Heliyon 10 (2024) e33841

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

On tricyclic graphs with maximum atom–bond sum–connectivity index

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A R T I C L E   I N F O

MSC:
05C07
05C09
05C90

Keywords:
Tricyclic graphs
Topological index
Extremal graph theory
Atom-bond sum-connectivity index

A B S T R A C T

The sum-connectivity, Randić, and atom-bond connectivity indices have a prominent place among those topological indices that depend on the graph's vertex degrees. The ABS (atom-bond sum-connectivity) index is a variant of all the aforementioned three indices, which was recently put forward. Let $T(n)$ be the class of all connected tricyclic graphs of order n. Recently, the problem of determining graphs from $T(n)$ having the least possible value of the ABS index was solved in (Zuo et al., 2024 [39]) for the case when the maximum degree of the considered graphs does not exceed 4. The present paper addresses the problem of finding graphs from $T(n)$ having the largest possible value of the ABS index for $n \geq 5$.

1. Introduction

In this paper, only finite and connected graphs are considered. By $V(G)$ and $E(G)$, we denote a graph G's vertex set and edge set, respectively. A (connected) graph is said to be a tricyclic graph if it has t vertices and $t+2$ edges. If a vertex μ of a graph G has degree one in G, then it is called a pendent vertex. A pendent edge is an edge incident to a pendent vertex. The maximum degree of a graph G is indicated by the symbol $\Delta(G)$. Let $N_\mu(G) = \{v \in V(G) : \mu v \in E(G)\}$. The degree of a vertex μ in a graph G is denoted by d_μ. The undefined terminology used in this article can be found in the graph theoretical books [8,9,14,35] and/or chemical-graph theoretical books [31,36].

A characteristic of a graph that is retained by isomorphism is called a graph invariant [14]. Real-valued graph invariants are often referred to as topological indices [36]. Topological indices are sometimes referred to as molecular descriptors [32,33]. The connectivity index, also known as the Randić (R) index, is considered the most researched and the most applied vertex-degree-based topological index. This index was developed by Milan Randić in 1975 [26] (see also [28]) as a graph-theoretical parameter for studying molecular branching of molecules. Details about the connectivity index can be found in the papers [18,19,27,29], the book [15] and related articles included therein.

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https://doi.org/10.1016/j.heliyon.2024.e33841
Received 13 October 2023; Received in revised form 24 June 2024; Accepted 27 June 2024

Available online 3 July 2024

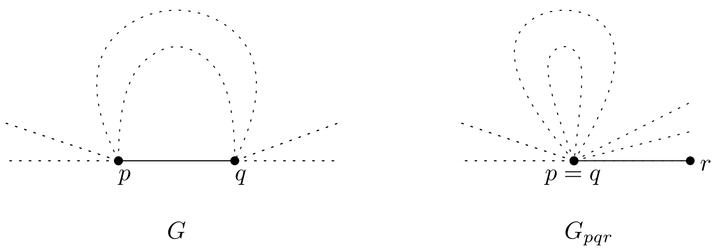

Fig. 1. The graphs G and G_{pqr} used in Lemma 1.

The popularity of the R index has resulted in the introduction of its numerous modified versions including the ABC (atom–bond connectivity) index [10,11] and SC (sum–connectivity index [38] (see also [21]). The ABS (atom–bond sum–connectivity) index [2] (a variant of the R index, SC index, and ABC index) is defined as

$$ABS(G) = \sum_{\mu\nu \in E(G)} \sqrt{1 - \frac{2}{d_\mu + d_\nu}}.$$

We remark here that the ABS index can be recovered from a more general topological index that appeared in [34].

Since the appearance of the ABS index, several researchers have been interested in solving the problems of extremal nature related to the ABS index. Ali et al. [4] investigated the extreme values of the ABS index of fixed-order unicyclic graphs and examined the chemical applications of this index. Nithya et al. [23] investigated the least value of the ABS index for the fixed-order (and fixed girth) unicyclic graphs and they also conducted the study on the chemical applications of the ABS index, and they found that the ABS index has significant potential in the structure-property modeling for cyclic compounds. Hu and Wang [16] reported some results involving the maximum values of the ABS index of trees. Ali et al. [5] initiated the study on the difference between ABS and ABC indices. Huang et al. [17] conducted a study on the ABS index of the molecular graphs of dendrimers. Gowtham and Gutman [13] initiated the study on the difference between ABS and SC indices. Extremal results on the ABS index of fixed-order trees with a given number of pendent vertices can be found in [22,25,30]. Additional details on the study of the ABS index can be found in the survey paper [3] and other recent papers [6,20,37].

In the present paper, we are concerned with the maximum value of the ABS index of fixed-order tricyclic graphs. The study of this kind of problem has been the topic of recent publications on the ABS index. The problem of determining the maximum and minimum values of the ABS index of fixed-order trees (graphs containing no cycles) was solved in [2]; also, a similar problem for the unicyclic graphs was solved in [4]. For other similar problems on the ABS index of unicyclic graphs, see [12,23,24]. The unique graph possessing the maximum ABS index among fixed-order bicyclic graphs was characterized in [1]. The problem of determining graphs having the least possible value of the ABS index among fixed-order trees, unicyclic graphs, bicyclic graphs, and tricyclic graphs of maximum degree at most 4 was addressed in [39]. Motivated by the recent developments in the theory of the extremum ABS index of cyclic graphs, we in the present paper address the problem of finding the maximum ABS index of fixed-order tricyclic graphs.

2. Results

Consider a non-trivial graph G and an edge $xy \in E(G)$. A graph G' is said to be obtained from G by "contracting xy" if G' can be formed from $G - y$ by joining x to every neighbor of y not already adjacent to x. The following result follows from Lemma 2.1 of [4].

Lemma 1. *Consider an n-order tricyclic graph G. Let p and q be a pair of non-pendent vertices of G such that $pq \in E(G)$ and they do not lie on a triangle. Let G_{pqr} be a graph obtained from G by contracting pq and inserting a new pendent vertex r adjacent to p, see Fig. 1. Then G_{pqr} is also an n-order tricyclic graph and $ABS(G_{pqr}) > ABS(G)$.*

Note that the least value of n for which at least one n-order tricyclic graph exists is 4. But, there is only one 4-order tricyclic graph, namely, the complete graph K_4. Hence, in what follows, we concentrate on tricyclic graphs of order at least 5.

Lemma 2. *If G is a graph with the maximum ABS index among all n-order tricyclic graphs then G is one of the graphs depicted in Figs. 2, 4 and 5, where $n \geq 5$.*

Proof. By Lemma 1, every non-pendent edge of G lies on a triangle. Hence the result holds. ■

By a cactus graph, we mean a graph whose every edge lies on at most one cycle. Note that the graphs H_1 and H_2 shown in Fig. 2 are cactus graphs. We will prove that

$$ABS(H_i) \leq ABS(G^\star(n,3)) \quad (1)$$

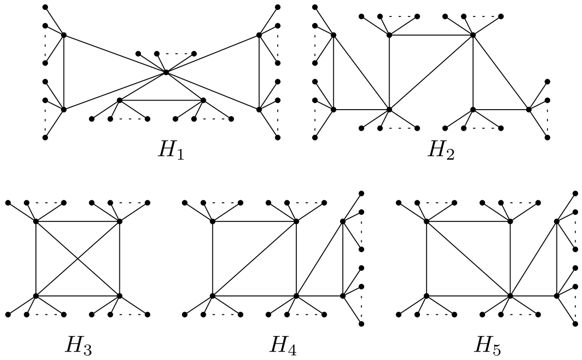

Fig. 2. The tricyclic graphs $H_1, H_2, \dots, H_5$.

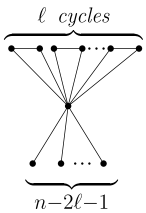

Fig. 3. The n-order cactus graph $G^\star(n,\ell)$ with ℓ cycles.

for each $i \in \{1,2\}$, where $G^\star(n,3)$ is the particular form of the graph depicted in Fig. 3 and each H_i has $n \geq 5$ vertices. Note that the ABS index of the graph $G^\star(n,\ell)$ shown in Fig. 3 is given as $ABS(G^\star(n,\ell)) = \Phi(n,\ell)$, where

$$\Phi(n,\ell) = \frac{\ell}{\sqrt{2}} + 2\ell\sqrt{\frac{n-1}{n+1}} + (n-2\ell-1)\sqrt{\frac{n-2}{n}}. \tag{2}$$

In order to show that Inequality (1) holds, we will prove a general form of this inequality. For this, we need some lemmas.

Lemma 3. [2] *If G is an n-order cactus graph containing no cycle, then*

$$ABS(G) \leq \Phi(n,0),$$

with equality if and only if $G \cong G^\star(n,0)$, where Φ is defined via (2) and the graph $G^\star(n,0)$ is a particular form of the graph shown in Fig. 3.

Observe that the class of n-order trees is the same as the class of n-order cactus graphs containing no cycle. Similarly, the class of n-order unicyclic graphs is the same as the class of n-order cactus graphs containing exactly one cycle.

Lemma 4. [4] *If G is an n-order cactus graph containing exactly one cycle, then*

$$ABS(G) \leq \Phi(n,1),$$

with equality if and only if $G \cong G^\star(n,1)$, where Φ is defined via (2) and the graph $G^\star(n,1)$ is a particular form of the graph shown in Fig. 3.

Lemma 5. [7] *Let $f(x,y) = \sqrt{\frac{x+y-2}{x+y}}$ with $x \geq 1$ and $y \geq 1$. Define*

$$g_s(x,y) = f(x+s,y) - f(x,y),$$

where $s > 0$. The function g_s is strictly decreasing in x and y.

Lemma 6. *Let $f(x,y) = \sqrt{\frac{x+y-2}{x+y}}$ with $x \geq 1$ and $y \geq 1$. Define*

$$\Upsilon(x,p) = p\,f(x,1) + (x-p)[f(x,2) - f(x-p,2)],$$

where $x \geq p+1 \geq 2$. The function Υ is strictly increasing in x.

Proof. The partial derivative Υ_x of Υ with respect to x is given as follows:

$$\Upsilon_x(x,p) = p f_x(x,1) + f(x,2) - f(x-p,2) + (x-p)[f_x(x,2) - f_x(x-p,2)],$$

where f_x denotes the partial derivative of f with respect to x. Note that

$$\frac{\partial}{\partial p}(f(x-p,2)) = -f_x(x-p,2) \quad \text{and} \quad \frac{\partial}{\partial p}\big(f_x(x-p,2)\big) = -f_{xx}(x-p,2).$$

Also, $f_{xx}(x,y) < 0$. Hence, the partial derivative Υ_{xp} of Υ_x with respect to p satisfies the following:

$$\begin{aligned}\Upsilon_{xp}(x,p) &= f_x(x,1) - f_x(x,2) + 2f_x(x-p,2) + (x-p)f_{xx}(x-p,2).\\ &> 2f_x(x-p,2) + (x-p)f_{xx}(x-p,2)\\ &= \frac{3}{(x-p+2)^{5/2}\sqrt{x-p}} > 0.\end{aligned}$$

Simple but lengthy calculations yield $\Upsilon_x(x,1) > 0$. Therefore,

$$\Upsilon_x(x,p) \geq \Upsilon_x(x,1) > 0,$$

as desired. ■

Theorem 1. *If G is an n-order cactus graph containing exactly ℓ cycle, then*

$$ABS(G) \leq \Phi(n,\ell),$$

with equality if and only if $G \cong G^{\star}(n,\ell)$, where Φ is defined via (2) and the graph $G^{\star}(n,\ell)$ is shown in Fig. 3.

Proof. The result will be proved using the induction on $n+\ell$. If $\ell = 0$ or $\ell = 1$, then the result follows from Lemma 3 or Lemma 4, respectively. Also, if $n = 5$ and $\ell \geq 2$, then the result again holds because there is exactly one 5-order cactus graph containing two cycles, namely, $G^{\star}(5,2)$. Now, assume that $n+\ell \geq 8$ with $n \geq 6$ and $\ell \geq 2$ such that the result holds for all p-order cactus graphs containing q cycles whenever $p+q < n+\ell$, where $p \geq 6$ and $q \geq 2$. In the rest of the proof, whenever any graph H different from G is under consideration such that $c \in V(H) \cap V(G)$, the notion d_c will represent the degree of c in G.

Case 1. The minimum degree of G is 1.
Let $w_0, u \in V(G)$ such that $w_0u \in E(G)$ with $d_{w_0} = 1$ and $N_G(u) = \{w_0, w_1, w_2, \dots, w_{x-1}\}$. Since $\ell \geq 2$, there exists some $i \in \{1,2,\dots,x-1\}$ such that $d_{w_i} \geq 2$. Suppose that $N_G(u)$ has exactly p vertices of degree 1. We may suppose that $d_{w_i} = 1$ for $0 \leq i \leq p-1$ and $d_{w_i} \geq 2$ for $p \leq i \leq x-1$. Let G' be the graph obtained from G by dropping $w_0, w_1, \dots, w_{p-1}$ (and all the edges incident with them). Evidently, G' has $n-p$ vertices and it contains exactly ℓ cycles, where $n-p \geq 5$ because $\ell \geq 2$. Hence, by the inductive hypothesis, we have

$$ABS(G') \leq \Phi(n-p,\ell), \tag{3}$$

with equality if and only if $G \cong G^{\star}(n-p,\ell)$. Now, observe that

$$ABS(G) - ABS(G') = p\sqrt{\frac{x-1}{x+1}} + \sum_{i=p}^{x-1}\left(\sqrt{\frac{d_{w_i}+x-2}{d_{w_i}+x}} - \sqrt{\frac{d_{w_i}+x-p-2}{d_{w_i}+x-p}}\right).$$

By Lemma 5, we have

$$\begin{aligned}ABS(G) - ABS(G') &= p\sqrt{\frac{x-1}{x+1}} + \sum_{i=p}^{x-1} g_p(d_{w_i}, x-p)\\ &\leq p\sqrt{\frac{x-1}{x+1}} + (x-p)\,g_p(2,x-p)\\ &= \Upsilon(x,p) \leq \Upsilon(n-1,p),\end{aligned} \tag{4}$$

where $\Upsilon(x,p)$ is defined in Lemma 6, and all the inequality signs in (4) are equality signs if and only if $x = n-1$ and $d_{w_i} = 2$ for every $i \in \{p,\dots,x-1\}$. By Mean-Value Theorem, there exists α_1 and α_2 such that

$$n-p-1 < \alpha_2 < n-p \leq n-1 < \alpha_1 < n$$

and

$$\begin{aligned}&\Phi(n-p,\ell) - \Phi(n,\ell) + \Upsilon(n-1,p)\\ =&(n-2\ell-p-1)\big(f(n,1) - f(n-1,1) + f(n-p-1,1) - f(n-p,1)\big)\end{aligned}$$

$$=(n-2\ell-p-1)\big(f_x(\alpha_1,1)-f_x(\alpha_2,1)\big), \tag{5}$$

where the function f is defined in Lemma 6 and f_x represents the partial derivative of f with respect to x. Since $f_{xx}(x,y)<0$ and $n-2\ell-p-1\geq 0$, from (5) we have

$$\Phi(n-p,\ell)-\Phi(n,\ell)+\Upsilon(n-1,p)\leq 0, \tag{6}$$

with equality if and only if $n-2\ell-p-1=0$. Now, in the present case, the desired result follows from (3), (4) and (6).

Case 2. The minimum degree of G is greater than 1.
Observe, in the current case, that there exist three vertices $z,z',z''\in V(G)$ on a cycle of G such that $z'z, z''z\in E(G)$, $d_z=2=d_{z'}$ and $d_{z''}=t\geq 3$.

Subcase 2.1. The vertices z' and z'' are not adjacent.
Consider the graph G'' obtained from G by dropping the vertex z and inserting the edge $z'z''$. Certainly, G'' has $n-1$ vertices and contains ℓ cycles. Hence, by the inductive hypothesis, we have

$$ABS(G'')\leq\Phi(n-1,\ell), \tag{7}$$

with equality if and only if $G\cong G^{\star}(n-1,\ell)$. Since $n\geq 6$, $n-2\ell-1\geq 0$ and $\ell\geq 2$, by using the fact that the function f is strictly increasing in both variables and utilizing (7), we have

$$\begin{aligned}
ABS(G)-\Phi(n,\ell)&=ABS(G'')+\frac{1}{\sqrt{2}}-\Phi(n,\ell)\\
&\leq\Phi(n-1,\ell)+\frac{1}{\sqrt{2}}-\Phi(n,\ell)\\
&=2\ell[f(1,n-1)-f(1,n)]+(n-2\ell-1)[f(1,n-2)-f(1,n-1)]\\
&\quad -f(1,n-2)+\frac{1}{\sqrt{2}}\\
&<-f(1,n-2)+\frac{1}{\sqrt{2}}<0.
\end{aligned}$$

Subcase 2.2. The vertices z' and z'' are adjacent.
Let $N_G(z'')=\{z,z',z_1,z_2,\dots,z_{r-2}\}$. Consider the graph G''' obtained from G by dropping the vertices z and z' (and all three edges incident with them). Certainly, G'' has $n-2$ vertices and it contains $\ell-1$ cycles. Hence, by the inductive hypothesis, we have

$$ABS(G''')\leq\Phi(n-2,\ell-1), \tag{8}$$

with equality if and only if $G\cong G^{\star}(n-2,\ell-1)$. Since $3\leq r\leq n-1$, it holds that

$$rf(r,2)-(r-2)f(r-2,2)\leq(n-1)f(n-1,2)-(n-3)f(n-3,2)$$

and hence by Lemma 5, we have

$$\begin{aligned}
ABS(G)-ABS(G''')&=\frac{1}{\sqrt{2}}+2f(r,2)+\sum_{i=1}^{r-2}\big(f(r,d_{z_i})-f(r-2,d_{z_i})\big)\\
&\leq\frac{1}{\sqrt{2}}+2f(r,2)+(r-2)\big(f(r,2)-f(r-2,2)\big)\\
&\leq\frac{1}{\sqrt{2}}+(n-1)f(n-1,2)-(n-3)f(n-3,2),
\end{aligned} \tag{9}$$

where all inequality signs in (9) become equality signs if and only if $r=n-1$ and $d_{z_i}=2$ for every $i\in\{1,2,\dots,r-2\}$. Now, there exist β_1 and β_2 such that

$$n-4<\beta_2<n-3<n-2<\beta_1<n-1$$

and

$$\begin{aligned}
&\Phi(n-2,\ell-1)-\Phi(n,\ell)+\frac{1}{\sqrt{2}}+(n-1)f(n-1,2)-(n-3)f(n-3,2)\\
&=(n-2\ell-1)\big(f(n-1,2)-f(n-2,2)-f(n-3,2)+f(n-4,2)\big)\\
&=(n-2\ell-1)\big(f_x(\beta_1,2)-f_x(\beta_2,2)\big)\leq 0,
\end{aligned} \tag{10}$$

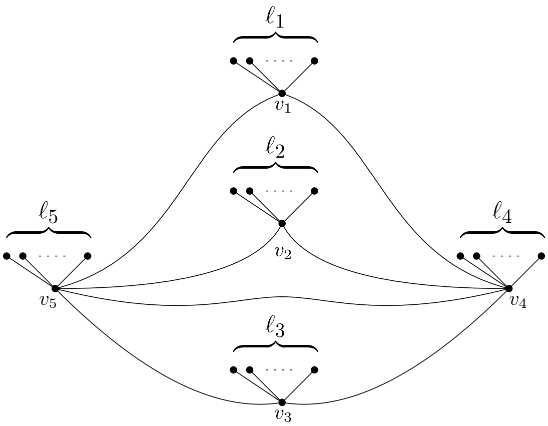

Fig. 4. The n-order tricyclic graph $B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$, where $\ell_4 \geq \ell_5$, $\sum_{i=1}^{5}\ell_i + 5 = n$, $\ell_1 \geq \max\{\ell_2,\ell_3\}$ and $\ell_i \geq 0$ for $i \in \{1,2,\ldots,5\}$.

where the equality in the last inequality holds if and only if $n - 2\ell - 1 = 0$. Now, in the current case, we obtain the desired result from (8), (9) and (10). ■

Theorem 1 implies the next result.

Corollary 1. *Inequality (1) holds. Equality in (1) holds if and only if $H_1 = G^\star(n,3)$.*

Let $K_{n-p}^{(p)}$ denote the graph formed by attaching p pendent vertices to a single vertex of the complete graph K_{n-p}. In [7], it was shown that $K_{n-p}^{(p)}$ uniquely attains the maximum value of the ABS index among all n-order graphs with p pendent vertices, provided that $p \leq n-3$. Hence, we have the following corollary:

Corollary 2. *If the graph H_3 depicted in Fig. 2 has $n \geq 4$ vertices then*

$$ABS(H_3) \leq ABS(K_4^{(n-4)}),$$

with equality if and only if $H_3 = K_4^{(n-4)}$.

Lemma 7. *Consider the function f defined in Lemma 6. If*

$$F(x_1,x_2,s,y) = f(x_1+s,y) - f(x_1,y) + f(x_2-s,y) - f(x_2,y),$$

where $x_1 \geq x_2 \geq s+1 \geq 2$ and $y \geq 1$, then F is strictly increasing in y.

Proof. We use the following notations concerning partial derivatives of f:

$$f_y = \frac{\partial f}{\partial y}, \quad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad f_{yxx} = \frac{\partial}{\partial x^2}\left(\frac{\partial f}{\partial y}\right).$$

Note that $f_{yxx} > 0$. Hence, by the mean value theorem, the partial derivative F_y of F with respect to y satisfies the following:

$$\begin{aligned} F_y(x_1,x_2,s,y) &= f_y(x_1+s,y) - f_y(x_1,y) - [f_y(x_2,y) - f_y(x_2-s,y)] \\ &= s[f_{yx}(c_1,y) - f_{yx}(c_2,y)] > 0, \end{aligned}$$

where $x_1 < c_1 < x_1+s$ and $x_2 - s < c_2 < x_2$. ■

Lemma 8. *Consider the graph $B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ shown in Fig. 4. If $\ell_5 \geq 1$ then*

$$ABS(B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)) < ABS(B_n(\ell_1,\ell_2,\ell_3,\ell_4+\ell_5,0)).$$

Proof. Take $G = B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ and $G' = B_n(\ell_1,\ell_2,\ell_3,\ell_4+\ell_5,0)$. Consider the function f defined in Lemma 6. By Lemma 7, we have

$$\begin{aligned}ABS(G) - ABS(G') &= \sum_{i=4}^{5} \ell_i[f(\ell_i+4,1) - f(\ell_4+4,1)]\\ &+ \sum_{i=1}^{3}[f(\ell_4+4,\ell_i) - f(\ell_4+4,\ell_i) + f(4,\ell_i) - f(4,\ell_i)]\\ &\le \sum_{i=4}^{5} \ell_i[f(\ell_i+4,1) - f(\ell_4+4,1)]\\ &+ 3[f(\ell_4+4,2) - f(\ell_4+4,2)] + 3[f(4,2) - f(4,2)]. \end{aligned} \tag{11}$$

Recall that f is strictly increasing in both variables and that $\ell_4 \ge \ell_5$. Now, define a function h of two real variables as follows

$$\begin{aligned}h(x_4,x_5) &= \sum_{i=4}^{5} x_i[f(x_i+4,1) - f(x_4+x_5+4,1)]\\ &+ 3[f(x_4+4,2) - f(x_4+x_5+4,2)] + 3[f(x_5+4,2) - f(4,2)],\end{aligned}$$

where $x_4 \ge x_5 \ge 1$. We observe that the function h is strictly decreasing in x_4 and thus $h(x_4,x_5) \le h(x_5,x_5) < 0$. Hence, from (11), we have $ABS(G) < ABS(G')$, as required. ■

Lemma 9. *Consider the graph $B_n(\ell_1,\ell_2,\ell_3,\ell_4,0)$ shown in Fig. 4 (with $\ell_5 = 0$). If $\ell_1 \ge 1$ then*

$$ABS(G) < ABS(G_1),$$

where $G = B_n(\ell_1,\ell_2,\ell_3,\ell_4,0)$ and $G_1 = B_n(0,0,0,\ell_1+\ell_2+\ell_3+\ell_4,0)$.

Proof. **Step 1.** Let G' be the graph formed from G by moving all the pendent edges incident with v_1 to v_4. Then the degree of v_1 in G' is 2 and the degree of v_4 in G' is $\ell_1+\ell_4+4$. Consider the function f defined in Lemma 6.

First, we assume that $\ell_4 \ge 1$. Since $\ell_1 \ge 1$ and f is strictly increasing in both variables, we have

$$\begin{aligned}ABS(G) - ABS(G') &= \sum_{i=2}^{3}[f(\ell_4+4,\ell_i+2) - f(\ell_1+\ell_4+4,\ell_i+2)]\\ &+ [f(\ell_4+4,4) - f(\ell_1+\ell_4+4,4)] + \ell_4[f(\ell_4+4,1) - f(\ell_1+\ell_4+4,1)]\\ &+ [f(\ell_1+2,4) - f(2,4)] + \ell_1[f(\ell_1+2,1) - f(\ell_1+\ell_4+4,1)]\\ &< [f(\ell_4+4,4) - f(\ell_1+\ell_4+4,4)] + \ell_4[f(\ell_4+4,1) - f(\ell_1+\ell_4+4,1)]\\ &+ [f(\ell_1+2,4) - f(2,4)] + [f(\ell_1+2,1) - f(\ell_1+\ell_4+4,1)]. \end{aligned} \tag{12}$$

Note that the right-hand side of (12) is negative for $\ell_1 \ge 1$ and $\ell_4 \ge 1$. Therefore, it follows that $ABS(G) < ABS(G')$.

Next, we assume that $\ell_4 = 0$. Then,

$$\begin{aligned}ABS(G) - ABS(G') &= \sum_{i=2}^{3}[f(4,\ell_i+2) - f(\ell_1+4,\ell_i+2)] + [f(4,4) - f(\ell_1+4,4)]\\ &+ [f(\ell_1+2,4) - f(2,4)] + \ell_1[f(\ell_1+2,1) - f(\ell_1+4,1)] < 0.\end{aligned}$$

Step 2. If $\ell_2 = 0$ then take $G'' = G'$ and if $\ell_2 \ne 0$ then let G'' be the graph formed from G' by moving all the pendent edges incident with v_2 to v_4. Hence, by Step 1, we have

$$ABS(G) < ABS(G') \begin{cases} = ABS(G'') & \text{when } \ell_2 = 0,\\ < ABS(G'') & \text{when } \ell_2 \ne 0.\end{cases}$$

Step 3. If $\ell_3 = 0$ then take $G''' = G''$ and if $\ell_3 \ne 0$ then let G''' be the graph formed from G'' by moving all the pendent edges incident with v_3 to v_4. Hence, by Steps 1 and 2, we have

$$ABS(G) < ABS(G') \le ABS(G'') \begin{cases} = ABS(G''') & \text{when } \ell_3 = 0,\\ < ABS(G''') & \text{when } \ell_3 \ne 0.\end{cases}$$

This completes the proof because $G''' = G_1$. ■

From Lemmas 8 and 9, the next result follows.

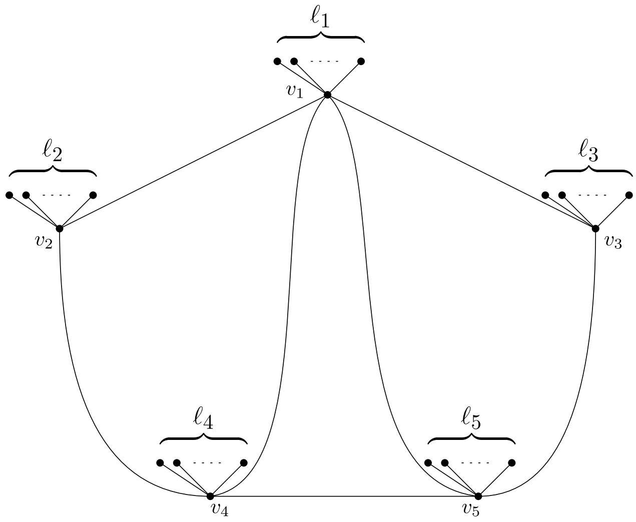

Fig. 5. The n-order tricyclic graph $G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$, where $\sum_{i=1}^{5}\ell_i+5=n$ and $\ell_i\geq 0$ for every $i\in\{1,2,\ldots,5\}$.

Corollary 3. *Consider the graph* $B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ *shown in Fig. 4. If*

$$\max\{\ell_1,\ell_2,\ell_3,\ell_5\}\geq 1$$

then

$$ABS(B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5))<ABS\left(B_n(0,0,0,n-5,0)\right).$$

By using the technique adopted to establish Corollary 3, we can prove the following lemma:

Lemma 10. *Consider the graph* $G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ *shown in Fig. 5. If*

$$\max\{\ell_2,\ell_3,\ell_4,\ell_5\}\geq 1$$

then

$$ABS(G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5))<ABS\left(G_n(n-5,0,0,0,0)\right).$$

Denote by $\hat{H}_5$ the particular form of the n-order graph H_5 (depicted in Fig. 2) such that the maximum degree of $\hat{H}_5$ is $n-1$. By using the technique adopted to establish Corollary 3, we can prove the following lemma:

Lemma 11. *Consider the graph* H_5 *depicted in Fig. 2. Then*

$$ABS(H_5)\leq ABS(\hat{H}_5),$$

with equality if and only if $H_5=\hat{H}_5$.

Lemma 12. *If* G *is an* n*-order tricyclic graph such that* $n\geq 5$ *and*

$$G\in\left\{G^\star(n,3),K_4^{n-4},\hat{H}_5,B_n(0,0,0,n-5,0),G_n(n-5,0,0,0,0)\right\},$$

then

$$ABS(G)\leq ABS\left(B_n(0,0,0,n-5,0)\right),$$

with equality if and only if $G=B_n(0,0,0,n-5,0)$.

Proof. The ABS index of each of the graphs belonging to the class

$$\left\{G^\star(n,3),K_4^{n-4},\hat{H}_5,B_n(0,0,0,n-5,0),G_n(n-5,0,0,0,0)\right\}$$

is calculated as follows:

$$ABS(G^{\star}(n,3)) = (n-7)\sqrt{\frac{n-2}{n}} + 6\sqrt{\frac{n-1}{n+1}} + \frac{3}{\sqrt{2}},$$

$$ABS(\hat{H}_5) = (n-6)\sqrt{\frac{n-2}{n}} + 4\sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n}{n+2}} + \frac{1}{\sqrt{2}} + 2\sqrt{\frac{3}{5}},$$

$$ABS\left(G_n(n-5,0,0,0,0)\right) = (n-5)\sqrt{\frac{n-2}{n}} + 2\sqrt{\frac{n-1}{n+1}} + 2\sqrt{\frac{n}{n+2}} + \sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{5}},$$

$$ABS\left(B_n(0,0,0,n-5,0)\right) = (n-5)\sqrt{\frac{n-2}{n}} + 3\sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n+1}{n+3}} + \sqrt{6},$$

$$ABS(K_4^{n-4}) = (n-4)\sqrt{\frac{n-2}{n}} + 3\sqrt{\frac{n}{n+2}} + \sqrt{6}.$$

Elementary comparisons of $ABS\left(B_n(0,0,0,n-5,0)\right)$ with each of the remaining four values quantities yield the required result. ■

Theorem 2. *Let G be an n-order tricyclic graph distinct from H_4 (shown in Fig. 2), with $n \geq 5$, then*

$$ABS(G) \leq ABS\left(B_n(0,0,0,n-5,0)\right),$$

with equality if and only if $G = B_n(0,0,0,n-5,0)$. In other words, among all n-order tricyclic graphs distinct from H_4, the graph $B_n(0,0,0,n-5,0)$ uniquely achieves the maximum ABS index for every $n \in \{5,6,7,\dots\}$.

Proof. Let $G^{\dagger}$ be a graph with the maximum ABS index among all n-order tricyclic graphs, where $n \geq 5$. By Lemma 2, $G^{\dagger}$ is of the form of the graphs depicted in Figs. 2, 4 and 5.

First, we consider the graphs shown in Fig. 2. By Corollary 1 and Lemma 12, we conclude that $G^{\dagger} \notin \{H_1, H_2\}$. From Corollary 2 and Lemma 12, it follows that $G^{\dagger} \neq H_3$. By the assumption of the theorem, $G^{\dagger} \neq H_4$. Finally, Lemmas 11 and 12 yield that $G^{\dagger} \neq \hat{H}_5$.

Next, we consider the graph $G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ shown in Fig. 5. Because of Lemmas 10 and 12, we have

$$ABS(G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)) \leq ABS\left(G_n(n-5,0,0,0,0)\right) < ABS\left(B_n(0,0,0,n-5,0)\right).$$

Hence, $G^{\dagger}$ is not of the form of $G_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$.

Last, we consider the graph $B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ shown in Fig. 4. By Corollary 3, we have

$$ABS(B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)) \leq ABS\left(B_n(0,0,0,n-5,0)\right),$$

with equality if and only if

$$B_n(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) = B_n(0,0,0,n-5,0).$$

Therefore, $G^{\dagger} = B_n(0,0,0,n-5,0)$. This proves the theorem ■

Funding statement

This research has been funded by Scientific Research Deanship, University of Ha'il - Saudi Arabia through project number RG-23 013.

CRediT authorship contribution statement

Sadia Noureen: Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Rimsha Batool:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Abeer M. Albalahi:** Writing – review & editing, Validation, Supervision, Formal analysis. **Yilun Shang:** Writing – review & editing, Validation, Supervision, Project administration. **Tariq Alraqad:** Writing – review & editing, Validation, Supervision, Methodology. **Akbar Ali:** Supervision, Project administration, Methodology.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yilun Shang, one of the coauthors of the present paper, is a section editor of the journal Heliyon where the present paper is being submitted. Other authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data is used in this study.

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