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On the Signless Laplacian ABC-Spectral Properties of a Graph

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Abstract: In the paper, we introduce the signless Laplacian ABC-matrix $\tilde{Q}(G) = \overline{D}(G) + \tilde{A}(G)$, where $\overline{D}(G)$ is the diagonal matrix of ABC-degrees and $\tilde{A}(G)$ is the ABC-matrix of G. The eigenvalues of the matrix $\tilde{Q}(G)$ are the signless Laplacian ABC-eigenvalues of G. We give some basic properties of the matrix $\tilde{Q}(G)$, which includes relating independence number and clique number with signless Laplacian ABC-eigenvalues. For bipartite graphs, we show that the signless Laplacian ABC-spectrum and the Laplacian ABC-spectrum are the same. We characterize the graphs with exactly two distinct signless Laplacian ABC-eigenvalues. Also, we consider the problem of the characterization of the graphs with exactly three distinct signless Laplacian ABC-eigenvalues and solve it for bipartite graphs and, in some cases, for non-bipartite graphs. We also introduce the concept of the trace norm of the matrix $\tilde{Q}(G) - \frac{tr(\tilde{Q}(G))}{n} I$, called the signless Laplacian ABC-energy of G. We obtain some upper and lower bounds for signless Laplacian ABC-energy and characterize the extremal graphs attaining it. Further, for graphs of order at most 6, we compare the signless Laplacian energy and the ABC-energy with the signless Laplacian ABC-energy and found that the latter behaves well, as there is a single pair of graphs with the same signless Laplacian ABC-energy unlike the 26 pairs of graphs with same signless Laplacian energy and eight pairs of graphs with the same ABC-energy.

Keywords: adjacency matrix; Laplacian (signless) matrix; ABC-matrix; Laplacian ABC-matrix; signless Laplacian ABC-matrix

MSC: 05C50; 05C92; 15A18

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1. Introduction

This paper delves into the analysis of connected, simple, and finite graphs. A graph $G = (V(G), E(G))$, denoted as G, encompasses a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$. The number of vertices in $V(G)$, denoted by n, defines the graph's order, while the number of edges in $E(G)$, denoted by m, signifies its size. The notation $u \sim v$ denotes adjacency between vertex u and vertex v, with the neighborhood $N(v)$ of a vertex v encompassing all vertices adjacent to $v \in V(G)$. The degree of a vertex v, denoted by $d_G(v)$ or simply d_v, reflects the count of vertices in its neighborhood. A graph is classified as r-regular if every vertex $v \in V(G)$ possesses a degree of r. The distance between two vertices represents the shortest path length, while the diameter of G denotes the maximum distance between any pair of vertices in the graph. For further graph theoretic notations and definitions, please refer to [1].

The adjacency matrix of a graph G, denoted as $A(G)$, is defined as follows:

$$A(G)_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Due to its real symmetric nature, the eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ of $A(G)$ are real and form the adjacency spectrum, or simply the spectrum, of G. The energy of the graph G, associated with its adjacency matrix, is defined as:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

The spectral parameter $\mathcal{E}(G)$ is an extensively studied metric, originating from theoretical chemistry for approximating the π-electron energy of hydrocarbons. For a detailed exploration of the energy $\mathcal{E}(G)$ associated with a graph G, see [2,3]. Further literature on the adjacency matrix $A(G)$ can be found in [4,5].

The ABC-matrix of a graph G is a square matrix of order n, defined as:

$$\tilde{A}(G) = (a_{ij})_{n\times n} = \begin{cases} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}} & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The sum of the absolute values of the eigenvalues of $\tilde{A}(G)$ is denoted by $E_{ABC}(G)$ and is called the ABC-energy of graph G, see [6]. The ABC-matrix introduced in [7] is associated with the atom-bond connectivity (ABC-index) of the graph G. The ABC-index, a degree-based topological index [8], is defined as the sum of weights $\sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}}$ over all edges v_iv_j in the graph G, given by:

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}}.$$

In [8], the ABC-index was shown to correlate with the heat of formation of alkanes. Gutman et al. [9] demonstrated that the ABC-index can reproduce the heat of formation with accuracy comparable to high-level *ab* initio and DFT (MP2, B3LYP) quantum chemical calculations. Further mathematical literature on the ABC-index and related results can be found in [10–15].

For any vertex $v_i \in V(G)$, we define the ABC-degree as the sum of certain weighted degrees over all vertices v_j adjacent to v_i. Mathematically, this is expressed as $\overline{d}vi = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}}$. Henceforth, to streamline notation, we simplify this to $\overline{d}i$, which conveniently corresponds to the i-th row sum of the ABC-matrix. This Laplacian ABC-matrix of G, first introduced in the seminal work by Yang et al. [16], is defined as $\mathrm{L}(G) = \overline{D}(G) - \mathrm{A}(G)$, where $\overline{D}(G) = \mathrm{diag}(\overline{d}1, \overline{d}2, \ldots, \overline{d}n)$ denotes the diagonal matrix consisting of the ABC-degrees of G. It is worth noting that every row sum of $\mathrm{L}(G)$ is precisely zero, a property that renders 0 as one of its eigenvalues. Additionally, owing to its construction, $\mathrm{L}(G)$ emerges as a real symmetric positive semi-definite matrix, endowing it with a set of eigenvalues referred to as the Laplacian ABC-eigenvalues of G. Denoted by η_i^{L} for $i = 1, 2, \ldots, n$, each of these eigenvalues is necessarily real, enabling us to organize them in descending order: $\eta_1^{\mathrm{L}} \geq \eta_2^{\mathrm{L}} \geq \cdots \geq \eta_{n-1}^{\mathrm{L}} > \eta_n^{\mathrm{L}} = 0$. This most significant eigenvalue, η_1^{L}, is of particular interest, often termed the Laplacian ABC-spectral radius. The entire set of eigenvalues of $\mathrm{L}(G)$ collectively constitutes the Laplacian ABC-spectrum of G. In situations where an eigenvalue η of the matrix M appears with multiplicity $\mu \geq 2$, it is conventionally denoted as $\eta^{[\mu]}$. The research conducted by the work [16] explored a plethora of properties associated with the matrix $\mathrm{L}(G)$, delving into characterizations of graphs possessing distinct Laplacian ABC-eigenvalues and deriving rigorous bounds for both the largest and second smallest Laplacian ABC-eigenvalues.

The Laplacian ABC-energy was introduced in [16] and is defined by

$$\mathcal{E}(\tilde{\mathrm{L}}(G)) = \sum_{i=1}^{n}\left|\eta_i^{\tilde{\mathrm{L}}} - \frac{2ABC(G)}{n}\right|. \quad (1)$$

For some recent works on Laplacian ABC-eigenvalues and related results, we refer to [17].

Motivated by the above works, we introduce the signless Laplacian ABC-matrix $\tilde{\mathrm{Q}}(G)$ for a graph G and is formally defined as $\tilde{\mathrm{Q}}(G) = \overline{D}(G) - \tilde{\mathrm{A}}(G)$. Clearly, the matrix $\tilde{\mathrm{Q}}(G)$ is a real symmetric non-negative matrix. We call the eigenvalues of $\tilde{\mathrm{Q}}(G)$ the signless Laplacian ABC-eigenvalues of G and we denote it by $\eta_1^{\tilde{\mathrm{Q}}} \geq \eta_2^{\tilde{\mathrm{Q}}} \geq \cdots \geq \eta_{n-1}^{\tilde{\mathrm{Q}}} \geq \eta_n^{\tilde{\mathrm{Q}}}$. The largest eigenvalue $\eta_1^{\tilde{\mathrm{Q}}}$ is called the signless Laplacian ABC-spectral radius of G. The multiset of eigenvalues of $\tilde{\mathrm{Q}}(G)$ is the signless Laplacian ABC-spectrum of G. In the rest of this paper, we focus on the spectral properties of the matrix $\tilde{\mathrm{Q}}(G)$. Formally, we discuss the following three problems for the matrix $\tilde{\mathrm{Q}}(G)$.

(i). Characterization of matrices with k distinct eigenvalues is an interesting but hard problem in matrix theory. We consider this problem for the matrix $\tilde{\mathrm{Q}}(G)$ when $k = 1, 2$, and $k = 3$. For $k = 1$ and $k = 2$, we completely solve this problem, see Theorem 1 and Corollary 2. For $k = 3$, we provide a partial solution to this problem by solving it for all bipartite graphs and some non-bipartite graphs, see Theorem 7.

(ii). The sum of all singular values of a matrix is called the trace norm of the matrix. One of the fundamental problems in matrix theory is to determine among a class of matrices the matrices that attain the minimum and the matrices that attain the maximum value for the trace norm. We consider this problem for the matrix $\tilde{\mathrm{Q}}(G) - \frac{2ABC(G)}{n}I_n$, where $ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i}d_{v_j}}}$ is the ABC-index of G, and obtain some upper and lower bounds for its trace norm. We characterize the graphs that attain these upper and lower bounds.

(iii). Another important problem regarding the trace norm of a matrix is that "matrices are determined by their trace norm". This problem has been studied for many graph matrices; here, we consider this problem for the trace norm of the matrix $\tilde{\mathrm{Q}}(G) - \frac{2ABC(G)}{n}I_n$. We show for graphs with at most 6 vertices, the trace norm of the matrix $\tilde{\mathrm{Q}}(G) - \frac{2ABC(G)}{n}I_n$ behaves well in differentiating between non-isomorphic graphs.

We denote the complete graph by K_n, the complete bipartite graph by $K_{a,b}$, the path of order n by P_n, and the cycle of order n by C_n, among others. For other notations and terminology from spectral graph theory, please refer to [5].

The remainder of the paper is structured as follows. In Section 2, we explore fundamental properties of $\tilde{Q}(G)$. Specifically, we demonstrate that for bipartite graphs, the signless Laplacian ABC-spectrum coincides with the corresponding Laplacian ABC-spectrum. Additionally, we provide a characterization of graphs with exactly two distinct signless Laplacian ABC-eigenvalues. Furthermore, we address the problem of characterizing graphs with exactly three distinct signless Laplacian ABC-eigenvalues, solving it for bipartite graphs and, in some instances, for non-bipartite graphs.

In Section 3, we introduce the concept of the signless Laplacian ABC-energy of a graph. We derive upper and lower bounds for the signless Laplacian ABC-energy and characterize extremal graphs for these bounds. Section 4 presents computational results for graphs of order up to 6. We compare the signless Laplacian energy and the ABC-energy, finding that the former behaves well as there is a single pair of graphs with the same signless Laplacian ABC-energy, unlike the 26 pairs of graphs with the same signless Laplacian energy and 8 pairs of graphs with the same ABC-energy.

2. Graphs with at Most Three Distinct Signless Laplacian ABC-Eigenvalues

In this section, we present fundamental results concerning the signless Laplacian ABC-eigenvalues. We derive the signless Laplacian ABC-spectrum for several well-known families of graphs. Additionally, we provide a complete solution to the problem of characterizing graphs with two distinct signless Laplacian ABC-eigenvalues. Furthermore, we explore the characterization of graphs with exactly three distinct signless Laplacian ABC-eigenvalues, solving it for bipartite graphs and, in certain instances, for non-bipartite graphs.

An intriguing problem in the spectral theory of graph matrices arises naturally:

Problem 1. *Let G be a connected graph of order $n \geq 2$ and let $M(G)$ be a graph matrix associated with G. For a positive integer k, where $1 \leq k \leq n$, characterize the graphs having exactly k distinct $M(G)$-eigenvalues.*

This problem has been extensively studied for various matrices such as the adjacency matrix, the normalized Laplacian matrix, and the distance matrix, among others, particularly for small values of k, as seen in [15,18–22]. It has applications in cooperative coordination [23,24]. Numerous papers in the literature address this problem for the mentioned matrices when $k \leq 4$, see [6,25–27] and references therein. Although generally challenging, solutions exist for small values of k. In this section, we explore this problem specifically for the signless Laplacian ABC-matrix for $k \leq 3$.

Consider any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, which can be interpreted as a function defined on $V(G)$, relating each v_i to x_i, i.e., $X(v_i) = x_i$ for all $i = 1, 2, \ldots, n$. Moreover, we observe that

$$X^T\tilde{Q}(G)X = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}\,(x_i + x_j)^2 = \sum_{i=1}^{n} \overline{d}_i x_i^2 + 2 \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} x_i x_j,$$

where $\overline{d}_i = \overline{d}_{v_i} = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i} d_{v_j}}}$. A real number ξ is a signless Laplacian ABC-eigenvalue with its associated eigenvector $X \neq 0$ if and only if X and for every $v_i \in V(G)$, we have

$$\xi X(v_i) = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}\big(X(v_i) + X(v_j)\big), \tag{2}$$

or equivalently

$$\xi X(v_i) - \overline{d}_i X(v_i) = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}\, X(v_j), \tag{3}$$

Equations (2) and (3) represent the (ξ, X)-eigen-equations for the signless Laplacian ABC-matrix.

We next present the following useful lemma.

Lemma 1. *Consider a connected graph G with an eigenvector $X = (x_1, x_2, \ldots, x_n)^T$ corresponding to the eigenvalue ξ of the signless Laplacian matrix. Suppose there exist vertices u and v in $V(G)$ such that $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Under this condition, it follows that the values assigned to vertices u and v by the eigenvector, denoted as x_u and x_v respectively, are equal.*

Proof. For $u, v \in V(G)$ with $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, it follows that $\overline{d}_u = \overline{d}_v$, so by Equation (3), we have

$$\xi x_u - \overline{d}_u x_u = \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} x_v + \sum_{v_j \in V(G), v_j \neq v, u} \sqrt{\frac{d_u + d_{v_j} - 2}{d_u d_{v_j}}}\, x_j,$$

and

$$\xi x_v - \overline{d}_v x_v = \sqrt{\frac{d_v + d_u - 2}{d_u d_v}} x_u + \sum_{v_j \in V(G), v_j \neq v,u} \sqrt{\frac{d_v + d_{v_j} - 2}{d_v d_{v_j}}}\, x_j$$

As $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, so $\sqrt{\frac{d_v+d_{v_j}-2}{d_v d_{v_j}}} = \sqrt{\frac{d_u+d_{v_j}-2}{d_u d_{v_j}}}$, for $v_j \neq u, v$. Therefore, it follows that

$$\xi(x_u - x_v) = \sqrt{\frac{d_v + d_u - 2}{d_u d_v}}(x_u - x_v),$$

which implies that $x_u = x_v$. □

Since for the graph $G \cong kK_2 \cup (n-k)K_1$, where $0 \leq k \leq \frac{n}{2}$, the signless Laplacian ABC-matrix is a zero matrix, it follows that the signless Laplacian ABC-spectrum of this graph is $\{0^{[n]}\}$ and so it has just one distinct signless Laplacian ABC-eigenvalue. On the other hand, suppose that G has one distinct signless Laplacian ABC-eigenvalue. Then, by Theorem 3, the matrix $\tilde{Q}(G)$ is positive definite when G is a non-bipartite graph and positive semi-definite when G is a bipartite graph. It follows that if G has one distinct signless Laplacian ABC-eigenvalue, then G must be bipartite (as if G is non-bipartite then $\tilde{Q}(G)$ being a positive matrix, which is positive definite, has at least two distinct eigenvalues by the Perron Frobinus Theorem). Now, using Theorem 3.2 of [16], we arrive at G having one distinct signless Laplacian ABC-eigenvalue if and only if $G \cong kK_2 \cup (n-k)K_1$. Thus, we have the following result.

Theorem 1. *A graph G of order n has one distinct signless Laplacian ABC-eigenvalue if and only if $G \cong kK_2 \cup (n-k)K_1$, where $0 \leq k \leq \frac{n}{2}$.*

The following well-known result of Brouwer and Heamers [4] provides a relationship between the number of distinct eigenvalues in a graph and its diameter.

Theorem 2 ([4])**.** *Consider a connected graph G with diameter D. The graph G possesses at least $D+1$ distinct eigenvalues for its adjacency matrix, at least $D+1$ distinct eigenvalues for its Laplacian matrix, and at least $D+1$ distinct eigenvalues for its signless Laplacian matrix.*

The proof, as presented in [4], establishes the universality of this result for any non-negative symmetric matrix $M = (m_{ij})$, indexed by the set of vertices of a graph G, where $m_{ij} > 0$ if and only if v_i is adjacent to v_j. As a direct consequence, we derive the following corollary.

Corollary 1. *For a graph G with diameter D and possessing k distinct eigenvalues for its signless Laplacian ABC-matrix, it follows that $k \geq D+1$.*

Another immediate implication of Corollary 1 is the subsequent result, which asserts that K_n is the sole graph exhibiting two distinct signless Laplacian ABC-eigenvalues.

Corollary 2. *For any connected graph G with order $n \geq 2$, G possesses precisely two distinct signless Laplacian ABC-eigenvalues if and only if $G \cong K_n$.*

Proof. The signless Laplacian ABC-matrix of K_n is

$$\begin{pmatrix} \sqrt{2n-4} & \frac{\sqrt{2n-4}}{n-1} & \cdots & \frac{\sqrt{2n-4}}{n-1} \\ \frac{\sqrt{2n-4}}{n-1} & \sqrt{2n-4} & \cdots & \frac{\sqrt{2n-4}}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2n-4}}{n-1} & \frac{\sqrt{2n-4}}{n-1} & \cdots & \sqrt{2n-4} \end{pmatrix},$$

and it can be easily verified that its spectrum consists of the eigenvalue $2\sqrt{2n-4}$ and the eigenvalue $\frac{n-2}{n-1}\sqrt{2n-4}$ with multiplicity $n-1$. Thus, K_n has exactly two distinct signless Laplacian ABC-eigenvalues.

Conversely, if G is a connected graph with exactly two distinct signless Laplacian ABC-eigenvalues, from Corollary 1, its diameter is 1 and hence G must be K_n. □

For the bipartite graphs, it is well-known that the Laplacian and signless Laplacian matrices have the same spectrum. Next, we show that Laplacian ABC and the signless Laplacian ABC-matrices also enjoy this property.

Theorem 3. *Let G be a bipartite graph of order $n \geq 2$. Then, $\tilde{Q}(G)$ and $\tilde{L}(G)$ are unitarily similar and share the same spectrum. Further, for connected bipartite graphs, the Laplacian ABC-spectral radius is simple with a positive eigenvector.*

Proof. As G is bipartite, $V(G)$ can be partitioned into two subsets V_1 and V_2 such that no two vertices in V_1 and V_2 are adjacent. Let $S = (s_{ii})_{n\times n}$ be the diagonal matrix, where s_{ii} is 1, if $v_i \in V_1$ and -1, otherwise. It is easy to see that $S = S^{-1}$. Since G is bipartite, its ABC-matrix can be written as $\tilde{A}(G) = \begin{pmatrix} O & -B \\ -B^T & 0 \end{pmatrix}$, where the matrix B represents the part of $\tilde{A}(G)$ corresponding to vertices in V_1 and V_2. We have

$$S\tilde{A}(G)S^{-1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} O & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} O & -B \\ -B^T & 0 \end{pmatrix} = -\tilde{A}(G).$$

Also, $S\overline{D} = \overline{D}S$. Therefore, we have $S\tilde{Q}(G)S^{-1} = S(\overline{D}(G) + \tilde{A}(G))S^{-1} = S\overline{D}(G)S^{-1} + S\tilde{A}(G)S^{-1} = SS^{-1}\overline{D}(G) - \tilde{A}(G) = \overline{D}(G) - \tilde{A}(G) = \tilde{L}(G)$. Thus, it follows that $\tilde{Q}(G)$ and $\tilde{L}(G)$ are unitarily similar and share the same spectrum. Since for connected bipartite graphs, the matrix $\tilde{Q}(G)$ is non-negative and irreducible, it follows by the Perron Frobenius Theorem and the above fact that the Laplacian ABC-spectral radius is a simple eigenvector corresponding to it is positive. This completes the proof. □

From the equation

$$X^T\tilde{Q}(G)X = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i}d_{v_j}}}\,(x_i + x_j)^2,$$

it is clear that the signless Laplacian ABC-matrix is a positive semi-definite matrix. So, all of its eigenvalues are non-negative. Concerning the least eigenvalue, we have the following result.

Theorem 4. *For a connected graph of order $n \geq 3$, the least eigenvalue of the signless Laplacian ABC-matrix of a connected graph is equal to 0 if and only if the graph is bipartite. Moreover, 0 is a simple eigenvalue.*

Proof. Let $R(G) = (r_{ie})$ be the vertex-edge incidence matrix of G with $r_{ie} = \left(\frac{d_i+d_j-2}{d_id_j}\right)^{\frac{1}{4}}$ whenever e is an edge between the i-th and j-th vertices and $r_{ie} = 0$, otherwise. It is easy to verify that $R(G)R(G)^T = \overline{D}(G) + \tilde{A}(G) = \tilde{Q}(G)$. Consider a real non-zero vector $X = (x_1, x_2, \ldots, x_n)^T$. We have that $\tilde{Q}(G)X = 0$ implies that $R^TX = 0$. The later equality gives that $(x_i + x_j)\left(\frac{d_i+d_j-2}{d_id_j}\right)^{\frac{1}{4}} = 0$, for every edge $v_iv_j \in E(G)$. Since G is a connected graph of order $n \geq 3$ from this last equality, we get $x_i = -x_j$, for every edge $v_iv_j \in E(G)$. Clearly, $x_i = -x_j$, for every edge $v_iv_j \in E(G)$ holds if G is a bipartite graph. Conversely, if G is a bipartite graph, then by Theorem 3, it is clear that 0 is a signless Laplacian ABC-eigenvalue of G.

Again, using Theorem 3, it follows that the multiplicity of the eigenvalue 0 is the same as its multiplicity as the Laplacian ABC-eigenvalue of G, which is one, as a G connected graph. This completes the proof. □

The following observation is immediate from Theorem 4.

Corollary 3. *Let G be a graph of order n. Let $\kappa_b(G)$ be the number of bipartite components of G. Then, the multiplicity of the eigenvalue 0 of the signless Laplacian ABC-matrix is equal to $\kappa_b(G)$.*

A subset $S \subseteq V(G)$ comprising of pairwise non-adjacent vertices is termed an independent set. If every pair of vertices in S is adjacent in G, S is referred to as a *clique*. The independence number of G denotes the cardinality of the largest independent set, while the clique number represents the cardinality of the largest clique in G.

Hereafter, we present a result facilitating the determination of certain signless Laplacian ABC-eigenvalues, under the condition that G exhibits a specific structure.

Theorem 5. *Consider a connected graph G with vertex set $V(G) = v_1, v_2, \ldots, v_n$. Let $S = v_1, v_2, \ldots, v_I$ be a subset of G such that $N(v_i) - v_j = N(v_j) - v_i$, for all $i, j \in 1, 2, \ldots, I$. Denote by d and d the degree and the ABC-degree of any vertex in S, respectively. Then, the following statements hold.*

(i) *If S forms a clique in G, then $d - \frac{\sqrt{2d-2}}{d}$ is an eigenvalue of the signless Laplacian ABC-matrix of G with multiplicity at least $I - 1$.*

(ii) *If S is an independent set in G, then d is an eigenvalue of the signless Laplacian ABC-matrix of G with multiplicity at least $I - 1$.*

Proof. Suppose that S is a clique in G. Let us label the vertices of G in such a way that the first S vertices $\{v_1, v_2, \ldots, v_I\}$ are the vertices in S. Thus, it gives that $d_1 = d_2 = \cdots = d_I = d$. This last equality gives us $\tilde{\mathrm{d}}_1 = \tilde{\mathrm{d}}_2 = \cdots = \tilde{\mathrm{d}}_I = \tilde{\mathrm{d}}$. Under this labeling, the signless Laplacian ABC-matrix of G can be written as

$$\tilde{\mathrm{L}}(G) = \left(\begin{array}{cccc|c} \tilde{\mathrm{d}} & \frac{\sqrt{2d-2}}{d} & \cdots & \frac{\sqrt{2d-2}}{d} & \\ \frac{\sqrt{2d-2}}{d} & \tilde{\mathrm{d}} & \cdots & \frac{\sqrt{2d-2}}{d} & B_{p\times(n-p)} \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{\sqrt{2d-2}}{d} & \frac{\sqrt{2d-2}}{d} & \cdots & \tilde{\mathrm{d}} & \\ \hline \multicolumn{4}{c|}{(B_{p\times(n-p)})^T} & C_{(n-p)\times(n-p)} \end{array}\right).$$

For $i = 2, 3, \ldots, I$, let $\mathcal{X}_{i-1} = \Big(-1, x_{i2}, x_{i3}, \ldots, x_{ip}, \underbrace{0, 0, 0, \ldots, 0}_{n-p}\Big)^T$ be the vector in $\mathbb{R}^n$ such that $x_{ij} = 1$ if $i = j$ and 0 otherwise. Recalling that the rows of B are identical, we get

$$\tilde{\mathrm{Q}}(G)\mathcal{X}_1 = \Big(-\big(\tilde{\mathrm{d}} - \tfrac{\sqrt{2d-2}}{d}\big), \quad \tilde{\mathrm{d}} - \tfrac{\sqrt{2d-2}}{d}, \quad 0, \quad \ldots \quad ,0, \quad 0, \quad \ldots, \quad ,0\Big)^T = \Big(\tilde{\mathrm{d}} - \frac{\sqrt{2d-2}}{d}\Big)\mathcal{X}_1.$$

Similarly, it can be verified that $\mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_{p-1}$ are the eigenvectors of $\tilde{\mathrm{Q}}(G)$ corresponding to eigenvector $\tilde{\mathrm{d}} - \frac{\sqrt{2d-2}}{d}$. This completes the proof of **(i)**.

Next, if S is an independent set, where each vertex shares the same neighborhood, we have $d_1 = d_2 = \cdots = d_I$. This last equality gives us $\tilde{\mathrm{d}}_1 = \tilde{\mathrm{d}}_2 = \cdots = \tilde{\mathrm{d}}_I = \tilde{\mathrm{d}}$. We first index the vertices in the independent set, so that the signless Laplacian ABC-matrix of G can be written as

$$\tilde{Q}(G) = \left(\begin{array}{cccc|c} \tilde{d} & 0 & \dots & 0 & \\ 0 & \tilde{d} & \dots & 0 & B_{p\times(n-p)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \tilde{d} & \\ \hline & (B_{p\times(n-p)})^T & & & C_{(n-p)\times(n-p)} \end{array}\right).$$

With the same set of eigenvectors $\mathcal{X}_i$'s, it can be verified that $\tilde{d}$ is the signless Laplacian ABC-eigenvalue of G with multiplicity $I-1$. □

Theorem 5 provides valuable insights into determining the signless Laplacian ABC-eigenvalues of several well-known families of graphs. The following result highlights some of these families.

Proposition 1. *Let G be a connected graph of order $n \geq 4$. Then, the following statements hold.*

(i) *The signless Laplacian ABC-spectrum of $K_{a,b}$, with $n = a+b$ and $a, b \geq 1$, is*

$$\left\{0, n\sqrt{\frac{n-2}{ab}}, \left(a\sqrt{\frac{n-2}{ab}}\right)^{[b-1]}, \left(b\sqrt{\frac{n-2}{ab}}\right)^{[a-1]}\right\}.$$

(ii) *The signless Laplacian ABC-spectrum of $K_{1,n-1}$ is*

$$\left\{0, n\sqrt{\frac{n-2}{n-1}}, \left(\sqrt{\frac{n-2}{n-1}}\right)^{[n-2]}\right\}.$$

(iii) *The signless Laplacian ABC-spectrum of $CS_{\omega,n-\omega}$ is*

$$\left\{\left((\omega-2)\frac{\sqrt{2n-4}}{n-1} + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}\right)^{[\omega-1]}, \left(\omega\sqrt{\frac{n+\omega-3}{\omega(n-1)}}\right)^{[n-\omega-1]},\right.$$
$$\left.\frac{1}{2}\left(2\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1) + n\sqrt{\frac{n+w-3}{(n-1)w}} \pm \sqrt{D}\right)\right\},$$

where $D = \left(2\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1) + n\sqrt{\frac{n+w-3}{(n-1)w}}\right)^2 - 8\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1)w\sqrt{\frac{n+w-3}{(n-1)w}}$.

(iv) *The signless Laplacian ABC-spectrum of $K_n - e$, where e is an edge, is*

$$\left\{(n-2)\sqrt{\frac{2n-5}{(n-1)(n-2)}}, \left(\frac{(n-4)\sqrt{2n-4}}{n-1} + 2\sqrt{\frac{2n-5}{(n-1)(n-2)}}\right)^{[n-4]},\right.$$
$$\left.\frac{1}{2(n-1)}\left(2(n-3)\sqrt{2(n-2)} - 2(n-1)\sqrt{\frac{2n-5}{n^2-3n+2}} + \sqrt{D}\right)\right\}.$$

where $D = 8n^3 - 64n^2 + 168n - 144 + 2(16n^2 - 2n^3 - 36n + 24)\sqrt{\frac{2(n-2)(2n-5)}{n^2-3n+2}} + \frac{n^2}{n^2-3n+2}(2n^3 - 9n^2 + 12n - 5)$.

(v) *The signless Laplacian ABC-spectrum of $S_n^+ \cong K_{1,n-1} + e$ is*

$$\left\{\sqrt{\frac{1}{2}}, \left(\sqrt{\frac{n-2}{n-1}}\right)^{[n-4]}, x_1, x_3, x_3\right\},$$

where $x_1 \geq x_2 \geq x_3$ are the zeros of the following polynomial

$$x^3 - x^2\left((n-2)\sqrt{\frac{n-2}{n-1}} + \frac{5}{\sqrt{2}}\right) + x\left(-2\sqrt{2}\sqrt{\frac{n-2}{n-1}} + 3n\sqrt{\frac{n-2}{2(n-1)}} + 2\right) - 2\sqrt{\frac{n-2}{n-1}},$$

and

$$x_1 \in (n-2, n-2),\ x_2 \in (2,3) \text{ and } x_3 \in (0,1).$$

Proof.

(i), (ii) Since $\tilde{Q}(G)$ and $\tilde{L}(G)$ are unitarily similar, and (i) and (ii) follows from [17].

(iii) As ω vertices of $CS_{\omega,n-\omega}$ form a clique in which any two vertices satisfy the condition in Theorem 5-(i), where $\tilde{d}_1 = \cdots = \tilde{d}_\omega = (\omega-1)\frac{\sqrt{2n-4}}{n-1} + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$. Thus, it follows that $(\omega-2)\frac{\sqrt{2n-4}}{n-1} + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$ is the signless Laplacian ABC-eigenvalue of $CS_{\omega,n-\omega}$ with multiplicity $\omega-1$. Also, the graph $CS_{\omega,n-\omega}$ has an independent set of $n-\omega$ vertices sharing the same neighborhood. So, $\omega\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$ is the signless Laplacian ABC-eigenvalue of $CS_{\omega,n-\omega}$ with multiplicity $n-\omega-1$. By Lemma 1, choosing the eigenvector $X = (\underbrace{x_1, x_1, \ldots, x_1}_{\omega\text{-times}}, \underbrace{x_2, x_2, \ldots, x_2}_{n-\omega\text{-times}})$. The eigenvector $\tilde{Q}(G)X = \xi X$, implies that

$$\begin{aligned}
\xi x_1 &= 2(\omega-1)\frac{\sqrt{2n-4}}{n-1} + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}x_1 + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}x_2,\\
\xi x_2 &= \omega\sqrt{\frac{n+\omega-3}{\omega(n-1)}}x_1 + \omega\sqrt{\frac{n+\omega-3}{\omega(n-1)}}x_2.
\end{aligned}$$

The solution of the above equations are

$$\frac{1}{2}\left(2\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1) + n\sqrt{\frac{n+w-3}{(n-1)w}} \pm \sqrt{D}\right)\Bigg\},$$

where $D = \left(2\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1) + n\sqrt{\frac{n+w-3}{(n-1)w}}\right)^2 - 8\sqrt{2}\sqrt{\frac{n-2}{n-1}}(w-1)w\sqrt{\frac{n+w-3}{(n-1)w}}$.

(iv) It follows from **(iii)**, with $\omega = n-2$.

(v) Let $v_1, v_2, v_3, v_4, \ldots, v_n$ be the vertex labeling of S_n^+, where $d_{v_1} = d_{v_2} = 2, d_{v_3} = n-1$ and remaining v_i's are vertices of degree $n-1$. Clearly, $\tilde{d}_1 = \tilde{d}_2 = \frac{2}{\sqrt{2}}$ and $\tilde{d}_4 = \cdots = \tilde{d}_n = \sqrt{\frac{n-2}{n-1}}$. By Theorem 5, $\frac{1}{\sqrt{2}}$ and $\sqrt{\frac{n-2}{n-1}}$ are the signless Laplacian eigenvalues S_n^+ with multiplicity 1 and $n-4$, respectively. By Lemma 1, choosing the eigenvector $X = (x_1, x_1, x_2, x_3, \ldots, x_n)$, the system of eigenequations are

$$\begin{aligned}
\xi x_1 &= \frac{3}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2,\\
\xi x_2 &= \frac{2}{\sqrt{2}}x_1 + \left(\frac{2}{\sqrt{2}} + (n-3)\sqrt{\frac{n-2}{n-1}}\right)x_2 + (n-3)\sqrt{\frac{n-2}{n-1}}x_3,\\
\xi x_3 &= \sqrt{\frac{n-2}{n-1}}x_2 + \sqrt{\frac{n-2}{n-1}}x_3.
\end{aligned}$$

The coefficient matrix of the right side of the above equations is

$$\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} + (n-3)\sqrt{\frac{n-2}{n-1}} & (n-3)\sqrt{\frac{n-2}{n-1}} \\ 0 & \sqrt{\frac{n-2}{n-1}} & \sqrt{\frac{n-2}{n-1}} \end{pmatrix}. \quad (4)$$

The characteristic polynomial of the above matrix is

$$p(x) = x^3 - x^2\left((n-2)\sqrt{\frac{n-2}{n-1}} + \frac{5}{\sqrt{2}}\right) + x\left(-2\sqrt{2}\sqrt{\frac{n-2}{n-1}} + 3n\sqrt{\frac{n-2}{2(n-1)}} + 2\right) - 2\sqrt{\frac{n-2}{n-1}}.$$

Let $x_1 \geq x_2 \geq x_3$ be the zeros of $p(x)$. Then, for $n \geq 4$, it can be easily seen that

$$p(0) < 0,\ p(1) > 0,\ p(2) > 0,\ p(3) < 0,\ p(n-2) < 0,\ p(n-1) > 0.$$

From the above calculations and by the intermediate value theorem, it follows that

$$x_1 \in (n-2, n-2),\ x_2 \in (2,3) \text{ and } x_3 \in (0,1).$$

□

The next result [26] states the distinct eigenvalues of an irreducible non-negative symmetric real matrix.

Theorem 6 ([26])**.** *Let M be an $n \times n$ irreducible non-negative symmetric matrix with real entries, and let a_1 be its maximum eigenvalue with the corresponding unit Perron–Frobenius eigenvector X. If M possesses k $(2 \leq k \leq n)$ distinct eigenvalues, then there exist $k-1$ real numbers $a_2, a_3, \ldots, a_n$ $(a_1 > a_2 > \cdots > a_n)$ such that*

$$\prod_{i=2}^{k}(M - a_i I_n) = \prod_{i=2}^{k}(a_1 - a_i)XX^T.$$

Moreover, $a_1 > a_2 > \cdots > a_k$ are precisely the k distinct eigenvalues of M.

The next observation is about the number of distinct signless Laplacian ABC-eigenvalues of a graph G.

Corollary 4. *Let G denote a connected graph of order $n \geq 3$, and let X represent the unit eigenvector corresponding to the signless Laplacian ABC-spectral radius $\eta_1^{\tilde{Q}}$. For k distinct signless Laplacian ABC-eigenvalues of G, where $2 \leq k \leq n$, there must exist $k-1$ real numbers $l_2, l_3, \ldots, l_k$ such that $\eta_1^{\tilde{Q}} > l_2 > l_3 > \cdots > l_k$. This condition holds true if and only if*

$$\prod_{i=2}^{k}\left(\tilde{Q}(G) - l_i I_n\right) = \prod_{i=2}^{k}(\eta_1^{\tilde{Q}} - l_i)XX^T.$$

These values $\eta_1^{\tilde{Q}}, l_2, l_3, \ldots, l_k$ precisely represent the k distinct signless Laplacian ABC-eigenvalues of G.

Proof. Note that the signless Laplacian $ABC(G)$ is an irreducible non-negative symmetric real matrix. Employing Theorem 6 for $\tilde{Q}(G)$, we have the desired result. □

Corollary 1 serves as a fundamental tool in characterizing graphs with distinct eigenvalues, playing a crucial role in solving Problem 1 for $k = 3$.

Corollary 5. *Let G be a connected graph of order $n \geq 3$. Let $\eta_1^{\tilde{Q}}$ be the signless Laplacian ABC-spectral radius of G with its associated unit eigenvector $X = (x_1, x_2, \ldots, x_n)^T$. Then, G has three distinct signless Laplacian ABC-eigenvalues $\eta_1^{\tilde{Q}} > \eta_2^{\tilde{Q}} > \eta_3^{\tilde{Q}}$ if and only if the following three conditions hold.*

(i) $\overline{d}_{v_i}^2 + \sum\limits_{v_j \in N(v_i)} \frac{(d_{v_i}+d_{v_j}-2)}{d_{v_i}d_{v_j}} - \overline{d}_{v_i}(\eta_1^{\tilde{Q}} + \eta_2^{\tilde{Q}}) = -\eta_2^{\tilde{Q}}\eta_3^{\tilde{Q}} + (\eta_1^{\tilde{Q}} - \eta_2^{\tilde{Q}})(\eta_1^{\tilde{Q}} - \eta_3^{\tilde{Q}})x_i^2$, *for every vertex v_i.*

(ii) $\sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}}(\overline{d}_{v_i} + \overline{d}_{v_j}) + \sum\limits_{v_k \in N(v_i)\cap N(v_j)} \sqrt{\frac{(d_{v_i}+d_{v_k}-2)d_{v_j}+d_{v_k}-2}{d_i d_k^2 d_{v_j}}} - (\eta_2^{\tilde{Q}} + \eta_3^{\tilde{Q}})\sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}} = (\eta_1^{\tilde{Q}} - \eta_2^{\tilde{Q}})(\eta_1^{\tilde{Q}} - \eta_2^{\tilde{Q}})x_i x_j - \eta_2^{\tilde{Q}}\eta_3^{\tilde{Q}}$, *for every pair of adjacent vertex v_i and v_j.*

(iii) $\sum\limits_{v_k \in N(v_i)\cap N(v_j)} \sqrt{\frac{(d_{v_i}+d_{v_k}-2)d_{v_j}+d_{v_k}-2}{d_i d_k^2 d_{v_j}}} = (\eta_1^{\tilde{Q}} - \eta_2^{\tilde{Q}})(\eta_1^{\tilde{Q}} - \eta_3^{\tilde{Q}})x_i x_j - \eta_2^{\tilde{Q}}\eta_3^{\tilde{Q}}$, *for every pair of non-adjacent vertex v_i and v_j.*

Proof. By Corollary 4, G has three distinct signless Laplacian ABC-eigenvalues $\eta_1^{\tilde{Q}}, \eta_2^{\tilde{Q}}$ and $\eta_3^{\tilde{Q}}$ if and only if the following equation holds

$$(\tilde{Q}(G))^2 - \tilde{Q}(G)(\eta_2^{\tilde{Q}} + \eta_3^{\tilde{Q}}) + \eta_2^{\tilde{Q}}\eta_3^{\tilde{Q}}I_n = (\eta_1^{\tilde{Q}} - \eta_2^{\tilde{Q}})(\eta_1^{\tilde{Q}} - \eta_3^{\tilde{Q}})XX^T.$$

Now, comparing the diagonal entries and the off-diagonal entries of the above equation, we get the desired result. □

Consider a matrix M partitioned into blocks, with Q representing the matrix whose entries are the average row sums (or column sums) of these blocks. This Q matrix is termed the quotient matrix. In cases where the row sums (or column sums) of each block in M are constant, the partition is deemed regular (or equitable), leading to Q being labeled a regular (or equitable) quotient matrix (as defined in [4]). While the eigenvalues of M generally interlace with those of Q, for partitions deemed regular, each eigenvalue (as discussed in [4,5]) of Q coincides with an eigenvalue of M.

Turning to graph theory, a bipartite graph is classified as (r,s)-semiregular if it comprises two partite sets X and Y, where the degree of each vertex in X is r and that of each vertex in Y is s. Meanwhile, a strongly regular graph with parameters (n, r, α, β), denoted as $srg(n, r, \alpha, \beta)$, is defined as an r-regular graph on n vertices, where each pair of adjacent vertices shares α common neighbors, and each pair of distinct non-adjacent vertices shares β common neighbors.

The following is another main theorem of this section and partially characterizes the connected graphs with exactly three distinct signless Laplacian ABC-eigenvalues.

Theorem 7. *Let G be a connected graph of order $n \geq 4$. Then, the following four statements hold:*

(i) *If the diameter of G is at least 3, then no graph has three distinct signless Laplacian ABC-eigenvalues.*

(ii) *For bipartite graphs, G has three distinct signless Laplacian ABC-eigenvalues if and only if it is either a star graph or a regular complete bipartite graph.*

(iii) *Complete multipartite graphs have three distinct signless Laplacian ABC-eigenvalues if and only if they are regular complete multipartite graphs.*

(iv) *Among unicyclic graphs, G has three distinct signless Laplacian ABC-eigenvalues if and only if it is either C_4 or C_5.*

(v) *Regular graphs ($G \neq K_n$) possess three distinct signless Laplacian ABC-eigenvalues if and only if they are strongly regular graphs.*

Proof.

(i) If the diameter of G is at least 3, then by Corollary 1, G has more than three distinct signless Laplacian ABC-eigenvalues.

(ii) For bipartite graphs, the signless Laplacian ABC-spectrum and the Laplacian ABC-spectrum coincide, and the result follows by (ii) of Theorem 2.5 of [17].

(iii) For a complete t-partite graph with $t \geq 3$, first suppose that $G \cong K_{\underbrace{p, p, \ldots, p}_{t\text{-times}}}$. Then there is t independent subsets sharing the same neighborhood such that each vertex has the same ABC-degree $\sqrt{2p(t-1)-2}$. So, by Theorem 5, we obtain the signless Laplacian ABC-eigenvalue $\sqrt{2p(t-1)-2}$ with multiplicity $pt-t$. By Lemma 1, choosing

$$X = (\underbrace{x_1, \ldots, x_1}_{p\text{-times}}, \underbrace{x_2, \ldots, x_2}_{p\text{-times}}, \ldots, \underbrace{x_t, \ldots, x_t}_{p\text{-times}}),$$

from the eigenequation $\tilde{Q}(G)X = \xi X$, the of system of equations is

$$\begin{aligned}
\xi x_1 &= \sqrt{2p(t-1)-2}x_1 + \frac{\sqrt{2p(t-1)-2}}{t-1}x_2 + \cdots + \frac{\sqrt{2p(t-1)-2}}{t-1}x_t, \\
\xi x_2 &= \frac{\sqrt{2p(t-1)-2}}{t-1}x_1 + \sqrt{2p(t-1)-2}x_2 + \cdots + \frac{\sqrt{2p(t-1)-2}}{t-1}x_t, \\
&\vdots \\
\xi x_t &= \frac{\sqrt{2p(t-1)-2}}{t-1}x_1 + \frac{\sqrt{2p(t-1)-2}}{t-1}x_2 + \cdots + \sqrt{2p(t-1)-2}x_t,
\end{aligned}$$

and the coefficient matrix of the right side of the above system of equations is

$$\begin{pmatrix}
\sqrt{2p(t-1)-2} & \frac{\sqrt{2p(t-1)-2}}{t-1} & \cdots & \frac{\sqrt{2p(t-1)-2}}{t-1} \\
\frac{\sqrt{2p(t-1)-2}}{t-1} & \sqrt{2p(t-1)-2} & \cdots & \frac{\sqrt{2p(t-1)-2}}{t-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\sqrt{2p(t-1)-2}}{t-1} & \frac{\sqrt{2p(t-1)-2}}{t-1} & \cdots & \sqrt{2p(t-1)-2}
\end{pmatrix}. \tag{5}$$

Now, it is easy to show that $2\sqrt{2p(t-1)-2}$ and $\frac{(t-2)\sqrt{2p(t-1)-2}}{t-1}$ are the eigenvalues of (5) with multiplicity 1 and $t-1$, respectively. This shows that $K_{p,p,\ldots,p}$ is the candidate graph with three distinct signless Laplacian ABC-eigenvalues. Next, for the complete multipartite graph $G \cong K_{p_1,p_2,\ldots,p_t}$, we show that G has more than three distinct signless Laplacian ABC-eigenvalues if all p_i's are different. To prove this, it is enough to show that the semi-regular complete multipartite graphs have more than three distinct signless Laplacian ABC-eigenvalues. Without loss of generality, assume that $G \cong K_{\underbrace{p, p, \ldots, p}_{l\text{-times}}, \underbrace{q, q \ldots, q}_{t-l\text{-times}}}$, $p \neq q$. We will show that G has more than three distinct signless Laplacian ABC-eigenvalues. Clearly, G is semiregular with two distinct degrees, say $d_1 = p(l-1) + (t-l)q$ and $d_t = lp + (t-l-1)q$. Also, the two distinct ABC-degrees are $\overline{d}_1 = (l-1)p\frac{\sqrt{2d_1-2}}{d_1} + (t-l)q\sqrt{\frac{d_1+d_t-2}{d_1 d_t}}$ and $\overline{d}_2 = (t-l-1)q\frac{\sqrt{2d_t-2}}{d_t} + lp\sqrt{\frac{d_1+d_t-2}{d_1 d_t}}$. By Theorem 5, $\overline{d}_1$ and $\overline{d}_2$ are the signless Laplacian

ABC-eigenvalues of G with multiplicity $l(p-1)$ and $(t-l)(q-1)$, respectively. The coefficient matrix of the eigenequation $\tilde{Q}(G)X = \xi X$, is

$$\left(\begin{array}{cccc|cccc}
\overline{d}_1 & \frac{\sqrt{2d_1-2}}{d_1} & \dots & \frac{\sqrt{2d_1-2}}{d_1} & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \\
\frac{\sqrt{2d_1-2}}{d_1} & \overline{d}_1 & \dots & \frac{\sqrt{2d_1-2}}{d_1} & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\sqrt{2d_1-2}}{d_1} & \frac{\sqrt{2d_1-2}}{d_1} & \dots & \overline{d}_1 & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \\
\hline
p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \overline{d}_t & \frac{\sqrt{2d_t-2}}{d_t} & \dots & \frac{\sqrt{2d_t-2}}{d_t} \\
p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \frac{\sqrt{2d_t-2}}{d_t} & \overline{d}_t & \dots & \frac{\sqrt{2d_t-2}}{d_t} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \dots & p\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & \frac{\sqrt{2d_t-2}}{d_t} & \frac{\sqrt{2d_t-2}}{d_t} & \dots & \overline{d}_t
\end{array}\right). \quad (6)$$

Choosing eigenvectors as in Theorem 5, it can be verified that $(l-2)p\frac{\sqrt{2d_1-2}}{d_1} + (t-l)q\sqrt{\frac{d_1+d_t-2}{d_1d_t}}$ with multiplicity $l-1$ and $(t-l-2)q\frac{\sqrt{2d_t-2}}{d_t} + lp\sqrt{\frac{d_1+d_t-2}{d_1d_t}}$ with multiplicity $t-l-1$ are the eigenvalues of (6). The other two eigenvalues of (6) are the eigenvalues of the following equitable matrix

$$\begin{pmatrix} 2(l-1)p\frac{\sqrt{2d_1-2}}{d_1} + (t-l)q\sqrt{\frac{d_1+d_t-2}{d_1+d_t}} & (t-l)q\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \\ lp\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & 2(t-l-1)q\frac{\sqrt{2d_t-2}}{d_t} + lp\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \end{pmatrix},$$

which has two distinct eigenvalues. Therefore, it follows that G has more than three distinct signless Laplacian ABC-eigenvalues.

(iv) If G is a unicyclic graph, then, as above, the diameter of G is 2. So, G must be one of the following: C_4, C_5, or S_n^+. By Proposition 1, the graph S_n^+ has more than three distinct signless Laplacian ABC-eigenvalues. The signless Laplacian ABC-spectrum of C_5 is

$$\left\{2.82843, (1.85123)^{[2]}, (0.270091)^{[2]}\right\},$$

and that of C_4 is

$$\left\{(2.82843)^{[2]}, (1.41421)^{[2]}, 0\right\},$$

and so the result follows in this case.

(v) If G is a k-regular graph, then $d_i = k$ for all i and so the signless Laplacian ABC-matrix $\tilde{Q}(G)$ becomes $\tilde{Q}(G) = \frac{\sqrt{2k-2}}{k}Q(G)$. This gives that $\tilde{Q}(G)$ has three distinct eigenvalues if and only if $Q(G)$ has three distinct eigenvalues. Now, using the fact that the regular graphs with three distinct signless Laplacian eigenvalues are precisely the strongly regular graphs, the result follows. This completes the proof.

□

Part (iii)–(v) of Theorem 7 provide insights suggesting that there might be more non-bipartite graphs with diameter 2 possessing three distinct signless Laplacian ABC-eigenvalues. Therefore, we propose the following problem:

Problem 2. *Completely characterize the non-bipartite graphs with diameter* 2 *and three distinct signless Laplacian ABC-eigenvalues.*

In the next result, we determine the signless Laplacian ABC-spectrum of the complete t-partite graph.

Proposition 2. *Let $G \cong K_{p_1,p_2,\ldots,p_t}$, be the complete t-partite graph of order n. Then, the signless Laplacian ABC-spectrum of G consists of the eigenvalues* $\sum_{i\neq j,j=1}^{t} p_j\sqrt{\frac{2n-p_i-p_j-2}{(n-p_i)(n-p_j)}}$ *with multiplicity $p_i - 1$, for $i = 1, 2, \ldots, t$ together with the eigenvalues of matrix* (7)

Proof. Let $G \cong K_{p_1,p_2,\ldots,p_t}$ be the complete t-partite graphs with at least one p_i greater or equal to 2 and let $\{v_{11}, v_{12}, \ldots, v_{1p_1}, v_{21}, v_{22}, \ldots, v_{2p_2}, \ldots, v_{t1}, v_{t2}, \ldots, v_{tp_t}, \}$ be the vertex indexing of G. Clearly, $d_{v_{11}} = d_{v_{12}} = \cdots = d_{v_{1p_1}} = d_1 = n - p_1$ and in general $d_i = n - p_i$, where $d_i = d_{v_{i1}} = d_{v_{i2}} = \cdots = d_{v_{ip_i}}$, for $i = 1, 2, \ldots, t$. Also, the ABC-degrees corresponding to d_i's are $\overline{d}_i = \sum_{i\neq j,j=1}^{t} p_j\sqrt{\frac{2n-p_i-p_j-2}{(n-p_i)(n-p_j)}}$ and by Theorem 5, $\overline{d}_i$ is the signless Laplacian ABC-eigenvalue of G with multiplicity $p_i - 1$, for $i = 1, 2, \ldots, t$. For the remaining t signless Laplacian eigenvalues of G, choosing X as in **(iii)** of Theorem 7, the coefficient matrix of eigenequation $\tilde{Q}(G)X = \xi X$ is

$$\begin{pmatrix} \overline{d}_1 & p_2\sqrt{\frac{d_1+d_2-2}{d_1d_2}} & \cdots & p_i\sqrt{\frac{d_1+d_i-2}{d_1d_i}} & \cdots & p_t\sqrt{\frac{d_1+d_t-2}{d_1d_t}} \\ p_1\sqrt{\frac{d_1+d_2-2}{d_1d_2}} & \overline{d}_2 & \cdots & p_i\sqrt{\frac{d_2+d_i-2}{d_2d_i}} & \cdots & p_t\sqrt{\frac{d_2+d_t-2}{d_2d_t}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_1\sqrt{\frac{d_1+d_i-2}{d_1d_i}} & p_2\sqrt{\frac{d_2+d_i-2}{d_2d_i}} & \cdots & \overline{d}_i & \cdots & p_t\sqrt{\frac{d_i+d_t-2}{d_id_t}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_1\sqrt{\frac{d_1+d_t-2}{d_1d_t}} & p_2\sqrt{\frac{d_2+d_t-2}{d_2d_t}} & \cdots & p_i\sqrt{\frac{d_2+d_i-2}{d_2d_i}} & \cdots & \overline{d}_t \end{pmatrix}. \tag{7}$$

□

Proposition 3. *Let $G \neq K_n$ be a complete t-multipartite graph of order n. Then, G has at least three distinct signless Laplacian ABC-eigenvalues and at most 2t distinct signless Laplacian ABC-eigenvalues.*

Proof. Let $G \cong K_{p_1,p_2,\ldots,p_t}$, be the complete t-partite graph of order $n = n_1 + n_2 + \cdots + n_t$. Since G is not a complete graph, therefore using part (iii) of Theorem 7, it follows that G has at least three distinct signless Laplacian ABC-eigenvalues. From the Proposition 2, it is clear that eigenvalues $\sum_{i\neq j,j=1}^{t} p_j\sqrt{\frac{2n-p_i-p_j-2}{(n-p_i)(n-p_j)}}$ each with multiplicity $p_i - 1$, for $i = 1, 2, \ldots, t$ gives at most t distinct signless Laplacian ABC-eigenvalues of G. Further, using the fact that the quotient matrix (7) is of order t and all the eigenvalues of a quotient matrix are simple, the result follows. □

3. Signless Laplacian *ABC*-Energy

In this section, we delve into the concept of the signless Laplacian ABC-energy of a graph G. Our aim is to establish precise bounds for this quantity.

Consider a matrix $M \in \mathbb{M}_{m\times n}(\mathbb{R})$. The positive square roots of the eigenvalues of MM^T are referred to as the *singular values*, denoted by $\sigma_i(M)$ (or simply σ_i), $i = 1, 2, \ldots, n$, of M. Ordering these singular values as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, we can explore various norms of M. For instance, the sum of the k largest singular values is denoted by $\|M\|_k = \sum_{i=1}^{k} \sigma_i$, where $1 \leq k \leq n$. These norms have distinct interpretations: for $k = 1$, $\|M\|_1 = \sigma_1$ represents the *spectral norm*; for $2 \leq k \leq n-1$, $\|M\|_k = \sum_{i=1}^{k} \sigma_i$ signifies the *Kay Fan k-norm*; and for $k = n$, $\|M\|_n = \sum_{i=1}^{n} \sigma_i$ is termed the *trace norm* of M. Notably, for symmetric matrices, the singular values coincide with the absolute values of their eigenvalues, making the trace norm equivalent to the sum of absolute eigenvalues. Nikiforov [3] introduced the concept of energy for a symmetric matrix M as the absolute sum of its eigenvalues. Inspired by this notion, we introduce the operator $\overline{Q} = \tilde{Q}(G) - \overline{\eta} I_n$, where $\overline{\eta} = \frac{1}{n}\sum_{i=1}^{n} \eta_i^{\tilde{Q}}$ represents the average of the signless Laplacian ABC-eigenvalues. Notably, $\overline{\eta} = \frac{1}{n} tr(\tilde{Q}) = \frac{1}{n}\sum_{i=1}^{n} \overline{d}_i = \frac{2ABC(G)}{n}$. It follows that $\overline{Q}$ is a real symmetric matrix with real eigenvalues Θ_i, $i = 1, 2, \ldots, n$. Formally, the signless Laplacian ABC-energy is defined as

$$\mathcal{E}(\tilde{Q}(G)) = \sum_{i=1}^{n} |\Theta_i| = \sum_{i=1}^{n} \left|\eta_i^{\tilde{Q}} - \overline{\eta}\right| = \sum_{i=1}^{n} \left|\eta_i^{\tilde{Q}} - \frac{2ABC(G)}{n}\right| = \sum_{i=1}^{n} \sigma_i(\overline{Q}). \tag{8}$$

Let σ' denote the largest positive integer satisfying $\eta_{\sigma'}^{\tilde{Q}} \geq \frac{2ABC(G)}{n}$. In other words, σ' indicates the number of signless Laplacian ABC-eigenvalues of G lying in $\left[0, \frac{2ABC(G)}{n}\right]$ and those lying in $\left[\frac{2ABC(G)}{n}, 2\overline{\Delta}\right]$, where $\overline{\Delta}$ denotes the maximum ABC-degree of G. Determining the distribution of eigenvalues for a given matrix poses an interesting and challenging problem in linear algebra. This problem has been extensively studied for various graph matrices, yielding numerous intriguing results. Similar to other graph matrices, the distribution of eigenvalues for the signless Laplacian ABC-matrix also warrants exploration.

Problem 3. *Among all connected graphs G of order n with a given parameter α, such as the number of edges, the independence number, the matching number, the chromatic number, the vertex covering number, the $ABC(G)$-index, etc., the task is to determine the number of signless Laplacian ABC-eigenvalues lying in the interval $[0, \alpha]$.*

It is possible to formulate the signless Laplacian ABC-energy in terms of Ky Fan k-norm of the signless Laplacian ABC-matrix.

Theorem 8. *Let G be a connected graph with n vertices, where $n \geq 3$, and let $ABC(G)$ denote its atom-bond connectivity index. We establish a relationship governing the signless Laplacian ABC-energy of G:*

$$\mathcal{E}(\tilde{Q}(G)) = 2\left(\sum_{i=1}^{\sigma'} \eta_i^{\tilde{Q}} - \frac{2\sigma' ABC(G)}{n}\right) = 2 \max_{1 \leq k \leq n} \left(\sum_{i=1}^{k} \eta_i^{\tilde{Q}} - \frac{2kABC(G)}{n}\right),$$

where $\sum_{i=1}^{k} \eta_i^{\tilde{Q}}$ denotes the sum of the first k largest Laplacian ABC-eigenvalues (Ky Fan k-norm) of G. The parameter σ' represents the number of signless Laplacian ABC-eigenvalues lying within the interval $\left[0, \frac{2ABC(G)}{n}\right]$.

Proof. Let σ' be the largest positive integer such that $\eta_{\sigma'}^{\tilde{Q}} \geq \frac{2ABC(G)}{n}$. Then, by the definition of signless Laplacian ABC-energy and the fact $2ABC(G) = \sum_{i=1}^{n} \eta_i^{\tilde{Q}}$, we have

$$\begin{aligned}
\mathcal{E}(\tilde{Q}(G)) &= \sum_{i=1}^{n} \left|\eta_i^{\tilde{Q}} - \frac{2ABC(G)}{n}\right| = \sum_{i=1}^{\sigma'} \left(\eta_i^{\tilde{Q}} - \frac{2ABC(G)}{n}\right) + \sum_{i=\sigma'+1}^{n} \left(\frac{2ABC(G)}{n} - \eta_i^{\tilde{Q}}\right) \\
&= \sum_{i=1}^{\sigma'} \eta_i^{\tilde{Q}} - \frac{4\sigma' ABC(G)}{n} + 2ABC(G) - \sum_{i=\sigma'+1}^{n} \eta_i^{\tilde{Q}} = 2\left(\sum_{i=1}^{\sigma'} \eta_i^{\tilde{Q}} - \frac{2\sigma' ABC(G)}{n}\right).
\end{aligned}$$

Next, we shall prove that $2\left(\sum_{i=1}^{\sigma'}\eta_i^{\tilde{Q}}-\frac{2\sigma' ABC(G)}{n}\right)=2\max_{1\le k\le n}\left(\sum_{i=1}^{k}\eta_i^{\tilde{Q}}-\frac{2kABC(G)}{n}\right)$. For $k>\sigma'$, we have

$$\begin{aligned}\sum_{i=1}^{k}\eta_i^{\tilde{Q}}-\frac{2kABC(G)}{n}&=\sum_{i=1}^{\sigma'}\eta_i^{\tilde{Q}}+\sum_{i=\sigma'+1}^{k}\eta_i-\frac{2kABC(G)}{n}\\&<\sum_{i=1}^{\sigma'}\eta_i^{\tilde{Q}}+(k-\sigma')\frac{2ABC(G)}{n}-k\frac{2ABC(G)}{n}\quad \text{as } \eta_i^{\tilde{Q}}<\frac{2ABC(G)}{n}, \text{for } i\ge\sigma'+1\\&=\sum_{i=1}^{\sigma'}\eta_i^{\tilde{Q}}-\frac{2\sigma' ABC(G)}{n}.\end{aligned}$$

Similarly, for $k\le\sigma'$, it can be easily verified that $\sum_{i=1}^{k}\eta_i^{\tilde{Q}}-k\frac{2ABC(G)}{n}\le\sum_{i=1}^{\sigma'}\eta_i^{\tilde{Q}}-\frac{2\sigma' ABC(G)}{n}$, that finishes the proof. □

The Frobinus norm of $\tilde{Q}(G)$ is $\|\tilde{Q}(G)\|_F^2=\sum_{i=1}^{n-1}(\eta_i^{\tilde{Q}})^2$. Also, the Frobinus norm of $\overline{Q}(G)=\tilde{Q}(G)-\frac{2ABC(G)}{n}I_n$ is

$$\begin{aligned}\|\overline{L}(G)\|_F^2=\sum_{i=1}^{n}\theta_i^2=\sum_{i=1}^{n}\left(\eta_i^{\tilde{Q}}-\frac{2ABC(G)}{n}\right)^2&=\sum_{i=1}^{n}(\eta_i^{\tilde{Q}})^2+\frac{4ABC(G)^2}{n^2}\sum_{i=1}^{n}.1-\frac{4ABC(G)}{n}\sum_{i=1}^{n}\eta_i^{\tilde{Q}}\\&=\sum_{i=1}^{n}(\eta_i^{\tilde{Q}})^2-\frac{4ABC(G)^2}{n}=\|\tilde{Q}(G)\|_F^2-\frac{4ABC(G)^2}{n}.\end{aligned}$$

In the next result, we show a lower bound for the signless Laplacian ABC-energy of G. We rely on the atom-bond connectivity index $ABC(G)$.

Corollary 6. *Suppose G is a connected graph and it has n vertices. Let $n\ge3$ and the atom-bond connectivity index is denoted by $ABC(G)$. We have*

$$\mathcal{E}(\tilde{Q}(G))\ge2\left(\eta_1^{\tilde{Q}}-\frac{2ABC(G)}{n}\right),$$

with equality if and only if $\sigma'=1$; and

$$\mathcal{E}(\tilde{Q}(G))\ge2\left(\frac{4ABC(G)}{n}-\eta_n^{\tilde{Q}}\right),$$

with equality if and only if $\sigma'=n-1$

Proof. Using Theorem 8 and the fact that $\sum_{i=1}^{k}\eta_i^{\tilde{Q}}=2ABC(G)-\sum_{i=k+1}^{n-1}\eta_i^{\tilde{Q}}$, the result follows. □

Corollary 6 provides valuable insights into the relationship between the extreme signless Laplacian ABC-eigenvalues and the signless Laplacian ABC-energy of a graph G. Specifically, it illustrates that any lower bound established for the largest signless Laplacian ABC-eigenvalue, $\eta_1^{\tilde{Q}}$, aids in deriving a lower bound for the overall signless Laplacian ABC-energy of G. Conversely, an upper bound on the smallest signless Laplacian ABC-eigenvalue, $\eta_n^{\tilde{Q}}$, facilitates the determination of a lower bound for the signless Laplacian ABC-energy of G. This observation underscores the importance of understanding the extremal behavior of signless Laplacian ABC-eigenvalues in characterizing the energy properties of graphs.

For a real vector $X = (x_i, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, the Rayleigh quotient theorem [28] yields that

$$\eta_1^{\tilde{Q}} = \sup_{X \neq 0} \frac{X^T \tilde{Q}(G) X}{X^T X} = \sup_{X \neq 0} \frac{\sum_{i=1}^{n} \overline{d}_i x_i^2 + 2 \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} x_i x_j}{\sum_{i=1}^{n} x_i^2}.$$

Taking $X = \mathbf{j}$, the all one-vector in the above expression, we get

$$\eta_1^{\tilde{Q}} \geq \frac{\mathbf{j}^T \tilde{Q}(G) \mathbf{j}}{\mathbf{j}^T \mathbf{j}} = \frac{\sum_{i=1}^{n} \overline{d}_i}{n} = \frac{4ABC(G)}{n}. \tag{9}$$

Equality occurs if and only if $\mathbf{j}$ is an eigenvector for $\tilde{Q}$ corresponding to eigenvalue $\eta_1^{\tilde{Q}}$. That is, $\tilde{Q}\mathbf{j} = \eta_1^{\tilde{Q}} \mathbf{j}$. This last equation shows that all the row sums of $\tilde{Q}$ are the same. Thus, equality occurs in 9 if and only if G is an ABC-regular graph. Therefore, we have the following result, which gives a lower bound for the signless Laplacian ABC-spectral radius in terms of the atom-bond connectivity index.

Theorem 9. *Let G be a connected graph of order $n \geq 3$. Then,*

$$\eta_1^{\tilde{Q}} \geq \frac{4ABC(G)}{n},$$

with equality if and only if G is a ABC-regular graph.

Theorem 9 together with the first part of Corollary 6 gives the following result.

Corollary 7. *Suppose that G is a connected graph having n vertices. We assume $n \geq 3$ and the atom-bond connectivity index is $ABC(G)$. We have*

$$\mathcal{E}(\tilde{Q}(G)) \geq \frac{4ABC(G)}{n},$$

with equality if and only if G is a ABC-regular graph with one positive ABC-eigenvalue.

Proof. The lower bound clearly follows from Theorem 9 and Corollary 6. We will consider the equality case. Suppose that equality occurs in both Theorem 9 and the first part of Corollary 6. Equality occurs in Corollary 6 if and only if G is a ABC-regular graph and equality occurs in Theorem 9 if and only if $\sigma' = 1$. Assume that G is a k-ABC-regular graph, then using the fact $\tilde{Q}(G) = kI + \tilde{A}(G)$, it follows that $\eta_i^{\tilde{Q}} = k + \eta_i^{\tilde{A}}$, for all i. Also, $2ABC(G) = \sum_{i=1}^{n} \overline{d}_i = kn$, gives that $\frac{2ABC(G)}{n} = k$. Therefore, for $\sigma' = 1$, we get $\eta_2^{\tilde{Q}} < \frac{2ABC(G)}{n}$, that is, $k + \eta_2^{\tilde{A}} < k$. This last inequality gives that $\eta_2^{\tilde{A}} < 0$. This completes the proof. □

It will be an interesting problem to characterize all graphs with one positive ABC-eigenvalue. Therefore, we leave the problem.

Problem 4. *Characterize all connected graphs with one positive ABC-eigenvalue.*

From Corollary 7, it is clear that the minimum value for the signless Laplacian ABC-energy $\mathcal{E}(\tilde{Q}(G))$ of a connected graph G is attained when G is a ABC-regular graph with one positive ABC-eigenvalue. We note that there are many ABC-regular graphs with one positive ABC-eigenvalue, namely the complete graph, the complete bipartite graph, etc.

Since for a bipartite graph it follows by Theorem 3 that $\eta_n^{\tilde{Q}} = \eta_n^{\tilde{L}} = 0$, we have the following observation from the second part of Theorem 6.

Corollary 8. *For a connected bipartite graph of order $n \geq 3$, we have*

$$\mathcal{E}(\tilde{Q}(G)) \geq \frac{8ABC(G)}{n},$$

with equality if and only if $\sigma' = n - 1$.

It is clear that for connected bipartite graphs, the lower bound given by Corollary 8 is better than the lower bound given by Corollary 7.

In the following theorem, we present an upper bound for the smallest signless Laplacian ABC-eigenvalue of a non-bipartite graph. We will rely on ABC-degrees again.

Theorem 10. *Suppose that G is connected and non-bipartite. If it has $n \geq 5$ vertices and let ABC-degrees be $\overline{d}_1 \geq \overline{d}_2 \geq \cdots \geq \overline{d}_n$, we have*

$$\eta_n^{\tilde{Q}} \leq \min_{v_iv_j \in E(G)} \left\{ \frac{\overline{d}_i + \overline{d}_j}{2} - \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} \right\}. \tag{10}$$

Equality occurs in (10) if and only if $\overline{d}_i = \overline{d}_j$ and $d_i = d_j$ or $\overline{d}_i = \overline{d}_i$ and $d_k = 2$, for all $v_k \in V(G) - \{v_i, v_j\}$. In particular, if $G \cong K_2 \vee H$, where H is a graph of order $n - 2$, then equality holds in (10).

Proof. For a non-zero vector $X = (x_1, x_2, \ldots, x_n)^T$, it follows by the Rayleigh–Ritz Theorem that

$$\eta_n^{\tilde{Q}} \leq \frac{X^T \tilde{Q}(G) X}{X^T X}, \tag{11}$$

Let v_i and v_j be two adjacent vertices in G. Taking $x_i = 1, x_j = -1$, and $x_k = 0$, for $k \neq i, j$ in inequality (11), after a simple calculation, we get that

$$\eta_n^{\tilde{Q}} \leq \frac{\overline{d}_i + \overline{d}_j}{2} - \sqrt{\frac{d_i + d_j - 2}{d_i d_j}},$$

with this inequality (10) now follows. Suppose that equality occurs in (10), then equality occurs in the Rayleigh–Ritz Theorem, giving that $X = (1, 0, \ldots, -1, 0, \ldots, 0)^T$ is an eigenvector of the matrix $\tilde{Q}(G)$ corresponding to the eigenvalue $\eta_n^{\tilde{Q}}$. For the vertex v_i, it follows from the equation $\tilde{Q}(G)X = \eta_n^{\tilde{Q}} X$ that $\eta_n^{\tilde{Q}} = \overline{d}_i - \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$. Similarly, for the vertex v_j, we get $\eta_n^{\tilde{Q}} = \overline{d}_j - \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$. These two equations together give that $\overline{d}_i = \overline{d}_2$. Let v_k be a vertex different from v_i and v_j. For this vertex, it follows from the equation $\tilde{Q}(G)X = \eta_n^{\tilde{Q}} X$ that $0 = \sqrt{\frac{d_k + d_i - 2}{d_k d_i}} - \sqrt{\frac{d_k + d_j - 2}{d_k d_j}}$. This last equality gives that $d_i = d_j$ or $d_k = 2$. Thus, it follows that equality occurs in (10) if and only if v_i and v_j are adjacent with $\overline{d}_i = \overline{d}_j$ and $d_i = d_j$ or $\overline{d}_i = \overline{d}_j$ and $d_k = 2$, for all $v_k \in V(G) - \{v_i, v_j\}$. For the graph $G = K_2 \vee H$, let v_1 and v_2 be the vertices of K_2 and $v_3, \ldots, v_n$ be the vertices of H. It is clear that $d_1 = n - 1 = d_2$ and

$\overline{d}_i = \overline{d}_j$. Therefore, by the Theorem statement equality occurs for this graph. This completes the proof. □

In the following result, we obtain a lower bound for the signless Laplacian ABC-energy in terms of the order n, the atom-bond connectivity index $ABC(G)$, and the ABC-degrees of the graph G.

Theorem 11. *Suppose that G is connected and non-bipartite with $n \geq 3$ vertices. Let the atom-bond connectivity index be $ABC(G)$. We have*

$$\mathcal{E}(\tilde{Q}(G)) \geq \frac{8ABC(G)}{n} - \min_{v_iv_j \in E(G)} \left\{ \overline{d}_i + \overline{d}_j - 2\sqrt{\frac{d_i + d_j - 2}{d_i d_j}} \right\}, \tag{12}$$

equality occurs if and only if $\sigma'(G) = n - 1$ and equality occurs in Theorem 10.

When it comes to regular and semi-regular bipartite graphs, the next relation between the signless Laplacian ABC-energy and the corresponding signless Laplacian energy $QE(G)$ is useful.

Theorem 12. *Let G be a connected graph with n vertices, where $n \geq 3$, and let $ABC(G)$ denote its atom-bond connectivity index. The following results hold:*

- **(i)** *If G is a r-regular graph, then the signless Laplacian ABC-energy of G is given by $\mathcal{E}(\tilde{Q}(G)) = \frac{\sqrt{2r-2}}{r} QE(G)$.*
- **(ii)** *For a (r,s)-semiregular bipartite graph G, the signless Laplacian ABC-energy is given by $\mathcal{E}(\tilde{Q}(G)) = \sqrt{\frac{r+s-2}{rs}} QE(G)$.*
- **(iii)** *If G has a vertex cover consisting only of vertices with degree 2, then its signless Laplacian ABC-energy is $\mathcal{E}(\tilde{Q}(G)) = \frac{1}{\sqrt{2}} QE(G)$.*

Proof. If G is a r-regular graph, then we have $\eta_i^{\tilde{Q}} = \frac{\sqrt{2r-2}}{r} q_i$, where q_i is the i-th signless Laplacian eigenvalue of G. Also, $2ABC(G) = \eta_1^{\tilde{Q}} + \eta_2^{\tilde{Q}} + \cdots + \eta_n^{\tilde{Q}} = \frac{\sqrt{2r-2}}{r}\Big(q_1 + q_2 + \cdots + q_{n-1}\Big) = \frac{\sqrt{2r-2}}{r} 2m$. The first part now follows from the definition of the signless Laplacian ABC-energy of G. Similarly, if G is a (r,s)-semiregular bipartite graph, then the second part follows by (2) of Theorem 3.1 in [16]. If G has vertex cover consisting of only the vertices of degree 2, then using the definition of vertex cover it follows that every edge of G has at least one end vertex of degree 2 and so $\tilde{Q} = \frac{1}{\sqrt{2}} Q(G)$. From this, we get $\eta_i^{\tilde{Q}} = \frac{1}{\sqrt{2}} q_i$, for all i. The result now follows in this case by using the fact $2ABC(G) = \eta_1^{\tilde{Q}} + \eta_2^{\tilde{Q}} + \cdots + \eta_n^{\tilde{Q}} = \frac{1}{\sqrt{2}}\Big(q_1 + q_2 + \cdots + q_{n-1}\Big) = \frac{1}{\sqrt{2}} 2m$. □

The following result from Fulton [29] is particularly useful.

Lemma 2. *Let A and B be real symmetric matrices of order n. For any positive integer $k \leq n$, the sum of the first k eigenvalues of $A + B$ is bounded above by the sum of the first k eigenvalues of A and B individually:*

$$\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

Recall that $\overline{d}_i = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$, representing the ABC-degree of the vertex $v_i \in V(G)$. A graph G is termed ABC-regular if all its vertices possess the same ABC-degree.

The subsequent result offers an upper bound for the signless Laplacian ABC-energy, expressed in terms of the ABC-degrees and the ABC-energy $E_{ABC}(G)$ of a graph.

Theorem 13. *Suppose that G is connected. It has $n \geq 3$ vertices and the ABC-degrees are $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n$. Assume that σ' is the number of signless Laplacian ABC-eigenvalues, which are no less than $\frac{2ABC(G)}{n}$. We have*

$$\mathcal{E}(\tilde{Q}(G)) \leq E_{ABC}(G) + 2\sum_{i=1}^{\sigma'} \left(\overline{d}_i - \frac{2ABC(G)}{n} \right).$$

The equality is true when G is ABC-regular.

Proof. We can use Lemma 2 and obtain

$$\tilde{Q}(G) = \overline{D}(G) + \tilde{A}(G),$$

where $\overline{D}(G) = \text{diag}(\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n)$ signifies the diagonal matrix of ABC-degrees. It is easy to see

$$\sum_{i=1}^{k} \eta_i^{\tilde{Q}}(G) \leq \sum_{i=1}^{k} \overline{d}_i + \sum_{i=1}^{k} \vartheta_i, \tag{13}$$

Here, $\vartheta_i(G)$ represents the i-th ABC-eigenvalue. Recall that σ' is the number of signless Laplacian ABC-eigenvalues which are no less than $\frac{2ABC(G)}{n}$. We know $1 \leq \sigma' \leq n-1$. It follows from the ABC-energy that

$$E_{ABC}(G) = 2 \max_{1 \leq j \leq n} \sum_{i=1}^{k} \vartheta_i(G) \geq 2\sum_{i=1}^{\sigma'} \vartheta_i(G).$$

Combining with the result 13, we know that

$$2\sum_{i=1}^{\sigma'} \eta_i^{\tilde{Q}}(G) \leq 2\sum_{i=1}^{\sigma'} \overline{d}_i + 2\sum_{i=1}^{\sigma'} \vartheta_i(G),$$

$$\text{that is,} \qquad 2\sum_{i=1}^{\sigma'} \eta_i^{\tilde{Q}} - \frac{4ABC(G)\sigma'}{n} \leq 2\sum_{i=1}^{\sigma'} \overline{d}_i + E_{ABC}(G) - \frac{4ABC(G)\sigma'}{n}.$$

Therefore, by Theorem 8, we obtain

$$E(\tilde{Q}(G)) \leq E_{ABC}(G) + 2\sum_{i=1}^{\sigma'} \left(\overline{d}_i - \frac{2ABC(G)}{n} \right).$$

When G is ABC-regular, the equality holds apparently. □

4. Some Computational Results

By using computations on graphs with at most degree six, we have seen that there is only one pair of ABC signless Laplacian equienergetic graphs, namely $\mathcal{E}(\tilde{Q}(C_6)) = \mathcal{E}(\tilde{Q}(K_6)) = 5.65685$, while they are not signless Laplacian equienergetic as $\mathcal{E}(Q(C_6)) = 8 \neq 10 = \mathcal{E}(Q(K_6))$. Further, there are 26 pairs of signless Laplacian equienergetic graphs, and none of them are ABC signless Laplacian equienergetic. The following table gives the families of signless Laplacian equienergetic and the ABC signless Laplacian equienergetic graphs up to order 6. The ordering of graphs is as in the computer package Mathematica.

From Table 1, we see that there are 26 pairs of graphs of order at most 6 that have the same signless Laplacian energy, while there is only one pair of ABC-signless Laplacian unenergetic graphs. From Table 2, we see that there are eight pairs of graphs of order at most 6 that have the same ABC-energy. Additionally, two ABC-equienergetic graphs need not be ABC-signless Laplacian equienergetic. Thus, for graphs of order at most 6, the signless Laplacian ABC-energy can be a better spectral graph invariant in differentiating between non-isomorphic graphs in comparison to the signless Laplacian energy and the ABC-energy.

Table 1. Graph pairs of order at most 6 with the same signless Laplacian energy but different signless Laplacian ABC-energy.

Graphs	G_{16}	G_{20}	G_{37}	G_{46}	G_{39}	G_{91}	G_{47}	G_{105}
$\mathcal{E}(Q(G))$	4.74955	4.74955	10.371	10.371	9.41123	9.41123	9.29779	9.29779
$\mathcal{E}(\tilde{Q}(G))$	5.20126	7.54456	7.42478	7.48084	6.806	6.11555	6.71135	6.43482
Graphs	G_{55}	G_{65}	G_{56}	G_{59}	G_{57}	G_{74}	G_{68}	G_{134}
$\mathcal{E}(Q(G))$	9.69854	9.69854	9.20775	9.20775	8.92299	8.92299	8.47214	8.47214
$\mathcal{E}(\tilde{Q}(G))$	5.95061	6.74134	6.35251	6.2184	6.17159	6.07797	5.99892	6.43732
Graphs	G_{81}	G_{117}	G_{94}	G_{108}	G_{99}	G_{121}	G_{100}	G_{110}
$\mathcal{E}(Q(G))$	8.22358	8.22358	9.2111	9.2111	8.45644	8.45644	8.85087	8.85087
$\mathcal{E}(\tilde{Q}(G))$	5.66487	5.81495	5.75093	5.54952	5.28173	6.59344	5.48126	5.50743
Graphs	G_{103}	G_{135}	G_{126}	G_{143}	G_{112}	G_{113}	G_{127}	G_{132}
$\mathcal{E}(Q(G))$	8.66667	8.66667	8.94427	8.94427	8	8	8	8
$\mathcal{E}(\tilde{Q}(G))$	6.07573	7.7517	6.65461	5.81316	5.65685	5.63128	4.89898	4.89898
Graphs	G_{82}	G_{98}	G_{107}	G_{83}	G_{106}	G_{122}		
$\mathcal{E}(Q(G))$	9.33333	9.33333	9.33333	10	10	10		
$\mathcal{E}(\tilde{Q}(G))$	6.39573	5.48786	6.59966	6.70656	5.65685	6.33052		

Table 2. Graph pairs of order at most 6 with same ABC-energy but different signless Laplacian ABC-energy.

Graphs	G_5	G_9	G_{15}	G_{29}	G_{34}	G_{127}	G_{106}	G_{112}
$E_{ABC}(G)$	2.82843	2.82843	3.4641	3.4641	4.89898	4.89898	5.65685	5.65685
$\mathcal{E}(\tilde{Q}(G))$	4.08248	2.82843	4.52548	5.88897	6.806	4.89898	5.65685	5.65685
Graphs	G_{118}	G_{121}	G_{107}	G_{135}	G_{107}	G_{142}	G_{135}	G_{142}
$E_{ABC}(G)$	4.8074	4.8074	4	4	4	4	4	4
$\mathcal{E}(\tilde{Q}(G))$	5.33878	6.59344	6.59966	7.7517	6.59966	4	7.7517	4

5. Conclusions

The concept of the ABC-matrix, introduced by Estrada, stems from its representation of the probability associated with visiting the nearest neighbor edge from either side of a given edge in a graph. This notion holds significance in molecular contexts, particularly in understanding the polarizing capacity of the bonds. The ABC-matrix finds a connection with the atom-bond connectivity index or the ABC-index. This relationship offers a potential avenue for utilizing matrix methods to address ABC-index-related problems. Much like the Laplacian and signless Laplacian matrices, which have proven invaluable in uncovering various graph properties. It is natural to extend this notion to include the Laplacian ABC-matrix and the signless Laplacian ABC-matrix for a graph. While the Laplacian ABC-matrix has been previously discussed in the literature, in this paper, we introduce the signless Laplacian ABC-matrix denoted as $\mathrm{Q}(G)$ for a graph G. This matrix serves as a weighted version of the signless Laplacian matrix of the graph and exhibits several spectral properties. Notably, for bipartite graphs, the signless Laplacian ABC-matrix shares the same spectrum as the Laplacian ABC-matrix. We establish that the matrix $\mathrm{Q}(G)$ is singular if and only if G is bipartite, with the multiplicity of eigenvalue 0 matching the number of bipartite components of G. We delve into the characterization of graphs with k distinct signless Laplacian ABC-eigenvalues, providing solutions for $k = 1, 2$, and a partial solution for $k = 3$. Notably, we demonstrate that complete graphs, complete bipartite graphs, and regular complete t-partite graphs are uniquely determined by their signless Laplacian ABC-spectrum. Additionally, we introduce the concept of signless Laplacian ABC-energy for a graph and establish some of its fundamental properties. Furthermore, we derive bounds for signless Laplacian ABC-energy and characterize the extremal graphs that attain these bounds.

The signless Laplacian ABC-matrix is a new graph matrix that enjoys the same basic properties as possessed by the signless Laplacian matrix of a graph. It will be natural to consider all the problems already considered for the signless Laplacian matrix for the signless Laplacian ABC-matrix. In particular, the following problems can be of interest in the future.

Problem 5. *Find the maximum value for the signless Laplacian ABC-spectral radius among all graphs of order n or among a given class of graphs of order n with some given parameter. Characterize the graph (graphs) that attains this maximum value.*

Problem 6. *Find the minimum value for the signless Laplacian ABC-spectral radius among all graphs of order n or among a given class of graphs of order n with some given parameter. Characterize the graph (graphs) that attains this minimum value*

Problem 7. *Find the maximum/minimum value for the signless Laplacian ABC-energy among all graphs of order n or among a given class of graphs of order n with some given parameter. Characterize the graph (graphs) that attains this extremal value*

Like the ABC-matrix and the signless Laplacian matrix, we can initially consider these problems for trees, unicyclic graphs, bicyclic graphs, etc.

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