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Resilient leaderless and leader-follower consensus over random networks through ℓ-hop communication

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Abstract

This paper studies the resilient consensus problems under ℓ-hop communication over directed random networks. We develop novel multi-hop protocols to tackle both leaderless and leader-follower consensus with a single leader when the multiagent system is under both attack from Byzantine agents and failure of communication edges. Our unified framework features a multiple-input multiple-output system design, and generalizes the weighted mean subsequence reduced algorithms, in which extremal values in the ℓ-hop neighborhood of a cooperative agent are censored based on the notion of minimum message cover. We establish conditions on dynamical network structures, under which secure consensus with vector states can be achieved in the sense of almost sure convergence.

Keywords: Multiagent system, multi-hop communication, random network, resilient consensus, vector-valued state.

1. Introduction

Coordinated control of distributed multi-agent systems has gained prominence over the past few decades. In such systems, reaching consensus in the states or outputs of agents is a fundamental problem, where local information exchange among agents is implemented through distributed consensus protocols [1]. Generally, consensus problems fall into two categories, namely the leaderless consensus and the leader-follower consensus. The former deals with systems without leaders and convergence is the main goal of the group of agents [1, 2, 3], whereas the latter requires all follower agents to track the leader's behavior asymptotically [4, 5, 6]. Various consensus strategies have been reported in the literature across the fields like control engineering, computer science, physical and social sciences [7, 8].

A challenging and crucial problem in consensus control is to design appropriate protocols to drive the agents securely to consensus in potential adversarial environments given the growing concerns regarding cyber security [9, 10]. Potential misbehaving agents can launch attacks on nodes and edges of the underlying communication network inhibiting the consensus ability of the cooperative agents. A graph-theoretical notion of network robustness has been introduced in [9] to facilitate the resilient consensus in the presence of malicious and Byzantine agents. On the basis of the weighted mean subsequence reduced (W-MSR) protocol, it is shown that $(2r+1)$-robustness is sufficient for resilient consensus if each cooperative agent has at most r malicious agents in its 1-hop neighborhood. The W-MSR algorithm is powerful as it effectively filters some largest and smallest values in each iteration step and has been generalized to tackle for example higher-order dynamics [11], observer-based control [12] and constrained systems [13].

Although network robustness is known to be essential for seeking resilient consensus under misbehaving agents [9], the network topological condition can be relaxed in some situations. Trusted nodes are introduced in [14], which are immune to malicious attacks. In [15], the author reduces the network connectivity condition by considering asymmetric interaction on the basis of a multiplex network framework. A multi-hop communication technique has been used in [16] to relay values in the network and an approximate Byzantine problem has been solved by using the concept of message cover. The idea of multi-hop communication is widely adopted in network security and computer communication [17, 18]. Recently, the original network robustness notion has been generalized to multi-hop network robustness in [19, 20, 21] and relevant issues such as time delay and asynchronous update have been factored in the W-MSR algorithm. However, the work [21] focuses on malicious agents and the breakdown of communication links is solely due to the W-MSR algorithm, which ignores the malicious information received from neighbors. In the multi-hop networks, the failure of links is of significant relevance posing a realistic harm to the collective behavior of the systems, which is a major motivation of our current work.

In this paper, we develop a general multi-hop resilient leaderless consensus framework that accommodates both misbehaving agents and edges. Unlike the previous consensus work [16, 20, 21] on multi-hop communication, the framework and the developed techniques in this paper are suitable for the threats modeled by Byzantine agents, which are often regarded as the worst scenario of malicious attacks. Different from the work [16, 20, 21], where only deterministic topology is considered, our results are built upon a rather general random network topology. Moreover, our system is based on discrete-time iteration, which is different from the continuous-time framework proposed in [19]. Vector states are used in this paper, which are not available in the previous scalar-valued protocols. We assume edges in the network can be removed randomly, and stochastic consensus is established by leveraging martingale convergence. It is worth mentioning that although some different stochastic resilient consensus strategies have been reported in the literature [22, 23, 24, 25], most of them are restrictive in terms of the control functions (typically weighted-average-like protocols) and only one-hop communication is investigated. The design of our consensus protocol favorably allows both linear and nonlinear control functions and accommodates multiple-input multiple-output systems.

Furthermore, we extend our framework to the multi-hop resilient leader-follower consensus scenario with a single leader. It is known that the W-MSR induced algorithms guarantee a consensus state within the convex hull of the initial states of cooperative agents. Since they do not differentiate between a leader and a misbehaving agent, agents cannot track a leader if its value is outside the convex hull. Due to the challenge, there has been very limited work in this direction. In [26], a sufficiently large set of leaders holding the same static reference value is introduced to ensure resilient reference tracking by using a sliding window based W-MSR approach. Resilient leader-follower consensus with one leader is studied in [27] for a continuous-time multiagent system, where the leader is a trusted node with the bound of its control input known by all cooperative followers. Under a similar setting, resilient tracking is also solved in [28] by introducing additional observer dynamics. An event-triggered method is proposed to solve the approximate Byzantine tracking problem in [30] for continuous-time dynamical agents. Compared with the above works [26, 27, 28, 30], the proposed leader-follower control method in this paper is more general and accommodates both node and edge faults. We note that the existing control strategies are not applicable to resilient leader-follower consensus under multi-hop communication.

We mention that consensus over random networks through multi-hop communication has real-world relevance. By using other nodes as relays, multi-hop routing is often more energy efficient than 1-hop communication in wireless sensor networks and mobile ad hoc networks [31]. In such networks, the random network setting captures the unreliable communication links due to numerous reasons like handover failures, signal attenuation, poor coverage, background noise, external block, multipath fading, etc. [32, 33]. A typical application scenario could be the consensus in sensor networks, which often experience random data packet dropouts due to some of the reasons above.

The main contribution of the work is summarized as follows. First, we present a multi-hop resilient consensus framework accommodating both misbehaving agents and random link failures. The framework is flexible and general in terms of agent states (vector-based), attack mode (Byzantine and multi-hop capability), control function (general forms beyond weighted average), and network topology (random directed graph without independence requirement). Second, we extend this framework to address multi-hop resilient leader-follower consensus problems with a single leader. Our framework allows more general environments (in terms of random environments and multi-hop communication) and at the same time eases restrictions in the existing works, where either a sufficiently large number of leaders or a trusted leader is required.

The rest of the paper is organized as follows. Section 2 presents the problem formulation. Sections 3 and 4 are devoted to leaderless and leader-follower consensus analyses, respectively. Simulations are shown in Section 5 and the conclusion is drawn in Section 6.

2. Problem formulation

2.1. Graph theory

We consider directed graphs (digraphs) in this paper and let $\mathbb{N}$ and $\mathbb{R}$ be the sets of non-negative and real numbers, respectively. For $t \in \mathbb{N}$, the communication topology among agents or nodes is described by a digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of agents, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A}(t) = (a_{ij}(t)) \in \mathbb{R}^{N\times N}$ is the random adjacency matrix. Here, $a_{ij}(t) > 0$ if $(j, i) \in \mathcal{E}(t)$, i.e., agent j can send information to agent i, and $a_{ij}(t) = 0$ otherwise. Suppose that $a_{ij}(t) > 0$ with probability $q_{ij}(t)$ and $a_{ij}(t) = 0$ with probability $1 - q_{ij}(t)$. We note that no independence regarding the edges appearance is assumed here, and $\mathcal{G}(t)$ is a general time-varying weighted digraph.

For two agents $i, j \in \mathcal{V}$, denote by $d_{ij}(t)$ the length of a shortest directed path in $\mathcal{G}(t)$ from j to i, where j is called the source and i the sink. Other agents (if any) in this path are internal agents. A path $(i_0, i_1, \cdots, i_k)$ from i_0 to i_k of length k is called a k-hop path. If $d_{ij}(t) = k$, denote by $\mathcal{P}_{ij}(t)$ the set of all k-hop paths from j to i in $\mathcal{G}(t)$. For example, an edge $(j, i) \in \mathcal{E}(t)$ gives rise to $d_{ij}(t) = 1$ and $\mathcal{P}_{ij}(t) = \{(j, i)\}$. For $k \in \mathbb{N}$, we refer to $\mathcal{N}_i^{k-}(t) = \{j \in \mathcal{V} : d_{ij}(t) = k\}$ and $\mathcal{N}_i^{k+}(t) = \{j \in \mathcal{V} : d_{ji}(t) = k\}$ as the k-hop in-neighborhood and the k-hop out-neighborhood of agent i, respectively. In particular, $\mathcal{N}_i^{0-}(t) = \mathcal{N}_i^{0+}(t) = \{i\}$. Moreover, for $\ell \in \mathbb{N}$, define the $(\leq \ell)$-hop in-neighborhood and the $(\leq \ell)$-hop out-neighborhood to be $\mathcal{N}_i^{\leq\ell-}(t) = \cup_{k=0}^{\ell}\mathcal{N}_i^{k-}(t)$ and $\mathcal{N}_i^{\leq\ell+}(t) = \cup_{k=0}^{\ell}\mathcal{N}_i^{k+}(t)$, respectively. Moreover, $|\mathcal{N}_i^{1-}(t)|$ and $|\mathcal{N}_i^{1+}(t)|$, respectively, are the in-degree and out-degree of node i in $\mathcal{G}(t)$, which are consistent with the conventional definition in graph theory.

In the leaderless scenario, the agents in $\mathcal{V} = \mathcal{C} \cup \mathcal{B}$ consists of two groups of agents: $\mathcal{C}$ containing the cooperative agents and $\mathcal{B}$ containing the Byzantine agents. Whereas in the leader-follower scenario, we have a similar partition $\mathcal{V} = \{l\} \cup \mathcal{C} \cup \mathcal{B}$, where l is the leader agent. The rules for these agents will be specified in the subsections below. Given the communication topology $\mathcal{G}(t)$, we define the following reachability and robustness under ℓ-hop communication slightly different from [20], where a robustness concept is defined to suit the more restrictive threat model therein (see also the comment at the end of Section 2.1).

Definition 1. Given $t, r, \ell \in \mathbb{N}$, a set of agents $\mathcal{S} \subseteq \mathcal{V}$ is called r-reachable under ℓ-hop communication w.r.t $\mathcal{B}$ if there is an agent $i \in \mathcal{S}$ such that it has no less than r independent ($\leq \ell$)-hop paths from sources outside S and no internal agents of these paths are in $\mathcal{B}$. Here, independence means no agents other than the sink i can appear more than once in these paths.

Definition 2. Given $t, r, \ell \in \mathbb{N}$, the graph $\mathcal{G}(t)$ is called r-robust under ℓ-hop communication w.r.t. $\mathcal{B}$ if for any pair of nonempty disjoint sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$, at least one of them is r-reachable under ℓ-hop communication w.r.t $\mathcal{B}$.

Definition 1 intuitively entails the existence of at least r sources in $\mathcal{N}_i^{\leq \ell -}(t) \backslash \mathcal{S}$, which emit independent paths ending at i and no internal agents of these paths are in $\mathcal{B}$. It is easy to see that the above definitions are equivalent to the original r-reachability and r-robustness concepts [9] when $\ell = 1$. In the leader-follower scenario, we introduce the following definition.

Definition 3. Given $t, r, \ell \in \mathbb{N}$, the graph $\mathcal{G}(t)$ is called leader-follower r-robust under ℓ-hop communication w.r.t. $\mathcal{B}$ if $|\mathcal{N}_l^{\leq \ell +}(t)| \geq r + 1$ and for any set $\mathcal{S} \subseteq \mathcal{V} \backslash \mathcal{N}_l^{\leq \ell +}(t)$, $\mathcal{S}$ is r-reachable under ℓ-hop communication w.r.t $\mathcal{B}$.

We define the locality of the Byzantine agents under ℓ-hop communication as follows. It naturally extends the locality concept for $\ell = 1$ [9].

Definition 4. Given $t, r, \ell \in \mathbb{N}$, a set $\mathcal{S} \subseteq \mathcal{V}$ is called r-local under ℓ-hop communication if $|\mathcal{S} \cap \mathcal{N}_i^{\leq \ell -}(t)| \leq r$ for any $i \in \mathcal{V} \backslash \mathcal{S}$.

Note that the work [20] only considers the so-called r-total model [9], meaning that there are at most r malicious agents in the entire network. Apparently, if the set $\mathcal{S}$ contains at most r agents, it is also r-local under ℓ-hop communication for any $\ell \in \mathbb{N}$.

2.2. System description

In the above network setup, the node set $\mathcal{V}$ is categorized into cooperative agents in C, Byzantine agents in $\mathcal{B}$ as well as the leader l in the leader-follower case. The Byzantine agents are notoriously deleterious as they have full knowledge regarding the network and can follow arbitrary strategies to undermine the system behavior [9, 10, 12, 13]. Their number and identity are not known to any cooperative agents. The cooperative agents, on the other hand, are those we have control over. The objective of resilient consensus problems is to design appropriate distributed protocols for cooperative agents to seek agreement between them irrespective of the disruption of potential misbehaving agents. The leader l is naturally assumed to be a cooperative agent.

For any agent $i \in \mathcal{V}$, let $x_i(t) \in \mathbb{R}^n$ be the state vector of agent i at time t. In the multi-hop communication network with a given parameter $\ell \in \mathbb{N}$ (i.e., the maximum communication capability), an agent i can transmit a message to any agent j in its ($\leq \ell$)-hop out-neighborhood $\mathcal{N}_i^{\leq \ell +}(t)$ along each path in $\mathcal{P}_{ji}(t)$ synchronously if all edges on that path are present. Along these paths, a cooperative internal agent will relay the message while a Byzantine agent may alter the information arbitrarily. Let $x_i^p(t) \in \mathbb{R}^n$ be the received message information of agent j sent from agent i along a path $p \in \mathcal{P}_{ji}(t)$. Along with $x_i^p(t) \in \mathbb{R}^n$, the agent j also receives the path information p. As in [16, 20], we assume that only the message information but not the path information can be forged by an internal agent. Clearly, if all internal agents in p are cooperative, $x_i^p(t) = x_i(t)$, i.e., agent j receives the originally sent state vector of source i along this path p.

At each time step $t \in \mathbb{N}$, a cooperative agent $i \in C$ executes the following three steps. 1) Send step, where agent i sends $x_i(t)$ to every agent $j \in \mathcal{N}_i^{\leq \ell +}(t)$ along each path in $\mathcal{P}_{ji}(t)$; 2) Collect step, where agent i collects information $x_j^p(t)$ for all $j \in \mathcal{N}_i^{\leq \ell -}(t)$ and $p \in \mathcal{P}_{ij}(t)$; 3) Update step, where the state vector obeys

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \tag{1}$$

with $u_i(t) \in \mathbb{R}$ being the control input of agent i to be designed later. For a malicious agent $i \in \mathcal{B}$, it could update its state arbitrarily as mentioned above. Here, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are system matrices and we assume the pair (A, B) is controllable. Define an invertible matrix $\Theta \in \mathbb{R}^{n\times n}$ as follows

$$\Theta = (B, AB, A^2B, \cdots, A^{n-1}B) \cdot \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}, \tag{2}$$

where the characteristic polynomial of A is $\det(\mu I_n - A) = \sum_{k=0}^{n} a_k \mu^{n-k}$ with $a_0 = 1$ and $I_n \in \mathbb{R}^{n\times n}$ being the identity matrix. The system (1) can be converted to the canonical form

$$y_i(t+1) = \hat{A}y_i(t) + \hat{B}u_i(t) \tag{3}$$

by using the similarity transform $x_i(t) = \Theta y_i(t)$ [34], where $\hat{A} = \Theta^{-1}A\Theta = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$ and $\hat{B} = \Theta^{-1}B = (1, 0, 0, \cdots, 0)^{\mathrm{T}} \in \mathbb{R}^n$, where T is the matrix transpose. The transformation from x_i to y_i through (3) is a standard linear algebra technique to normalize the system matrix and simplify the interrelationship between components in the state vector. From (3), we observe that the first component of y_i becomes a 'principal' component, which is the only component receives the control input u_i. All the other components of y_i are just a time shift. This facilitates the design of the ℓ-hop resilient consensus strategy (see Section 2.3 and (8) below) as we can focus on the first component.

A Byzantine agent $i \in \mathcal{B}$ can also be viewed as following the above three steps except that 1) it can send any different values to different agents in $\mathcal{N}_i^{\leq \ell +}(t)$ in Send step; 2) it can collect information from any agent in the Collect step; and 3) it can use any control input $u_i(t)$ of (1) in Update step.

Write $y_i(t) = (y_{i,n-1}(t), y_{i,n-2}(t), \cdots, y_{i,0}(t))^{\mathrm{T}} \in \mathbb{R}^n$ for $i \in \mathcal{V}$. We define a scalar

$$v_i(t) = (b_0, b_1, \cdots, b_{n-1}) \cdot y_i(t) \tag{4}$$

encapsulating the values of its state vector, where the coefficients $\{b_k\}_{k=1}^{n-1}$ are taken such that the polynomial $\sum_{k=0}^{n-1} b_k \mu^{n-k-1}$ of μ is Schur stable with $b_0 = 1$. It is known [35] e.g., if $b_0 > b_1 > b_2 > \cdots > b_{n-1} > 0$, the above polynomial is Schur stable. Here, we write the components in y_i starting from $y_{i,0}$ instead of $y_{i,1}$ for better matching the above polynomial and (4).

For any path $p \in \mathcal{P}_{ji}(t)$, we define the converted received state vector from source i at sink j as $y_i^p(t) = \Theta^{-1}x_i^p(t) \in \mathbb{R}^n$, where Θ is defined by (2). Write $y_i^p(t) = (y_{i,n-1}^p(t), y_{i,n-2}^p(t), \cdots, y_{i,0}^p(t))^{\mathrm{T}} \in \mathbb{R}^n$ for $i \in \mathcal{V}$ accordingly, and the encoded value analogous to (4) is given by

$$v_i^p(t) = (b_0, b_1, \cdots, b_{n-1}) \cdot y_i^p(t). \tag{5}$$

In the leader-follower scenario, the leader l has no in-neighbors and hence at time t it sends $x_l(t)$ to every agent $j \in \mathcal{N}_i^{\leq \ell +}(t)$ and updates its state vector using (1) with $u_l(t) = \varphi(t) : \mathbb{N} \to \mathbb{R}$ satisfying the following Assumption 1. The encoded values $v_l(t)$ and $v_l^p(t)$ are defined as in (4) and (5) accordingly.

Assumption 1. There exists some vector $x_\varphi \in \mathbb{R}^n$ such that $\|x_l(t) - x_\varphi\| = o(t^{-1})$ as t tends to infinity, where $\|\cdot\|$ is the Euclidean norm.

An example is to take $\varphi(t) = 0$, which yields $x_l(t+1) = Ax_l(t)$. If all eigenvalues of A have absolute values less than one, the convergence is exponentially fast in light of the Jordan normal form. Hence, Assumption 1 holds. This assumption is more general than the static reference signal used in [26] and it does not require any shared information of signal bounds of l assumed in [27, 28]. Note that the convergence here is deterministic since the leader l is not influenced by follower agents.

Remark 1. We have assumed that (1) is a single-input system for simplicity. If $u_i(t) \in \mathbb{R}^m$ for some $m \in \mathbb{N}$, then we can take $u_i(t) = Cx_i(t) + c\hat{u}_i(t)$ for some $C \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$ and transfer (1) to the following single input system

$$x_i(t+1) = (A + BC)x_i(t) + Bc\hat{u}_i(t), \tag{6}$$

where $\hat{u}_i(t) \in \mathbb{R}$ becomes the new input value. When (A, B) is controllable, it is shown in [29] that the matrix C can be defined as below such that the pair $(A + BC, Bc)$ is also controllable for any $c \in \mathbb{R}^m$ satisfying $Bc \neq 0_n$:

$$C := FG^{-1}, \tag{7}$$

where $G := (B_1, AB_1, \cdots, A^{r_1-1}B_1, B_2, AB_2, \cdots, A^{r_2-1}B_2, \cdots, B_k, AB_k, \cdots, A^{r_k-1}B_k) \in \mathbb{R}^{n\times n}$ with $B_1, B_2, \cdots, B_k$ $(k \leq m)$ being the distinct columns in B, $\sum_{i=1}^k r_i = n$, $r_i \geq 1$ for $1 \leq i \leq k$, and $F := (F_1, F_2, \cdots, F_n) \in \mathbb{R}^{m\times n}$ with $F_{a_j} = e_{j+1}$ for $1 \leq j \leq k-1$, $a_j = \sum_{i=1}^j r_i$, and $F_a = 0_m$ for $a \in \{1, 2, \cdots, n\}\backslash\{a_1, a_2, \cdots, a_{k-1}\}$. Here, $e_{j+1} \in \mathbb{R}^m$ is the $(j+1)$-th unit vector for $1 \leq j \leq k-1$.

2.3. *ℓ-hop resilient consensus strategies*

Our ℓ-hop resilient consensus strategies rely on a concept of minimum message cover introduced in the multi-hop communication network [16]. Given a collection of paths $\mathcal{P}(t)$ in $\mathcal{G}(t)$ sharing the same sink $i \in \mathcal{V}$, a message cover $\mathcal{M}(t) \subseteq \mathcal{V}$ for $\mathcal{P}(t)$ is defined as a set of agents whose removal would disconnect every path $p \in \mathcal{P}(t)$. We assume $i \notin \mathcal{M}(t)$. Clearly, for every path $p \in \mathcal{P}(t)$, $\mathcal{M}(t)$ contains at least an internal agent or the source agent of p. A minimum message cover is one that has the minimum cardinality.

In the leaderless scenario, given $t, r, \ell \in \mathbb{N}$, the Update step in Section 2.2 is implemented as follows. Each agent $i \in C$ calculates the values $\{v_j^p(t)\}_{j\in\mathcal{N}_i^{\leq \ell -}(t),\ p\in\mathcal{P}_{ij}(t)}$ using (5) based on the information it receives at the Collect step. The values are sorted in a list $L_i(t)$ from the largest to the smallest. We first check the values that are larger than $v_i(t)$ and let the minimum message cover for the set of associated paths be $\mathcal{M}_i^>(t)$. If $|\mathcal{M}_i^>(t)| \leq r$, all these values that are larger than $v_i(t)$ are removed from $L_i(t)$; Otherwise, we remove the values starting from the largest in $L_i(t)$ downwards such that the corresponding minimum message cover has cardinality r but the removal of one more agent would increase the cardinality to $r+1$. Analogously, we then check the values that are smaller than $v_i(t)$ and let the minimum message cover for the set of associated paths be $\mathcal{M}_i^<(t)$. If $|\mathcal{M}_i^<(t)| \leq r$, all these values that are smaller than $v_i(t)$ are removed from $L_i(t)$; Otherwise, we remove the values starting from the smallest in $L_i(t)$ upwards such that the corresponding minimum message cover has cardinality r but the removal of one more agent would increase the cardinality to $r+1$. Denote by $\mathcal{R}_i(t) \subseteq \cup_{j\in\mathcal{N}_i^{\le\ell-}}\mathcal{P}_{ij}(t)$ the set of paths corresponding to the removed values in $L_i(t)$. A pseudo algorithm is shown in Table 1. The control input in (1) is designed as

$$\begin{aligned} u_i(t) =& (a_1-b_1,\cdots,a_{n-1}-b_{n-1},a_n)\cdot y_i(t)+v_i(t)\\ &+\theta_i(t)\sum_{p\in\left(\cup_{j\in\mathcal{N}_i^{\le\ell-}(t)}\mathcal{P}_{ij}(t)\right)\setminus\mathcal{R}_i(t)} b_{ij}^p(t) f_{ij}(v_j^p(t),v_i(t)), \end{aligned} \tag{8}$$

where $0<\theta_i(t)<\left(\sum_{p\in\left(\cup_{j\in\mathcal{N}_i^{\le\ell-}(t)}\mathcal{P}_{ij}(t)\right)\setminus\mathcal{R}_i(t)} b_{ij}^p(t)\right)^{-1}$ and $b_{ij}^p(t)=a_{ii_1}(t)a_{i_1i_2}(t)\cdots a_{i_{k-1}j}(t)$ with $p=(j,i_{k-1},\cdots,i_2,i_1,i)\in\mathcal{P}_{ij}(t)$ being a k-hop path with some $k\le\ell$. Here, the function $f_{ij}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies (i) $f_{ij}(v_1,v_2)=0$ if and only if $v_1=v_2$ and (ii) there exist two constants $0<\alpha_i\le\beta_i\le 1$ such that

$$\alpha_i \le \frac{f_{ij}(v_1,v_2)}{v_1-v_2} \le \beta_i \tag{9}$$

for $v_1\neq v_2$.

Remark 2. The function $f_{ij}(\cdot,\cdot)$ in (9) is flexible as long as it satisfies the conditions (i) and (ii). This includes not only the typical choice $f_{ij}(v_1,v_2)=v_1-v_2$ but also $f_{ij}(v_1,v_2)=g_{ij}(v_1)-g_{ij}(v_2)$ for any monotonically increasing Lipschitz function $g_{ij}(\cdot)$ with the Lipschitz coefficient no greater than 1. For instance, $g_{ij}(\cdot)=(\cdot)^c$ for an odd integer c or $g_{ij}(\cdot)=\mathrm{sgn}(\cdot)(\cdot)^c$ for an even integer c over the interval $[-1,1]$, where sgn is the signum function. Another example is $g_{ij}(\cdot)=\arctan(\cdot)$ over $\mathbb{R}$. These nonlinear functions are often crucial to achieving higher performance such as finite-time and fixed-time consensus [39, 40] although the choices of the function $f_{ij}(\cdot,\cdot)$ do not seem to have immediate impact in the current setting.

Next, in the leader-follower scenario, given $t,r,\ell\in\mathbb{N}$, we can follow the same procedure as in the leaderless scenario above. The only exception is that any potential values sent from the source l should be retained when deleting values from the list $L_i(t)$.

Several remarks are in order.

Remark 3. In the proposed protocols, each cooperative agent $i\in\mathcal{C}$ has the knowledge of $\mathcal{N}_i^{\le\ell+}(t)$ and $\mathcal{N}_i^{\le\ell-}(t)$, which essentially embodies the ℓ-hop communication capability. Although our network $\mathcal{G}(t)$ is random, the communication capability is governed by a mechanism that is implemented in the same way as in a deterministic network, e.g. [16]. For instance, the ℓ-hop communication capability depends on the nature of human behavior in social acquaintance networks [41] and the bot's tracking ability in cybersecurity [42], both of which are independent of the specific network topology, be it random or deterministic. In the leader-follower scenario, a cooperative agent i can tell whether the message is from the leader l because the path information is not altered by our assumption in Section 2.2. This is in fact the minimum requirement because if l is not identifiable, its value may be removed at all times, and hence the leader-follower consensus is impossible. Moreover, since ℓ is a finite (and typically small) number, minimum message covers can be easily calculated [16]. Although the minimum message covers need to be calculated at each iteration with $O(\ell n)$ time, the sorting in $L_i(t)$ takes $O(n\ln n)$ time dominating the workload. Therefore, our algorithms are of similar complexity comparable to classical W-MSR algorithms [9, 13].

Remark 4. Recall that the adjacency weights $\{a_{ij}(t)\}$ are time-varying and random. So do the entries $\{b_{ij}^p(t)\}$. Nevertheless, the gain $\theta_i(t)$ in (8) can be conveniently determined as a non-random

Input: $v_i(t)$, $\{v_j^p(t)\}_{j \in \mathcal{N}_i^{\leq \ell-}(t), p \in \mathcal{P}_{ij}(t)}$
Output: $\mathcal{R}_i(t)$
let $\xi = |\mathcal{N}_i^{\leq \ell-}(t)|$ and $\mathcal{R}'_i(t) = \emptyset$
sort $\{v_j^p(t)\}_{j \in \mathcal{N}_i^{\leq \ell-}(t), p \in \mathcal{P}_{ij}(t)}$ in a descending order as $v_{i_\xi}(t) \geq \cdots \geq v_{i_2}(t) \geq v_{i_1}(t)$
if $|\mathcal{M}_i^{>}(t)| \leq r$
    for $\eta = \xi, \xi - 1, \cdots, 1$
        if $v_{i_\eta}(t) > v_i(t)$
            insert i_η into $\mathcal{R}'_i(t)$
        end if
    end for
else
    for $\eta = \xi, \xi - 1, \cdots, 1$
        $flag = 0$
        if $v_{i_\eta}(t) > v_i(t)$
            $flag = 1$
            insert i_η into $\mathcal{R}'_i(t)$, calculate $\mathcal{M}_i^{>}(t)$
        end if
        if $flag = 0$ or $|\mathcal{M}_i^{>}(t)| \leq r$
            exit for
        end if
    end for
end if
if $|\mathcal{M}_i^{<}(t)| \leq r$
    for $\eta = 1, 2, \cdots, \xi$
        if $v_{i_\eta}(t) < v_i(t)$
            insert i_η into $\mathcal{R}'_i(t)$
        end if
    end for
else
    for $\eta = 1, 2, \cdots, \xi$
        $flag = 0$
        if $v_{i_\eta}(t) < v_i(t)$
            $flag = 1$
            insert i_η into $\mathcal{R}'_i(t)$, calculate $\mathcal{M}_i^{<}(t)$
        end if
        if $flag = 0$ or $|\mathcal{M}_i^{<}(t)| \leq r$
            exit for
        end if
    end for
end if
$\mathcal{R}_i(t)$ is the set of paths corresponding to $\mathcal{R}'_i(t)$

Table 1: Processing flow for a cooperative node $i \in C$.

function. For instance, if there is $\alpha(t) \geq 1$ such that $a_{ij}(t) \leq \alpha(t)$ holds for all $i, j \in \mathcal{V}$, we can choose $\theta_i(t) \in (0, (\alpha(t)^{\ell} \big| \cup_{j\in\mathcal{N}_i^{\leq \ell-}(t)} \mathcal{P}_{ij}(t)\big|)^{-1})$.

Remark 5. We note that a cooperative agent $i \in C$ cannot identify whether an agent $j \in \mathcal{N}_i^{\leq \ell-}(t)$ is cooperative or Byzantine since the identities of the Byzantine agents are not available to it. Intuitively, without this identification information the best thing a cooperative agent can possibly do is to remove some extremal values of $\{v_j^p(t)\}_{j\in\mathcal{N}_i^{\leq \ell-}(t),\ p\in\mathcal{P}_{ij}(t)}$, which hopefully could block some Byzantine sources. In fact, we can prove that if $v_j^p(t)$ is close to $v_i(t)$, then $x_j^p(t)$ is close to $x_i(t)$, and vice versa. This would support our intuition and be in line with the above filtering strategy. More specifically, for a vector $M = (m_1, \cdots, m_n)^{\mathrm{T}} \in \mathbb{R}^n$, let $\|M\|_1 := \sum_{i=1}^n |m_i|$ represent the vector 1-norm; for a matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$, let $\|M\|_1 := \max_{1\leq j\leq n} \sum_{i=1}^n |m_{ij}|$ represent the matrix 1-norm (with a slight abuse of notation), which is the operator norm induced by the vector 1-norm. It follows from (4) and (5) that

$$b_{n-1}\|y_i(t) - y_j^p(t)\|_1 \leq |v_i(t) - v_j^p(t)| \leq \|y_i(t) - y_j^p(t)\|_1 \tag{10}$$

since we can choose $1 > b_1 > b_2 > \cdots > b_{n-1} > 0$ as commented underneath (4). Since Θ defined in (2) is an invertible matrix and recall the relations $x_i(t) = \Theta y_i(t)$ and $x_i^p(t) = \Theta y_i^p(t)$, we have

$$\begin{aligned} \frac{1}{\|\Theta^{-1}\|_1}\|y_i(t) - y_j^p(t)\|_1 \leq& \|x_i(t) - x_j^p(t)\|_1 \\ \leq& \|\Theta\|_1\|y_i(t) - y_j^p(t)\|_1. \end{aligned} \tag{11}$$

3. Resilient leaderless consensus analysis

In this section, we present the resilient leaderless consensus result for the system (1) under ℓ-hop communication over $\mathcal{G}(t)$. Specifically, we aim to show secure almost surely convergence satisfying the following two conditions: (i) $x_i(t)$ remains bounded for any $i \in C$ and $t \in \mathbb{N}$; and (ii) there exists a vector $\hat{x} \in \mathbb{R}^n$ such that $\mathrm{P}(\lim_{t\to\infty} x_i(t) = \hat{x}) = 1$ for any $i \in C$ and initial states $\{x_i(0)\}_{i\in\mathcal{V}}$, where P is the probability operator.

Remark 6. The condition that the state $x_i(t) \in \mathbb{R}^n$ for a cooperative agent i remains in a bounded region for $t \in \mathbb{N}$ is often referred to as the safety condition in the literature except that most of them only consider the scalar case $n = 1$; see e.g. [5, 9, 12, 20, 21]. Clearly, if there is a $c > 0$ such that $\|x_i(t)\|_1 \leq c$, then each dimension of $x_i(t)$ remains in the interval $[-c, c]$. Moreover, we mention that the vector-based safety condition presents a potential advantage, where cluster consensus of the agents' states may be accommodated. In [43], a novel matrix-weighted protocol is proposed to realize the cluster consensus in vector-valued multiagent systems. A vector-based safety condition would be feasible to withstand misbehaving agents in that scenario.

Theorem 1. Consider the random digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with $\mathcal{V} = C \cup \mathcal{B}$, where the cooperative agents in C update their states following the above ℓ-hop leaderless resilient consensus strategy, and the set $\mathcal{B}$ of Byzantine agents is r-local under ℓ-hop communication. If (A, B) is controllable and $\mathrm{P}(\mathcal{G}(t)$ is $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}) > 0$ for $t \in \mathbb{N}$, then resilient leaderless consensus is achieved almost surely.

Proof. We will first show that the states of cooperative agents are bounded and then show the almost sure convergence.

(i) Given $t \in \mathbb{N}$, in view of (1) and (8) we obtain that for each agent $i \in C$,

$$y_{i,n-1}(t+1) = -\sum_{k=1}^{n-1} b_k y_{i,n-k}(t) + v_i(t) + \theta_i(t) \sum_{p \in \left(\cup_{j \in \mathcal{N}_i^{\leq \ell-}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} b_{ij}^p(t) f_{ij}(v_j^p(t), v_i(t)), \tag{12}$$

and

$$y_{i,n-2}(t+1) = y_{i,n-1}(t), \cdots, y_{i,1}(t+1) = y_{i,2}(t), \quad y_{i,0}(t+1) = y_{i,1}(t). \tag{13}$$

where we have noted $y_i(t) = (y_{i,n-1}(t), y_{i,n-2}(t), \cdots, y_{i,0}(t))^{\mathrm{T}}$. Using (4) and (13) we have

$$v_i(t+1) = y_{i,n-1}(t+1) + \sum_{k=1}^{n-1} b_k y_{i,n-k}(t) = v_i(t) + \theta_i(t) \cdot \sum_{p \in \left(\cup_{j \in \mathcal{N}_i^{\leq \ell-}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} b_{ij}^p(t) f_{ij}(v_j^p(t), v_i(t)), \tag{14}$$

where we have applied (12) in the last equality. By the choice of $\theta_i(t)$ and the condition (9) for f_{ij}, we know that $v_i(t+1)$ is bounded from the above and the below by convex combinations of $v_i(t)$ and $v_j^p(t)$ for $p \in (\cup_{j \in \mathcal{N}_i^{\leq \ell-}(t)} \mathcal{P}_{ij}(t)) \setminus \mathcal{R}_i(t)$.

Define $v_{\max}(t) = \max_{i \in C} v_i(t)$ and $v_{\min}(t) = \min_{i \in C} v_i(t)$. Since the set $\mathcal{B}$ is r-local under ℓ-hop communication, any $v_j^p(t) \notin [v_{\min}(t), v_{\max}(t)]$ will not be used in the update (14) in light of the ℓ-hop leaderless resilient consensus strategy. Therefore, $v_i(t+1) \in [v_{\min}(t), v_{\max}(t)]$ by the above comment of convex combinations. In other words, $v_{\min}(t) \leq v_{\min}(t+1) \leq v_{\max}(t+1) \leq v_{\max}(t)$ holds for any $t \in \mathbb{N}$. Although the agents in $\mathcal{G}(t)$ possess random states, this relationship holds in a deterministic manner.

Write $z_i(t) = (y_{i,n-2}(t), y_{i,n-3}(t), \cdots, y_{i,0}(t))^{\mathrm{T}} \in \mathbb{R}^{n-1}$ for $i \in C$ and hence $y_i(t) = (y_{i,n-1}(t), z_i(t)^{\mathrm{T}})^{\mathrm{T}}$. It follows from (4) and (13) that

$$z_i(t+1) = \tilde{A} z_i(t) + \tilde{B} v_i(t), \tag{15}$$

where $\tilde{A} = \begin{pmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_{n-1} \\ b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\tilde{B} = (1, 0, 0, \cdots, 0)^{\mathrm{T}} \in \mathbb{R}^{n-1}$. It is easy to see that the matrix $\tilde{A}$ is Schur stable because its characteristic polynomial $\sum_{k=0}^{n-1} b_k \mu^{n-k-1}$ is Schur stable by our choice of the coefficients $\{b_k\}_{k=0}^{n-1}$ in Section 2.2. An application of the input-to-state stability theory [36, Lemma 3.5] to (15) yields the boundedness of $z_i(t)$ for $i \in C$ and $t \in \mathbb{N}$. By (13), $y_{i,n-1}(t)$ and consequently the vector $y_i(t)$ are both bounded. The boundedness of $x_i(t)$ then follows from the linear transform $x_i(t) = \Theta y_i(t)$.

(ii) In this step, we aim to show the almost sure convergence for the cooperative agents based on the limit theory of martingales. To this end, we note that $\mathcal{F}(t) = \sigma(\{v_i(0)\}_{i \in \mathcal{V}}, \{v_i(1)\}_{i \in \mathcal{V}}, \cdots,$

$\{v_i(t)\}_{i\in\mathcal{V}}$) for $t \in \mathbb{N}$ is a filtration. It follows from the step (i) that $\mathrm{E}(|v_{\max}(t) - v_{\min}(t)|) < \infty$ and $\mathrm{E}(v_{\max}(t+1) - v_{\min}(t+1)|\mathcal{F}(t)) \le v_{\max}(t) - v_{\min}(t)$ for $t \in \mathbb{N}$. This implies $v_{\max}(t) - v_{\min}(t) \ge 0$ is a super-martingale with respect to $\mathcal{F}(t)$. Doob's martingale convergence theorem (see e.g. Theorem 4.2.12 in [37, p. 193]) indicates that there must exist a random variable $v \ge 0$ satisfying

$$\mathrm{P}\big(\lim_{t\to\infty}(v_{\max}(t) - v_{\min}(t)) = v\big) = 1. \tag{16}$$

The variable v can be proved to be zero with probability 1. If this is not true, we have $v > 0$ with positive probability. For any $t \in \mathbb{N}$, the cooperative agents in $\mathcal{C}$ belong to one and only one of the following three categories: $\mathcal{V}_1(t) = \{i \in \mathcal{C} : v_i(t) = v_{\max}(t)\}$, $\mathcal{V}_2(t) = \{i \in \mathcal{C} : v_i(t) = v_{\min}(t)\}$ and $\mathcal{V}_3(t) = \mathcal{C}\backslash(\mathcal{V}_1(t)\cup\mathcal{V}_2(t))$. Evidently, $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ are two mutually exclusive nonempty sets. By assumption, $\mathrm{P}(\mathcal{G}(t)$ is $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}) > 0$. Hence, with positive probability, $\mathcal{V}_1(t)$ or $\mathcal{V}_2(t)$ is $(2r+1)$-reachable under ℓ-hop communication w.r.t. $\mathcal{B}$. Without loss of generality, we assume $\mathcal{V}_1(t)$ is $(2r+1)$-reachable under ℓ-hop communication w.r.t. $\mathcal{B}$ with positive probability. By Definition 1, the probability that there is some agent $i_0 \in \mathcal{V}_1(t)$ such that it has at least $2r+1$ independent $(\le \ell)$-hop paths originated from outside $\mathcal{V}_1(t)$ avoiding any Byzantine agents as internal agents will be positive. Denote by $\mathcal{H}$ the random event that i_0 has at least $r+1$ independent $(\le \ell)$-hop paths originated from outside $\mathcal{V}_1(t)$ avoiding any Byzantine agents as internal and source agents. Since $\mathcal{B}$ is r-local under ℓ-hop communication, the probability $\mathrm{P}(\mathcal{H}) > 0$. Any agent outside $\mathcal{V}_1(t)$ has a value less than $v_{\max}(t)$. Conditional on the event $\mathcal{H}$, by our proposed leaderless resilient consensus strategy, at least one of $r+1$ source agents of the above independent paths will be used in the state update at time step $t+1$ and any value greater than $v_{\max}(t)$ will not be used in the update. Consequently, with positive probability, the value $v_{i_0}(t+1)$ will be pulled down. Using an analogous argument, we conclude the gap $v_{\max} - v_{\min} \to 0$ in probability as t tends to infinity. This would be at odds with (16) and the assumption that $v > 0$ with positive probability. Hence, we have shown the limit is $v = 0$.

From the step (i) we know that $v_{\max}(t)$ and $v_{\min}(t)$ are bounded and monotonic sequences. Hence, there exists a constant $w \in \mathbb{R}$ such that $v_i(t) \to w$ as $t \to \infty$ almost surely for any $i \in \mathcal{C}$ by (16) and $v = 0$. Define the gap $\varepsilon_i(t) = z_i(t) - w/(1 + \sum_{k=1}^{n-1} b_k)1_{n-1} \in \mathbb{R}^{n-1}$ for each $i \in \mathcal{C}$. Recall that $z_i(t)$ satisfies (15). We will first show the vector $\varepsilon_i(t)$ is vanishing almost surely.

Note that

$$\begin{aligned}
&\varepsilon_i(t+1)\\
=&\tilde{A}z_i(t) + \tilde{B}v_i(t) - \frac{w}{1+\sum_{k=1}^{n-1} b_k}1_{n-1}\\
=&\tilde{A}\varepsilon_i(t) + \frac{w}{1+\sum_{k=1}^{n-1} b_k}\tilde{A}1_{n-1} + \tilde{B}v_i(t)\\
&-\frac{w}{1+\sum_{k=1}^{n-1} b_k}1_{n-1}\\
=&\tilde{A}\varepsilon_i(t) - \tilde{B}w + \tilde{B}v_i(t)
\end{aligned} \tag{17}$$

in view of the definitions of $\tilde{A}$ and $\tilde{B}$. Write $\delta_i(t) = v_i(t) - w$ for $i \in \mathcal{C}$. We have $\delta_i(t) \to 0$ almost surely as $t \to \infty$, and

$$\varepsilon_i(t+1) = \tilde{A}\varepsilon_i(t) + \tilde{B}\delta_i(t) \tag{18}$$

by invoking (17). From the step (i) we know that $\tilde{A}$ is stable. Using the Lyapunov stability result [38, Theorem 5.D5], for any positive definite $Q \in \mathbb{R}^{(n-1)\times(n-1)}$, there must exist a positive definite

$R \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $Q = R - \tilde{A}^{\mathrm{T}} R \tilde{A}$. Define a function $g_i(t) = \varepsilon_i(t)^{\mathrm{T}} R \varepsilon_i(t) \geq 0$ for $i \in C$. Along the trajectory of (18),

$$
\begin{aligned}
&g_i(t+1) - g_i(t)\\
=&\varepsilon_i(t+1)^{\mathrm{T}} R \varepsilon_i(t+1) - \varepsilon_i(t)^{\mathrm{T}}(Q + \tilde{A}^{\mathrm{T}} R \tilde{A})\varepsilon_i(t)\\
=&2\varepsilon_i(t)^{\mathrm{T}} \tilde{A}^{\mathrm{T}} R \tilde{B} \delta_i(t) + \delta_i(t)^2 \tilde{B}^{\mathrm{T}} R \tilde{B} - \varepsilon_i(t)^{\mathrm{T}} Q \varepsilon_i(t).
\end{aligned} \tag{19}
$$

Since $\delta_i(t)$ is vanishing almost surely, the term $2\varepsilon_i(t)^{\mathrm{T}} \tilde{A}^{\mathrm{T}} R \tilde{B} \delta_i(t) + \delta_i(t)^2 \tilde{B}^{\mathrm{T}} R \tilde{B}$ is also vanishing almost surely. We argue that $g_i(t) \to 0$ almost surely as $t \to \infty$. If this does not hold, there exists a constant $\gamma > 0$ such that for any $t \in \mathbb{N}$ there is $t_1 \geq t$ satisfying $\varepsilon_i(t_1)^{\mathrm{T}} Q \varepsilon_i(t_1) > \gamma$ in view of the positive definiteness of Q and R and the definition of $g_i(t)$. By (19), there exists $t_2 \geq t_1$ satisfying $g_i(t_2 + 1) - g_i(t_2) \leq -\gamma/2$. Hence, we can find a sequence of time steps such that this inequality holds. This is at odds with $g_i(t) \geq 0$. Hence, we conclude $g_i(t) \to 0$ almost surely as $t \to \infty$. Using the positive definiteness of R, we obtain $\varepsilon_i(t) \to 0_{n-1}$ almost surely as $t \to \infty$ for $i \in C$.

By construction, we have $z_i(t) \to w/(1 + \sum_{k=1}^{n-1} b_k)1_{n-1} \in \mathbb{R}^{n-1}$ for $i \in C$ as $t \to \infty$. By (13), $y_i(t) \to w/(1 + \sum_{k=1}^{n-1} b_k)1_n \in \mathbb{R}^n$ almost surely. Finally, we conclude for any $i \in C$, $x_i(t) \to w/(1 + \sum_{k=1}^{n-1} b_k)\Theta 1_n$ almost surely as t tends to infinity. This completes the proof. □

Remark 7. Unlike many existing consensus problems under packet losses/link failures, we do not require a sufficiently large $q_{ij}(t)$ thanks to the W-MSR algorithm for resilient consensus. With the martingale convergence, the probabilistic robustness condition in Theorem 1 specifies how $q_{ij}(t)$ should behave - the positivity is more important than the magnitude.

Remark 8. The probabilistic robustness condition in Theorem 1 can be converted to a deterministic one, which would be easier to verify. In fact, note that the digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ is finite for any $t \in \mathbb{N}$. If the weights in its adjacency matrix $\mathcal{A}(t)$ are independent, the robustness condition is tantamount to requiring the digraph $(\mathcal{V}, \tilde{\mathcal{E}}(t))$ being $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}$, where $\tilde{\mathcal{E}}(t) = \{(i, j) \in \mathcal{V} \times \mathcal{V} : q_{ij}(t) > 0\}$. Furthermore, if the edge probabilities are constant with respect to time, namely, $q_{ij}(t) \equiv q_{ij}$, then the robustness condition simplifies to requiring the digraph $(\mathcal{V}, \tilde{\tilde{\mathcal{E}}})$ being $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}$, where $\tilde{\tilde{\mathcal{E}}} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : q_{ij} > 0\}$.

Remark 9. In our threat model, we assume that the identity of agents in $\mathcal{B}$ is not known to the cooperative agents, which is in line with the existing works regarding resilient consensus; e.g. [9, 10, 11, 12, 13]. In this worst situation, we may verify the robustness of $\mathcal{G}(t)$ under ℓ-hop communication w.r.t. any set $\mathcal{S} \subseteq \mathcal{V}$ with $|\mathcal{S}| = r$ (since $\mathcal{B}$ is r-local under ℓ-hop communication). If some Byzantine agents are identifiable, we could reduce the testing scenarios of $\mathcal{S}$ accordingly.

4. Resilient leader-follower consensus analysis

The resilient leader-follower consensus for the system (1) under ℓ-hop communication over $\mathcal{G}(t)$ is considered in this section. Formally, the objective is to show secure almost surely convergence satisfying the following two conditions: (i) $x_i(t)$ remains bounded for any $i \in C$ and $t \in \mathbb{N}$; and (ii) $\mathrm{P}(\lim_{t\to\infty} x_i(t) = x_\varphi) = 1$ for any $i \in C$ and initial states $\{x_i(0)\}_{i\in\mathcal{V}}$, where x_φ is given in Assumption 1.

Theorem 2. Consider the random digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with $\mathcal{V} = C \cup \mathcal{B} \cup \{l\}$, where the cooperative agents in C update their states following the above ℓ-hop leaderless resilient consensus strategy, and the set $\mathcal{B}$ of Byzantine agents is r-local under ℓ-hop communication. If Assumption 1 holds, (A, B) is controllable and $\mathrm{P}(\mathcal{G}(t)$ is leader-follower $(2r +$

1)-robust under ℓ-hop communication w.r.t. $\mathcal{B}$) > 0 for $t \in \mathbb{N}$, then resilient leader-follower consensus is achieved almost surely.

Proof. The proof contains two parts: the first part establishes the boundedness for states of cooperative agents and the second part shows the almost sure convergence to x_φ.

(i) Fix any $t \in \mathbb{N}$ and $i \in C$, we obtain (12) and (13) by using (1) and (8) and noting $y_i(t) = (y_{i,n-1}(t), y_{i,n-2}(t), \cdots, y_{i,0}(t))^{\mathrm{T}}$ as in Theorem 1. Combining (4), (12) and (13) yields (14). By the conditions of $\theta_i(t)$ and f_{ij}, we know that $v_i(t+1)$ is bounded from the above and the below by convex combinations of $v_i(t)$ and $v_j^p(t)$ for $p \in (\cup_{j\in\mathcal{N}_i^{\leq\ell-}(t)} \mathcal{P}_{ij}(t))\backslash\mathcal{R}_i(t)$.

Denote by $v_{\max}(t) = \max_{i\in C\cup\{l\}} v_i(t)$ and $v_{\min}(t) = \min_{i\in C\cup\{l\}} v_i(t)$ the two extreme values among cooperative agents. Consider any $i \in C$. Since the set $\mathcal{B}$ is r-local under ℓ-hop communication, there are no more than r values $v_j^p(t)$ for $j \in \mathcal{N}_i^{\leq\ell-}(t)$ (excluding any values possibly coming from source l) taking values outside $[v_{\min}(t), v_{\max}(t)]$ at time $t+1$. In light of our ℓ-hop leader-follower resilient consensus strategy, any such values from the ℓ-hop in-neighbors of i will be scrapped in the update (14) with two potential exceptions: (a) there exist $j \in \mathcal{N}_i^{\leq\ell-}(t)$ and $p \in \mathcal{P}_{ij}(t)$ such that $v_{\max}(t) < v_j^p(t+1) \leq v_l(t+1)$; and (b) there exist $j \in \mathcal{N}_i^{\leq\ell-}(t)$ and $p \in \mathcal{P}_{ij}(t)$ such that $v_{\min}(t) > v_j^p(t+1) \geq v_l(t+1)$. Thanks to Assumption 1 and the transformation (4), there must exit $\delta(t) \in [0, o(t^{-1})]$ satisfying

$$\begin{aligned} v_{\min}(t) - \delta(t) \leq & v_{\min}(t+1) \leq v_{\max}(t+1) \\ \leq & v_{\max}(t) + \delta(t) \end{aligned} \tag{20}$$

for sufficiently large $t \in \mathbb{N}$. The inequalities in (20) hold true irrespective of the random edges in the network.

For $i \in C$, we write $z_i(t) = (y_{i,n-2}(t), y_{i,n-3}(t), \cdots, y_{i,0}(t))^{\mathrm{T}} \in \mathbb{R}^{n-1}$ and $y_i(t) = (y_{i,n-1}(t), z_i(t)^{\mathrm{T}})^{\mathrm{T}}$. By (4) and (13) we obtain the same system (15). Clearly, $v_i(t)$ is still bounded in view of (20). Applying the input-to-stability theory as in Theorem 1, we conclude the proof for boundedness of $\{x_i(t)\}_{i\in C, t\in\mathbb{N}}$.

(ii) In this step, we will show the almost sure tracking behavior for all cooperative agents in C. Define $h(t) = v_{\max}(t) - v_{\min}(t) - 2t\delta(t)$. As in Theorem 1, $\mathcal{F}(t) = \sigma(\{v_i(0)\}_{i\in\mathcal{V}}, \{v_i(1)\}_{i\in\mathcal{V}}, \cdots, \{v_i(t)\}_{i\in\mathcal{V}})$ for $t \in \mathbb{N}$ is a filtration. By (20), $\mathrm{E}(|h(t)|) < \infty$ and $\mathrm{E}(h(t+1)|\mathcal{F}(t)) \leq h(t)$ for $t \in \mathbb{N}$. This implies that $h(t)$ is a super-martingale with respect to $\mathcal{F}(t)$. The limit theory of martingales [37, p. 193] again suggests that there exists a random variable $v \geq 0$ satisfying

$$P(\lim_{t\to\infty} h(t) = \lim_{t\to\infty}(v_{\max}(t) - v_{\min}(t)) = v) = 1 \tag{21}$$

by invoking the condition of $\delta(t)$.

We claim that the limit v in (21) is zero with probability 1. If this is not true, we have $v > 0$ with positive probability. For any $t \in \mathbb{N}$, the cooperative agents in $C \cup \{l\}$ belong to one and only one of the following three categories: $\mathcal{V}_1(t) = \{i \in C \cup \{l\} : v_i(t) = v_{\max}(t)\}$, $\mathcal{V}_2(t) = \{i \in C \cup \{l\} : v_i(t) = v_{\min}(t)\}$ and $\mathcal{V}_3(t) = (C \cup \{l\})\backslash(\mathcal{V}_1(t) \cup \mathcal{V}_2(t))$. Apparently, $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ are two mutually exclusive nonempty sets. In the following, we examine three separate situations on the basis of the location of l: (a) $l \in \mathcal{V}_1(t)$, (b) $l \in \mathcal{V}_2(t)$ and (c) $l \in \mathcal{V}_3(t)$.

(a) Firstly, suppose that the set $\mathcal{V}_2(t) \cap \mathcal{N}_l^{1+}(t) \neq \emptyset$. Hence, there is some agent $i_0 \in \mathcal{V}_2(t) \cap \mathcal{N}_l^{1+}(t)$. In view of the ℓ-hop leader-follower resilient consensus strategy and the edge randomness, $v_l(t)$ will be used in the update (14) at time $t+1$ for agent i_0 with positive probability. Condition on this, our proposed algorithm ensures i_0 will not use any value outside $[v_{\min}(t), v_{\max}(t)]$ in its update at $t+1$ since $\mathcal{B}$ is r-local under ℓ-hop communication. Recall that in (14) $v_{i_0}(t+1)$ is bounded from both ways by convex combinations, and hence $v_{i_0}(t+1) \geq v_{i_0}(t)$ and with positive probability $v_{i_0}(t+1) > v_{i_0}(t)$.

On the other hand, suppose $\mathcal{V}_2(t) \cap \mathcal{N}_l^{1+}(t) = \emptyset$. It is clear $\mathcal{V}_2(t)$ is $(2r+1)$-reachable under ℓ-hop communication w.r.t. $\mathcal{B}$ with positive probability by invoking the condition P($\mathcal{G}(t)$ is leader-follower $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}$) > 0. By Definition 1, the probability that there is some agent $i_0 \in \mathcal{V}_2(t)$ such that it has at least $2r+1$ independent ($\leq \ell$)-hop paths originated from outside $\mathcal{V}_2(t)$ avoiding any Byzantine agents as internal agents will be positive. Denote by $\mathcal{H}$ the random event that i_0 has at least $r+1$ independent ($\leq \ell$)-hop paths originated from outside $\mathcal{V}_2(t)$ avoiding any Byzantine agents as internal and source agents. Since $\mathcal{B}$ is r-local under ℓ-hop communication, the probability P($\mathcal{H}$) > 0. Any agent outside $\mathcal{V}_2(t)$ has a value greater than $v_{\min}(t)$. Conditional on the event $\mathcal{H}$, by the ℓ-hop leader-follower resilient consensus strategy, at least one of $r+1$ source agents of the above independent paths will be used in the state update at time $t+1$ and any value lower than $v_{\min}(t)$ will not be used in the update. Therefore, $v_{i_0}(t+1) \geq v_{i_0}(t)$ and with positive probability $v_{i_0}(t+1) > v_{i_0}(t)$.

(b) The case of $l \in \mathcal{V}_2(t)$ can be proved like (a) by considering two scenarios $\mathcal{V}_1(t) \cap \mathcal{N}_l^{1+}(t) \neq \emptyset$ and $\mathcal{V}_1(t) \cap \mathcal{N}_l^{1+}(t) = \emptyset$. We know there is some $i_0 \in \mathcal{V}_1(t)$ such that $v_{i_0}(t+1) \leq v_{i_0}(t)$ and with positive probability $v_{i_0}(t+1) < v_{i_0}(t)$.

(c) We first consider the two situations $\mathcal{V}_2(t) \cap \mathcal{N}_l^{1+}(t) \neq \emptyset$ and $\mathcal{V}_2(t) \cap \mathcal{N}_l^{1+}(t) = \emptyset$. Arguing similarly as in (a), there exists some agent $i_0 \in \mathcal{V}_2(t)$ such that $v_{i_0}(t+1) \geq v_{i_0}(t)$ and with positive probability $v_{i_0}(t+1) > v_{i_0}(t)$. We then study the two scenarios $\mathcal{V}_1(t) \cap \mathcal{N}_l^{1+}(t) \neq \emptyset$ and $\mathcal{V}_1(t) \cap \mathcal{N}_l^{1+}(t) = \emptyset$. Likewise, we can show there is some $i_1 \in \mathcal{V}_1(t)$ such that $v_{i_1}(t+1) \leq v_{i_1}(t)$ and with positive probability $v_{i_1}(t+1) < v_{i_1}(t)$.

Since $\mathcal{G}(t)$ is finite, the above discussions (a), (b) and (c) indicate the gap $v_{\max} - v_{\min} \to 0$ in probability as t tends to infinity. This would conflict with (21) and the assumption that $v > 0$ with positive probability. Hence, we have proved the claim $v = 0$.

From the step (i) we know that $v_{\max}(t)$ and $v_{\min}(t)$ are bounded and monotonic sequences. Thus, there exists a constant $w \in \mathbb{R}$ such that $v_i(t) \to w$ as $t \to \infty$ almost surely for any $i \in \mathcal{C} \cup \{l\}$ by (21) and $v = 0$. It can be seen from (4) that w is determined as

$$w = (\Theta^{-1}x_\varphi)_{n-1} + \sum_{k=1}^{n-1} b_k (\Theta^{-1}x_\varphi)_{n-1-k}, \tag{22}$$

where $\Theta^{-1}x_\varphi = ((\Theta^{-1}x_\varphi)_{n-1}, (\Theta^{-1}x_\varphi)_{n-2}, \cdots, (\Theta^{-1}x_\varphi)_0)^{\mathrm{T}} \in \mathbb{R}^n$. Define the gap $\varepsilon_i(t) = z_i(t) - w/(1+\sum_{k=1}^{n-1} b_k)1_{n-1} \in \mathbb{R}^{n-1}$ for every agent $i \in \mathcal{C}$. Recall that $z_i(t)$ still satisfies (15) here. Applying the Lyapunov stability theory as in the proof of Theorem 1 gives rise to the almost sure convergence of $\varepsilon_i(t) \to 0_{n-1}$ as $t \to \infty$ for any $i \in \mathcal{C}$.

Therefore, we have $z_i(t) \to w/(1+\sum_{k=1}^{n-1} b_k)1_{n-1} \in \mathbb{R}^{n-1}$ for $i \in \mathcal{C}$ as $t \to \infty$. By (13), $y_i(t) \to w/(1+\sum_{k=1}^{n-1} b_k)1_n \in \mathbb{R}^n$ almost surely. By involving (22), the following limit holds for each cooperative agent $i \in \mathcal{C}$:

$$\begin{aligned} x_i(t) &\to \frac{w}{(1+\sum_{k=1}^{n-1} b_k)}\Theta 1_n \\ &= \Theta 1_n \frac{(b_0, b_1, b_2, \cdots, b_{n-1})\Theta^{-1}x_\varphi}{(1+\sum_{k=1}^{n-1} b_k)} = \Theta\Theta^{-1}x_\varphi = x_\varphi \end{aligned} \tag{23}$$

almost surely as $t \to \infty$. This means the almost sure tracking has been realized and the proof is complete. □

Remark 10. We note that the comments in Remarks 7-9 for leader-less consensus can also be applied here. Namely, if the weights in the adjacency matrix $\mathcal{A}(t)$ are independent, the probabilistic robustness condition in Theorem 2 can be converted to requiring the digraph $(\mathcal{V}, \tilde{\mathcal{E}}(t))$ being leader-follower $(2r+1)$-robust under ℓ-hop communication w.r.t. $\mathcal{B}$, where $\tilde{\mathcal{E}}(t) = \{(i,j) \in \mathcal{V} \times \mathcal{V} : q_{ij}(t) > 0\}$. Moreover, when $\mathcal{B}$ is unknown or partly unknown, we can verify the robustness condition of $\mathcal{G}(t)$ with respect to every possible set $\mathcal{S} \subseteq \mathcal{V}$ with $|\mathcal{S}| = r$.

5. An illustrative example

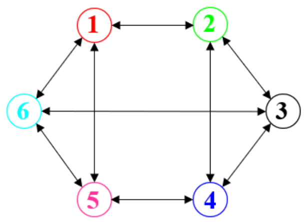

Figure 1: A network topology $\mathcal{G}$ over $N = 6$ agents with $\mathcal{B} = \{1\}$.

Consider a communication network $\mathcal{G}$ with $\mathcal{V} = \{1, 2, \cdots, 6\}$ and $\mathcal{B} = \{1\}$; see Fig. 1. We set $C = \mathcal{V} \backslash \mathcal{B}$ for the leaderless scenario; $l = 2$ and $C = \mathcal{V} \backslash (\mathcal{B} \cup \{l\})$ in the leader-follower scenario. The digraph $\mathcal{G}(t)$ is created by including each directed edge (j, i) with probability $q_{ij}(t) = 0.4$ (for $i < j$) and $q_{ij}(t) = 0.2$ (for $i > j$) independently and the weights $a_{ij}(t)$ is set as 1 if (j, i) is present and 0 otherwise. We note that $\mathcal{G}$ is not 3-robust under 1-hop communication. This can be seen by taking $\mathcal{S}_1 = \{1, 6\}$ and $\mathcal{S}_2 = \{2, 3, 4, 5\}$. However, it is 3-robust under 2-hop communication w.r.t. any set $\mathcal{S}$ with $|\mathcal{S}| = 1$. This demonstrates the significance of considering multi-hop communication in sparser networks.

Let $n = 2$. Write $x_i(t) = (x_{i,1}(t), x_{i,2}(t))^{\mathrm{T}}$ for $i \in \mathcal{V}$. The system (1) is chosen as

$$x_i(t+1) = \begin{pmatrix} 0.3 & 1 \\ 0 & 0.5 \end{pmatrix} x_i(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i(t), \tag{24}$$

where $u_1(t) = 0.5\sin(t/4)$ and $u_i(t)$ for $i \in C$ is given by (8) with $a_1 = -0.8$, $a_2 = 0.15$, $b_1 = 0.6$, $\theta_i(t) = 0.1$. At time t, the Byzantine agent 1 sends its state $x_1(t)$ to 2, $x_1(t) + \begin{pmatrix} \sin(t/4) \\ 1/2 \end{pmatrix}$ to 5, and $x_4(t) + \begin{pmatrix} \sin(t/20) \\ -\sin(t/20) \end{pmatrix}$ to 6. In the leader-follower case, we take $u_2(t) = \varphi(t) = -0.35$. Moreover, we take the function $f_{ij}(v_1, v_2) = v_1 - v_2$ in (8) for all i and j. Note that the matrix pair (A, B) is controllable and $x_\varphi = (-1, -0.7)^{\mathrm{T}}$. Set $r = 1$ and $\ell = 2$. It is direct to check that all conditions in Theorem 1 for the leaderless situation are satisfied. In the leader-follower situation, $\mathcal{G}$ is leader-follower 3-robust under 2-hop communication w.r.t. any set $\mathcal{S}$ with $|\mathcal{S}| = 1$. All conditions in Theorem 2 also hold.

We take initial conditions randomly in the range $[-3, 3]$. Fig. 2 shows the leaderless consensus in panels (a) and (b) and Fig. 3 shows leader-follower consensus in panels (a) and (b) as one would expect. In particular, the cooperative agents are able to track the leader agent 2 and converge to x_φ.

In addition, we show the convergence time $T := \min\{t : \max_{i \in C} \|x_i(t+1) - x_i(t)\| < 10^{-3}\}$ in the insets of Fig. 2(b) and Fig. 3(b) for different link probability $q := q_{ij}(t)$ for $i > j$ from $q = 0.2$ to 1. We observe that larger edge probabilities contribute to faster consensus in general, which complements our previous discussion (c.f. Remark 7).

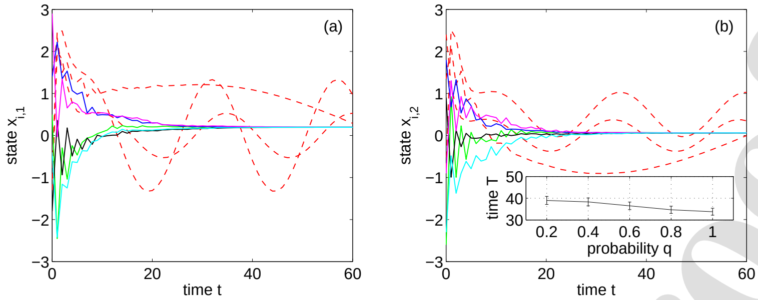

Figure 2: Resilient leaderless consensus of the first component (a) and the second component (b). The color code is in line with Fig. 1. The red dashed lines indicate the values sent by agent 1. The insets show the consensus time averaged over 100 independent runs with error bars indicating standard deviations.

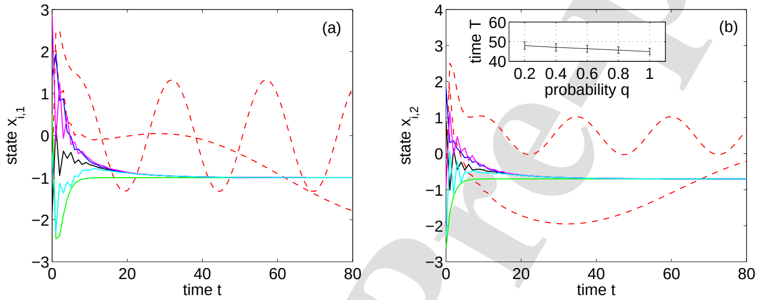

Figure 3: Resilient leader-follower consensus of the first component (a) and the second component (b). The color code is in line with Fig. 1. The red dashed lines indicate the values sent by agent 1. The insets show the consensus time averaged over 100 independent runs with error bars indicating standard deviations.

6. Conclusion

This paper presents a unified framework for achieving both resilient leaderless and leader-follower consensus under ℓ-hop communication over a general random network. We show that the multi-hop communication scheme is valuable when the multiagent systems are subject to both the attack of misbehaving agents and the failure of communication links. Some interesting problems for future study include issues related to asynchronous message transmission and communication delays. We will also investigate whether a weaker topology condition can resist milder malicious attacks.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: