frac{2(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta}{A(\beta)}\left[\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}dx+\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\right] \\ &= \left(\frac{(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta}{A(\beta)}\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}dx\right)+\left(\frac{(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta}{A(\beta)}\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\right).\end{aligned}$$

We obtain

$$\begin{aligned}&\frac{2(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta(a_2^p-a_1^p)}{pA(\beta)}\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \\ &\le \left(\frac{(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta}{A(\beta)}\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}dx\right)+\left(\frac{(1-\beta)}{A(\beta)k^{1-p}}\phi(k)+\frac{\beta}{A(\beta)}\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\right).\end{aligned}$$

Substituting $\psi(x)=\frac{\phi(x)}{x^{1-p}}$ in the above equation, we obtain

$$\begin{aligned}&\frac{2(1-\beta)}{A(\beta)}\psi(k)+\frac{\beta(a_2^p-a_1^p)}{pA(\beta)}\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \\ &\le \left(\frac{(1-\beta)}{A(\beta)}\psi(k)+\frac{\beta}{A(\beta)}\int_{a_1}^{k}\psi(x)dx\right)+\left(\frac{(1-\beta)}{A(\beta)}\psi(k)+\frac{\beta}{A(\beta)}\int_{k}^{a_2}\psi(x)dx\right).\end{aligned}\tag{3.5}$$

Through the application of the Caputo-Fabrizio fractional integral definition, we obtain the following result.

$$\begin{aligned}&\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}} \\ &\le \frac{pA(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}_{a}^{CF}I^{\beta}\psi\right)(k)+\left({}_{b}^{CF}I^{\beta}\psi\right)(k)-\frac{2(1-\beta)}{A(\beta)}\right],\end{aligned}\tag{3.6}$$

which is the right-hand side of the theorem.

To prove the right-hand side of the theorem, we use the right-hand side of Hermite-Hadamard type inequality of p-convex function,

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}dx \le \frac{\phi(a_1)+\phi(a_2)}{2}. \tag{3.7}$$

By multiplying both side of (3.7) with $\dfrac{\beta(a_2^p - a_1^p)}{2pA(\beta)}$ and adding $\dfrac{2(1-\beta)}{A(\beta)k^{1-p}}\phi(k)$ we have

$$\begin{aligned}&\left[\frac{1-\beta}{A(\beta)}\psi(k)+\frac{\beta}{A(\beta)}\int_{a_1}^{k}\psi(x)dx\right]+\left[\frac{1-\beta}{A(\beta)}\psi(k)+\frac{\beta}{A(\beta)}\int_{k}^{a_2}\psi(x)dx\right]\\&\le \frac{2(1-\beta)}{A(\beta)}\psi(k)+\frac{\beta(a_2^p - a_1^p)}{2pA(\beta)}\left[\phi(a_1)+\phi(a_2)\right].\end{aligned} \tag{3.8}$$

Now, employing the definition of the Caputo-Fabrizio fractional integral, we derive the right-hand side of the theorem. Namely,

$$\begin{aligned}&\frac{pA(\beta)}{\beta(a_2^p - a_1^p)}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\right)(k)+\left({}^{CF}_{a_2}I^{\beta}\psi\right)(k)-\frac{2(1-\beta)}{A(\beta)}\right]\\&\le \frac{\phi(a_1)+\phi(a_2)}{2},\end{aligned} \tag{3.9}$$

which completes the proof. ■

Remark 3.2 *If we put $p = 1$ in (3.1) we will obtain Hermite-Hadamard type inequality via Caputo-Fabrzio fractional integral for classical convex function; see [19, Theorem 2].*

Theorem 3.3 *Let $\phi, \xi : I \to \mathbb{R}$ be two p-convex function, where $p > 0$. If $\phi\xi \in L([a_1, a_2])$, then following inequality holds*

$$\begin{aligned}&\frac{2pA(\beta)}{\beta(a_2^p - a_1^p)}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\xi\big(k\right)\big)+\left({}^{CF}I^{\beta}_{a_2}\psi\xi\big(k\right)\big)-\frac{2(1-\beta)}{A(\beta)}\xi(k)\psi(k)\right]\\&\le \frac{2}{3}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2),\end{aligned} \tag{3.10}$$

with $\psi(x) = \dfrac{\phi(x)}{x^{1-p}}$ and $M(a_1, a_2) = \phi(a_1)\xi(a_1) + \phi(a_2)\xi(a_2)$,
$N(a_1, a_2) = \phi(a_1)\xi(a_2) + \phi(a_2)\xi(a_1)$, and $k' \in [a_1, a_2]$, and $A(\beta) > 0$ is normalization function.

Proof. Let ϕ, ξ are p-convex functions with $p > 0$, then

$$\begin{aligned}&\phi(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}} \le \theta a_1^p + (1-\theta)a_2^p,\\&\forall a_1, a_2 \in I, \theta \in [0,1]\\&\xi(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}} \le \theta a_1^p + (1-\theta)a_2^p,\\&\forall a_1, a_2 \in I, \theta \in [0,1].\end{aligned} \tag{3.11}$$

Upon multiplication of the inequalities on both sides, we have the following expression.

$$\begin{aligned}&\phi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\xi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\\&\theta^2\phi(a_1)\xi(a_1)+(1-\theta)^2\phi(a_2)\xi(a_2)+\theta(1-\theta)\left[\phi(a_1)\xi(a_2)+\phi(a_2)\xi(a_1)\right].\end{aligned} \tag{3.12}$$

Integrating the (3.12) with respect to θ over $[0,1]$ and making change of variable we get

$$\begin{aligned}&\frac{p}{a_1^p-a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)\xi(x)}{x^{1-p}}dx\\&\leq\frac{2}{3}\left[\phi(a_1)\xi(a_1)+\phi(a_2)\xi(a_2)\right]+\frac{1}{3}\left[\phi(a_1)\xi(a_2)+\phi(a_2)\xi(a_1)\right].\end{aligned} \tag{3.13}$$

This implies in turn that,

$$\frac{p}{a_2^p-a_1^p}\int_{a_1}^{a_2}\frac{\phi(x)}{x^{1-p}}\xi(x)dx\leq\frac{2}{3}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2). \tag{3.14}$$

By multiplying the above equation with $\frac{\beta(a_2^p-a_1^p)}{2pA(\beta)}$ and $\frac{2(1-\beta)\phi(k)\xi(k)}{A(\beta)k^{1-p}}$ and using the definition of Caputo-Fabrizio fractional integral we get

$$\begin{aligned}&\frac{\beta}{A(\beta)}\left[\int_{a_1}^{k}\psi(x)\xi(x)dx+\int_{k}^{a_2}\psi(x)\xi(x)\right]+\frac{2(1-\beta)}{A(\beta)}\psi(x)\xi(x)\\&\frac{\beta(a_2^p-a_1^p)}{2pA(\beta)}\left[\frac{2}{3}M(a_1,a_2)+N(a_1,a_2)\right]+\frac{2(1-\beta)}{A(\beta)}\psi(x)\xi(x).\end{aligned} \tag{3.15}$$

By using the definition of Caputo-Fabrizio fractional integral we get

$$\begin{aligned}&\frac{2pA(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\xi\big(k\big)\right)+\left({}^{CF}I^{\beta}_{a_2}\psi\xi\big(k\big)\right)-\frac{2(1-\beta)}{A(\beta)}\xi(k)\psi(k)\right]\\&\leq\frac{2}{3}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2),\end{aligned} \tag{3.16}$$

This completes the proof. ■

Remark 3.4 *If we put $p=1$ in Theorem 3.3, then we will obtain Theorem 3 in [19].*

Theorem 3.5 *Let $\phi,\xi:I\subset\mathbb{R}\to\mathbb{R}$ be two p-convex function. If $\phi\xi\in L([a_1,a_2])$ with $L([a_1,a_2])$ being the set of integrable functions, then*

$$\begin{aligned}&2\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}}\xi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}}-\frac{p}{a_2^p-a_1^p}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\xi\right)(k)+\left({}^{CF}I^{\beta}_{a_2}\psi\xi\right)(k)\right]\\&+\frac{(1-\beta)\psi(k)\xi)k}{\beta(a_2^p-a_1^p)}\leq\frac{2}{3}M(a_1,a_2)+\frac{4}{3}N(a_1,a_2)\end{aligned} \tag{3.17}$$

holds with $\psi(x)=\frac{\phi(x)}{x^{1-p}}$.

Proof. Let ϕ and ξ be p-convex functions. For $\theta = \dfrac{1}{2}$ we have

$$\phi\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \le \frac{\phi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}+\phi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}}{2} \tag{3.18}$$

and

$$\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}} \le \frac{\xi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}+\xi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}}{2}. \tag{3.19}$$

Multiplying the above inequalities (3.18) and (3.19) on both sides, we have

$$\begin{aligned}
&\phi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}} \le \\
&\frac{1}{4}\left[\phi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}\xi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}+\phi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\xi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\right]+\\
&\frac{1}{4}\left[\phi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}\xi((1-\theta)a_2^p+\theta a_1^p)^{\frac{1}{p}}+\phi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\xi(\theta a_2^p+(1-\theta)a_1^p)^{\frac{1}{p}}\right]\\
&\le \frac{1}{4}\left[\phi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}\xi((1-\theta)a_1^p+\theta a_2^p)^{\frac{1}{p}}+\phi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\xi(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\right]\\
&+\frac{1}{2}\left[\theta(1-\theta)\left[\phi(a_1)\xi(a_1)+\phi(a_2)\xi(a_2)\right]+(1-\theta)^2\phi(a_1)\xi(a_2)+\theta^2\phi(a_2)\xi(a_1)\right].
\end{aligned} \tag{3.20}$$

By integrating the preceding inequality with respect to θ over the interval $[0,1]$ and performing a change of variable, we obtain the following expression.

$$\begin{aligned}
&\phi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\\
&\le \frac{p}{2(a_2^p-a_1^p)}\int_{a_1}^{a_2}\frac{\phi(x)\xi(x)}{x^{1-p}}dx+\frac{1}{3}\left[\phi(a_1)\xi(a_1)+\phi(a_2)\xi(a_2)\right]+\frac{2}{3}\left[\phi(a_1)\xi(a_2)+\phi(a_2)\xi(a_1)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&4\phi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\\
&\le \frac{p}{2(a_2^p-a_1^p)}\int_{a_1}^{a_2}\frac{\phi(x)\xi(x)}{x^{1-p}}dx+\frac{4}{3}M(a_1,a_2)+\frac{8}{3}N(a_1,a_2).
\end{aligned} \tag{3.21}$$

By multiplying the above inequality with $\dfrac{\beta(a_2^p-a_1^p)}{2pA(\beta)}$ and subtracting $\dfrac{2(1-\beta)\phi(k)\xi(k)}{A(\beta)k^{1-p}}$ on both sides we have,

$$\begin{aligned}
&\frac{2\beta(a_2^p-a_1^p)}{pA(\beta)}\phi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}-\frac{\beta}{A(\beta)}\left[\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}\xi(x)dx+\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}\xi(x)dx\right]\\
&-\frac{2(1-\beta)\phi(k)\xi(k)}{k^{1-p}A(\beta)}\\
&\le \frac{\beta(a_2^p-a_1^p)}{2pA(\beta)}\left[\frac{4}{3}M(a_1,a_2)+\frac{8}{3}N(a_1,a_2)\right]-\frac{2(1-\beta)\phi(k)\xi(k)}{k^{1-p}A(\beta)}.
\end{aligned} \tag{3.22}$$

Utilizing the definition of the Caputo-Fabrizio fractional integral, we arrive at the following

$$\begin{aligned}
&\frac{2\beta(a_2^p-a_1^p)}{pA(\beta)}\phi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\xi\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}-\frac{\beta}{A(\beta)}\left[\left({}_{a_1}^{CF}I^{\beta}\psi\xi\right)(k)+\left({}^{CF}I_{a_2}^{\beta}\psi\xi\right)(k)\right]\\
&\le\frac{\beta(a_2^p-a_1^p)}{2pA(\beta)}\left[\frac{4}{3}M(a_1,a_2)+\frac{8}{3}N(a_1,a_2)\right]-\frac{2(1-\beta)\psi(k)\xi(k)}{A(\beta)}. \qquad (3.23)
\end{aligned}$$

Multiplying the inequality with $\dfrac{pA(\beta)}{\beta(a_2^p-a_1^p)}$ on both sides we obtain

$$\begin{aligned}
&\frac{2pA(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}_{a_1}^{CF}I^{\beta}\psi\xi(k)\right)+\left({}^{CF}I_{a_2}^{\beta}\psi\xi(k)\right)-\frac{2(1-\beta)}{A(\beta)}\xi(k)\psi(k)\right]\\
&\le\frac{2}{3}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2). \qquad (3.24)
\end{aligned}$$

This completes the proof. ■

Remark 3.6 *If we put $p=1$ in Theorem 3.5, then we will obtain Theorem 4 in [19].*

4 Some new inequalities with Caputo-Fabrizio fractional operator

We will first generalize the lemma and then proceed to articulate the theorem using the lemma as a foundation.

Lemma 4.1 *Let $\phi: I\to\mathbb{R}$ be a differentiable mappings on $I^o, a_1, a_2\in I$ with $a_1<a_2$. If $\phi'\in L_1[a_1,a_2]$ and $\beta\in[0,1]$, then :*

$$\begin{aligned}
&\frac{a_2^p-a_1^p}{2p}\int_0^1(1-2\theta)(\theta a_1^p+(1-k\theta)a_2^p)^{\frac{1}{p}-1}\phi'(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}d\theta-\frac{2(1-\beta)\psi(k)}{\beta(a_2^p-a_1^p)}\\
&=\frac{\phi(a_1)+\phi(a_2)}{2}-\frac{pA(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}_{a_1}^{CF}I^{\beta}\psi\right)(k)+\left({}^{CF}I_{a_2}^{\beta}\psi\right)(k)\right], \qquad (4.1)
\end{aligned}$$

with $\psi(x)=\dfrac{\phi(x)}{x^{1-p}}$ holds, where $p>0$, $k\in[a_1,a_2]$ and $B(\beta)>0$ is a normalization function.

Proof. Let $\phi: I\to\mathbb{R}$ be a p-convex function with $p>0$. It is easy to see that from Lemma 2.3

$$\begin{aligned}
&\int_0^1(1-2\theta)(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}-1}\phi'(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}d\theta\\
=\quad&\frac{p}{a_2^p-a_1^p}\left(\frac{\phi(a_1)+\phi(a_2)}{2}\right)-\frac{2p^2}{a_2^p-a_1^p}\left(\int_{a_1}^{k}\frac{\phi(x)}{x^{1-p}}dx+\int_{k}^{a_2}\frac{\phi(x)}{x^{1-p}}dx\right). \qquad (4.2)
\end{aligned}$$

Multiplying the above inequality with $\frac{\beta(a_2^p - a_1^p)^2}{2p^2 A(\beta)}$ and subtracting $\frac{2(1-\beta)\phi(k)}{A(\beta)k^{1-p}}$,

$$
\begin{aligned}
&\frac{\beta(a_2^p - a_1^p)^2}{2p^2 A(\beta)} \int_0^1 (1-2\theta)(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}-1} \phi'(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}} d\theta - \frac{2(1-\beta)\phi(k)}{A(\beta)k^{1-p}} \\
&= \frac{a_2^p - a_1^p}{2pA(\beta)} (\phi(a_1) + \phi(a_2)) - \frac{\beta}{A(\beta)} \int_{a_1}^{a_2} \frac{\phi(x)}{x^{1-p}} dx - \frac{2(1-\beta)\phi(k)}{A(\beta)k^{1-p}} \\
&= \frac{a_2^p - a_1^p}{2pA(\beta)} (\phi(a_1) + \phi(a_2)) - \left(\frac{(1-\beta)\phi(k)}{A(\beta)k^{1-p}} + \frac{\beta}{A(\beta)} \int_{a_1}^{k} \frac{\phi(x)}{x^{1-p}} dx \right) \\
&- \left(\frac{(1-\beta)\phi(k)}{A(\beta)k^{1-p}} + \frac{\beta}{A(\beta)} \int_{k}^{a_2} \frac{\phi(x)}{x^{1-p}} dx \right) \\
&= \frac{a_2^p - a_1^p}{2pA(\beta)} (\phi(a_1) + \phi(a_2)) - \left[\left({}_{a_1}^{CF} I^{\beta} \psi \right)(k) + \left({}^{CF} I_{a_2}^{\beta} \psi \right)(k) \right]. \qquad (4.3)
\end{aligned}
$$

By solving the above equation we derive

$$
\begin{aligned}
&\frac{a_2^p - a_1^p}{2p} \int_0^1 (1-2\theta)(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}-1} \phi'(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}} d\theta - \frac{2(1-\beta)\psi(k)}{\beta(a_2^p - a_1^p)} \\
&= \frac{\phi(a_1) + \phi(a_2)}{2} - \frac{pA(\beta)}{\beta(a_2^p - a_1^p)} \left[\left({}_{a_1}^{CF} I^{\beta} \psi \right)(k) + \left({}^{CF} I_{a_2}^{\beta} \psi \right)(k) \right]. \qquad (4.4)
\end{aligned}
$$

The proof is then complete. ∎

Remark 4.2 *If we put $p = 1$ in Theorem 4.1, then we will obtain Lemma 2 in [19].*

Theorem 4.3 *Let $\phi : I \to R$ be a different iable mapping on I^o and $|\phi'|$ be p-convex on $[a_1, a_2]$ where $a_1, a_2 \in I$ with $a_1 < a_2, p > 0$. If $\phi' \in L_1[a_1, a_2]$ and $\beta \in [0, 1]$, then:*

$$
\begin{aligned}
&\left| \frac{\phi(a_1) + \phi(a_2)}{2} + \frac{2(1-\beta)\psi(k)}{\beta(a_2^p - a_1^p)} - \frac{pA(\beta)}{\beta(a_2^p - a_1^p)} \left[\left({}_{a_1}^{CF} I^{\beta} \psi \right)(k) + \left({}^{CF} I_{a_2}^{\beta} \psi \right)(k) \right] \right| \\
&\leq \frac{a_2^p - a_1^p}{2p} \left[C_1(a_1, a_2) \left| \phi'(a_1) \right| + C_2(a_1, a_2) \left| \phi'(a_2) \right| \right] \qquad (4.5)
\end{aligned}
$$

holds, where $C_1(a_1, a_2) = \int_0^{\frac{1}{2}} \frac{(1-2\theta)\theta}{[(\theta a_1^p + (1-\theta)a_2^p)]^{\frac{1}{p}-1}} dt + \int_{\frac{1}{2}}^1 \frac{(2\theta-1)\theta}{[(\theta a_1^p + (1-\theta)a_2^p)]^{\frac{1}{p}-1}} d\theta,$
and
$C_2(a_1, a_2) = \int_0^{\frac{1}{2}} \frac{(1-2\theta)(1-\theta)}{[(\theta a_1^p + (1-\theta)a_2^p)]^{\frac{1}{p}-1}} dt + \int_{\frac{1}{2}}^1 \frac{(2\theta-1)(1-\theta)}{[(\theta a_1^p + (1-\theta)a_2^p)]^{\frac{1}{p}-1}} d\theta.$

Proof. Let $\phi : I \to \mathbb{R}$ be p-convex function. From Lemma 4.1 we have

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} + \frac{2(1-\beta)\psi(k)}{\beta(a_2^p - a_1^p)} - \frac{A(\beta)}{\beta(a_2^p - a_1^p)}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\right)(k) + \left({}^{CF}I^{\beta}_{a_2}\psi\right)(k)\right]\right| \\
&\le \frac{a_2^p - a_1^p}{2p}\int_0^1 \frac{\left(|1-2\theta|\left|\phi'(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}}\right|\right)}{\left|(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}-1}\right|}d\theta \\
&\le \frac{a_2^p - a_1^p}{2p}\int_0^1 \frac{\left(|1-2\theta|\left(\theta\left|\phi'(a_1)\right| + (1-\theta)\left|\phi'(a_2)\right|\right)\right)}{\left|(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}-1}\right|}d\theta \\
&= \frac{a_2^p - a_1^p}{2p}\left[\int_0^{\frac{1}{2}} \frac{(1-2\theta)\theta\left|\phi'(a_1)\right|}{\left[(\theta a_1^p + (1-\theta)a_2^p)\right]^{\frac{1}{p}-1}}d\theta + \int_{\frac{1}{2}}^1 \frac{(2\theta-1)\theta\left|\phi'(a_1)\right|}{\left[(\theta a_1^p + (1-\theta)a_2^p)\right]^{\frac{1}{p}-1}}d\theta\right] \\
&+ \frac{a_2^p - a_1^p}{2p}\left[\int_0^{\frac{1}{2}} \frac{(1-2\theta)(1-\theta)\left|\phi'(a_2)\right|}{\left[(\theta a_1^p + (1-\theta)a_2^p)\right]^{\frac{1}{p}-1}}d\theta + \int_{\frac{1}{2}}^1 \frac{(2\theta-1)(1-\theta)\left|\phi'(a_2)\right|}{\left[(\theta a_1^p + (1-\theta)a_2^p)\right]^{\frac{1}{p}-1}}dt\theta\right] \\
&= \frac{a_2^p - a_1^p}{2p}\left[C_1(a_1,a_2)\left|\phi'(a_1)\right| + C_2(a_1,a_2)\left|\phi'(b)\right|\right]. \qquad (4.6)
\end{aligned}$$

The proof is complete. ■

Remark 4.4 *If we put $p = 1$ in Theorem 4.3, then we will obtain Theorem 5 in [19].*

In [20], Iscan gave a refinement of Holder integral inequality as follows.

Theorem 4.5 (Holder Iscan integral inequality [20]) . *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ϕ and ψ are real functions defined on interval $[a_1, a_2]$ and if $|\phi|^q$ and $|\psi|^q$ are integrable functions on $[a_1, a_2]$ then*

$$\begin{aligned}
\int_{a_1}^{a_2} |\phi(x)\psi(x)|\,dx \le \frac{1}{a_2 - a_1}&\left\{\left(\int_{a_1}^{a_2}(a_2 - x)\,|\phi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}(a_2 - x)\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right\} + \\
\frac{1}{a_2 - a_1}&\left\{\left(\int_{a_1}^{a_2}(x - a_1)\,|\phi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}(x - a_1)\,|\psi(x)|^q\,dx\right)\right\}^{\frac{1}{q}}. \qquad (4.7)
\end{aligned}$$

Theorem 4.6 *Let $\phi : I \to \mathbb{R}$ be differentiable positive mappings on I^o and $|\phi'|^q$ is p-convex function on $[a_1, a_2]$ where $p > 1, p^{-1} + q^{-1} = 1, a_1, a_2 \in I$ with $a_1 < a_2$. if $\phi' \in L_1[a_1, a_2]$ and $\beta \in [0, 1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2} + \frac{2(1-\beta)\psi(k)}{\beta(a_2^p - a_1^p)} - \frac{pA(\beta)}{\beta(a_2^p - a_1^p)}\left[\left({}^{CF}_{a_1}I^{\beta}\psi\right)(k) + \left({}^{CF}I^{\beta}_{a_2}\psi\right)(k)\right]\right| \\
&\le \frac{a_2^p - a_1^p}{2p}\left(C_3(a_1,a_2)\right)^{\frac{1}{p}}\left(\frac{|\phi'(a_1)|^q + |\phi'(a_1)|^q}{2}\right)^{\frac{1}{q}}, \qquad (4.8)
\end{aligned}$$

where

$$C_3(a_1,a_2) = \left(\int_0^1 \frac{|1-2\theta|^p}{\left|(\theta a_1^p + (1-\theta)a_2^p)^{\frac{1}{p}-1}\right|^p}\right)^{\frac{1}{p}}$$

and $k \in [a_1, a_2]$ and $A(\beta) > 0$ is a normalization function.

Proof. Using the same method as that of the above theorem and Lemma 4.1, the Holder inequality and p-convexity of $|\phi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\phi(a_1)+\phi(a_2)}{2}+\frac{2(1-\beta)\psi(k)}{\beta(a_2^p-a_1^p)}-\frac{A(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}_{a_1}^{CF}I^{\beta}\psi\right)(k)+\left({}^{CF}I_{a_2}^{\beta}\psi\right)(k)\right]\right| \\
&\leq \frac{a_2^p-a_1^p}{2p}\int_0^1 \frac{\left(|1-2\theta|\left|\phi'(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\right|\right)}{\left|(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}-1}\right|}d\theta \\
&\leq \frac{a_2^p-a_1^p}{2p}\left(\int_0^1 \frac{|1-2\theta|^p}{\left|(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}-1}\right|^p}\right)^{\frac{1}{p}}\left(\int_0^1\left|\phi'(\theta a_1^p+(1-\theta)a_2^p)^{\frac{1}{p}}\right|^q\right)^{\frac{1}{q}} \\
&= \frac{a_2^p-a_1^p}{2p}\left(C_3(a_1,a_2)\right)^{\frac{1}{p}}\left(\frac{|\phi'(a_1)|^q+|\phi'(a_1)|^q}{2}\right)^{\frac{1}{q}}. \qquad (4.9)
\end{aligned}$$

We complete the proof. ■

Remark 4.7 *If we insert $p=1$ in Theorem 4.6, then we reproduce Theorem 6 in [19].*

5 Applications to special means

Means play a significant role in both applied and pure mathematics, particularly in numerical approximation, where they are utilized extensively. In literature, they are ordered in the following way

$$H \leq G \leq L \leq I \leq A \qquad (5.1)$$

The arithmetic mean of a_1, a_2 with $a_1 \neq a_2$: is defined as

$$A = A(a_1,a_2) = \frac{a_1+a_2}{2}, a_1, a_2 \in \mathbb{R}. \qquad (5.2)$$

The generalized logarithmic mean is defined as:

$$L = L_r^r(a_1,a_2) = \frac{a_2^{r+1}-a_1^{r+1}}{(r+1)(a_2-a_1)}, r \in \mathbb{R}\setminus\{-1\}, a_1, a_2 \in \mathbb{R}, a_1 \neq a_2. \qquad (5.3)$$

$$L_p(a_1,a_2) = \left[\left(\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)}\right)^{\frac{1}{p}}\right], p \in \mathbb{R}\setminus\{-1\}. \qquad (5.4)$$

Proposition 5.1 *Assume $a_1, a_2 > 0$ and $a_1 < a_2$. Then*

$$M_p(a_1,a_2) \leq L_{p-1}^{1-p}(a_1,a_2)\cdot L_p^p(a_1,a_2) \leq A \qquad (5.5)$$

holds for $p \in (-\infty, 1)\backslash\{0\}$, where

$$M_p(a_1,a_2) = \left[\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}}\right].$$

Proof. From Theorem 3.1 we have

$$\begin{aligned}
&\phi\left[\left(\frac{a_1^p+a_2^p}{2}\right)\right]^{\frac{1}{p}}\\
&\leq \frac{pA(\beta)}{\beta(a_2^p-a_1^p)}\left[\left({}_{a_1}^{CF}I^{\beta}\psi\right)(k)+\left({}_{a_2}^{CF}I^{\beta}\psi\right)(k)-\frac{2(1-\beta)}{B(\beta)}\psi(k)\right]\\
&\leq \frac{\phi(a_1)+\phi(a_2)}{2}, \qquad (5.6)
\end{aligned}$$

where $\psi(x)=\dfrac{\phi(x)}{x^{1-p}}$ holds.

Setting $\phi(x)=x, \beta=1$ and $A(\beta)=B(1)=1$ in the above theorem, we obtain

$$\begin{aligned}
\left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{1}{p}} &\leq \frac{p(a_2-a_1)}{a_2^p-a_1^p}\left[\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)}\right]\\
&\leq \frac{a_1+a_2}{2}. \qquad (5.7)
\end{aligned}$$

We have the following

$$L_p(a_1,a_2)=\left(\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)}\right)^{\frac{1}{p}}.$$

Replacing p with $p-1$, we have

$$L_{p-1}(a_1,a_2)=\left(\frac{a_2^p-a_1^p}{p(a_2-a_1)}\right)^{\frac{1}{p-1}}.$$

This implies that

$$L_{p-1}^{p-1}(a_1,a_2)=\frac{a_2^p-a_1^p}{p(a_2-a_1)}.$$

By using these means we finally obtain

$$M_p(a_1,a_2)\leq L_{p-1}^{1-p}(a_1,a_2)\cdot L_p^p(a_1,a_2)\leq A. \qquad (5.8)$$

■

By using Theorem 4.3 in Section 4, we have some further results regarding the special means.

Proposition 5.2 *Let $a_1,a_2\in\mathbb{R}^+, a_1<a_2$, then*

$$\left|A(a_1^2,a_2^2)-pL_{p+1}^{p+1}(a_1^p,a_2^p)\right|\leq\frac{a_2^p-a_1^p}{p}\left[\left|a_1\right|C_1(a_1,a_2)+\left|a_2\right|C_2(a_1,a_2)\right]. \qquad (5.9)$$

Proof. In Theorem 4.3, if we set $\phi(x)=x^2$ with $\beta=1$ and $A(\beta)=A(1)=1$, then we obtain the result immediately. ■

Remark 5.3 *If we insert $p=1$ in Proposition 5.2 then we will obtain Proposition 1 in [19].*

Proposition 5.4 *Let $a_1, a_2 \in \mathbb{R}^+$ and $a_1 < a_2$. Then*

$$\left|A(a_1^n, a_2^n) - pL_{n-1+p}^{n-1+p}(a_1^p, a_2^p)\right| \leq \frac{a_2^p - a_1^p}{2p}\left[\left|a_1^{n-1}\right| C_1(a_1, a_2) + \left|a_2^{n-1}\right| C_2(a_1, a_2)\right]. \quad (5.10)$$

Proof. In Theorem 4.3, if we set $\phi(x) = x^n, n \in \mathbb{N}$ with $\beta = 1$ and $B(\beta) = B(1) = 1$, then we obtain the result immediately. ■

Remark 5.5 *If we insert $p = 1$ in Proposition 5.4, then we will obtain Proposition 2 in [19].*

Author Contribute

All authors contributed equally to this paper.

Data Amiability

All data required for this paper is included within this paper.

Competing interests

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