An agent based model for opinion dynamics with random confidence threshold

Yilun Shang

SUTD-MIT International Design Center

Singapore University of Technology and Design

20 Dover Drive, Singapore 138682

Email: shylmath@hotmail.com; Tel: 65-90374465

Abstract

Although the idea of introducing heterogeneous confidence levels has been around for some time, its implications for the influence of the confidence thresholds on the consensus formation have not yet been sufficiently elaborated. In this paper, we propose a bounded confidence opinion model with random confidence threshold D associated with potential opinion interaction. Initial opinions are uniformly continuously distributed on the interval $[0, 1]$. Using techniques developed in interacting particle systems, we identify the critical confidence threshold to be $\mathrm{E}(D) = 1/2$ in the sense of almost surely convergence regardless of the distribution of D, if the underlying communication graph is the real line $\mathbb{Z}$. Numerical simulations are performed to illustrate our results. Noticed are some interesting dependency of the rate of consensus and sharpness of phase transition on the distribution of D. Additionally, numerical experiments show that our obtained results are applicable to Barabási-Albert scale-free networks, implying a universality of the obtained critical confidence threshold. It is hoped that the results could lead to new insights in understanding the mechanisms and dynamics of those group-level collective behaviors arising from micro-level decision-making patterns.

Keywords: opinion dynamics, consensus, bounded confidence, phase transition, agent-based simulation.

1 Introduction

The spreading of opinions through a population, as a macroscopic collective social phenomenon, has become a popular research topic and received an increasing attention in social psychology, anthropology, physics, economics and mathematics [1, 2, 3]. The analysis of the underlying mechanism of opinion dynamics can help understand (in a bottom-up manner) the dynamics of, e.g., collective decision making, shifts in preferences, minority opinion survival, rise of extremism, political changes, emergence of fads and the like. In an agent-based model of opinion dynamics, each agent has an opinion which might be described by a continuous or discrete variable. Relationships between agents, such as their family relation and acquaintance, are usually represented by means of a social network. Agents can influence each others' opinions through the connections existing between them by exercising some common rule of opinion updating. Sometimes even simple opinion update rules might lead to rather rich and complicated dynamics [4, 5, 6].

According to whether the variable that represents the opinion of an agent is discrete or continuous, opinion dynamic models are classified into discrete opinion dynamics model and continuous opinion dynamics model. Most established examples for discrete models include the voter model [7], the Galam majority rule model [8], the Sznajd model [9] and the social impact model [10]. A unifying frame for discrete opinion dynamics was also provided in [11]. Among continuous models, bounded confidence (BC) models have received significant attention. The two most cited BC models are Hegselmann-Krause (HK) [12, 13] and Deffuant-Weisbuch (DW) [14, 15]. In these models, the interactions are nonlinear: an agent only interacts with those whose opinions are close to its own under a given confidence threshold. The threshold can be interpreted as an uncertainty/tolerance, or a bounded confidence, around the opinion (see [16] for a comprehensive survey on BC models).

The previous studies of continuous opinion model have been mainly performed considering uniform agents, i.e., the confidence thresholds for all agents are equal. Such homogeneous models have been widely studied in literatures due to its simplicity. For example, it was shown [14, 17] through a large number of simulations that, over various networks, be them complete graphs, lattices, or scale-free networks, there exists a universal critical confidence threshold value for the classical DW model, above which complete consensus is reached (i.e., a single cluster emerges) while below which opinions diverge (i.e., two or more clusters are observed).

However, the agents in reality are diverse in their wealth and social status/power, and hence, have different influence on others [6, 18]. The Fukushima Daiichi nuclear accident in March 2011 brought about a panic buying of iodized salt and iodine tablets in many countries across the Asian and Pacific region, which was mostly caused by the rumor that iodized salt could help prevent the human body from absorbing radioactive materials [19, 20]. Regarding to whether believe or disbelieve the rumor, the attitude of an agent is in the range of a bounded opinion domain. It is reasonable to assume that non-uniform confidence thresholds exist due to disparity of people's knowledge, experience, personality, etc. In such scenarios, a heterogeneous model is more appropriate for the opinion evolution with agent-dependent confidence levels. In [21], Lorenz went about the heterogeneous HK model in terms of interactive Markov chains. A heterogeneous DW and HK model was proposed in [22]. It was experimentally shown that a society of open-minded agents (with relatively large confidence threshold) and closed-minded agents (with relatively small confidence threshold) can find consensus even when both thresholds are significantly below the critical threshold of confidence of a homogeneous society. The delicate influence of heterogeneous fractions, non-uniform initial opinion distribution and group size on the evolution of opinions in the heterogeneous HK model was systematically investigated in [23] via computer simulation.

In this work, we consider the evolution of continuous opinions of heterogeneous agents, and aim at extending this line of research by introducing random confidence thresholds. Helbing and Yu [24] argue that randomness in opinion dynamics—and more broadly the collective phenomena emerging from social systems—plays a key functional role. Albeit extremely difficult due to the nonlinear interaction in BC models, rigorous analyses are conducted by Lanchier [25] and Häggström [26] recently using probabilistic frameworks. They successfully identified the critical value of confidence threshold for consensus formation in the DW model on the real line $\mathbb{Z}$ to be $1/2$, given the initial opinions are distributed uniformly on $[0, 1]$. General initial opinions are addressed in parallel very recently by Shang [27]. In these works, each agent is homogeneous and has the same (non-random) confidence level. In the Fukushima-triggered panic buying case, for example, an agent may have a low or high confidence level on the opinion that iodized salt could be an antidote for radiation. The results in [25, 26, 27] do not apply when it comes to describing the dynamics of groups of heterogeneous agents.

The objective of this paper is to build a heterogeneous BC model on networks from a probabilistic point of view and explore analytically the influence of random confidence thresholds on the convergence of the opinion dynamics. We model the agents as an interacting particle system (see e.g. [28]) on a graph G with bounded degrees. On each edge of G we assign independently a Poisson process which governs the meeting time and a random variable D indicating the confidence threshold. Based on the framework developed in [26, 27], we identify the critical confidence threshold to be $\mathrm{E}(D) = 1/2$, regardless of the concrete distribution of D, provided the threshold has finite mean $\mathrm{E}(D)$ and the underlying communication graph G is taken as $\mathbb{Z}$. When $\mathrm{E}(D) > 1/2$, we show that the opinions converge to the average of the initial configuration with probability one. As a consequence, our results disprove the conjecture put forward in the pioneering work [14], where the authors had reservation on whether the critical phenomenon could really happen for broadly/flatly distributed D in the same way as for deterministic (or sharply distributed) D.

On the other hand, agent-based computer simulations are viewed as an innovative way of simulating many social behaviors grounded on minimal rules, simplicity and emergence [29, 30]. In this work, to illustrate our theoretical results, we perform extensive agent-based simulations with D satisfying a variety of unimodal and bimodal distributions for agents distributed on large rings. A systematic difference in the rate of convergence is discerned: opinions reach consensus faster with sharply distributed D than with broadly distributed or bimodal D. An interesting implication behind could be that the existence of a handful of closed-minded agents might notably hinder the progress of reaching consensus in a community. A profound relation between the sharpness of opinion phase transition and the distribution of D is also revealed via simulation. Moreover, we experimentally show that our obtained analytical results are in effect applicable to synthetic networks constructed through Barabási-Albert (BA) scheme [34]. A power law degree distribution is the signature of these networks, which are very different from regular (or purely random) networks: one should expect a few vertices to be very highly connected, and the vast majority to have smaller degree than the average; rather than almost all vertices have the same degree. We contend that the observed universality of the critical confidence threshold $\mathrm{E}(D) = 1/2$ with respect to different network structures (e.g., characterized by degree distributions) may find its origins in the underlying opinion dynamics, as analyzed in [17] for homogeneous agents.

The rest of the paper is organized as follows. In Section 2, we describe the continuous opinion model with random confidence threshold. Our main theoretical results concerning the critical confidence value and consensus formation are presented in Section 3. Results of computer simulation for the opinion models with different random confidence thresholds on rings as well as BA networks are reported in Section 4. Section 5 concludes this paper and suggests directions for future work. In the Appendix, we briefly review the sharing a drink (SAD) scheme proposed by Häggström [26], which is the foundation of theoretical development in this work.

2 Proposed continuous opinion model

Consider a graph $G = (V, E)$ with vertex set V signifying the agents in a population and edge set E representing the potential social interactions among agents. At time $t = 0$, each agent $u \in V$ keeps opinion $X_0(u)$ uniformly and independently with a continuous opinion space of $[0, 1]$. Each edge $e = \{u, v\} \in E$ is independently assigned a unit rate Poisson process as well as an i.i.d. continuous random variable D_e taking values in $(0, 1)$ (or equivalently, $[0, 1]$ with the natural extension). Let D be a random variable with the same distribution of D_e. Denote by $X_t(u)$ the opinion value of agent $u \in V$ at time $t \geq 0$. The above collection of Poisson processes dictates the meeting time and hence the dynamics of the agents. Specifically, when at some time t the Poisson event occurs at edge $e = \{u, v\}$ such that the pre-meeting opinions of the two agents are $X_{t-}(u) := \lim_{s \to t-} X_s(u)$ and $X_{t-}(v) := \lim_{s \to t-} X_s(v)$, we set

$$X_t(u) = \begin{cases} X_{t-}(u) + \mu(X_{t-}(v) - X_{t-}(u)) & \text{if } |X_{t-}(u) - X_{t-}(v)| \leq D_e; \\ X_{t-}(u) & \text{otherwise,} \end{cases} \tag{1}$$

and

$$X_t(v) = \begin{cases} X_{t-}(v) + \mu(X_{t-}(u) - X_{t-}(v)) & \text{if } |X_{t-}(u) - X_{t-}(v)| \leq D_e; \\ X_{t-}(v) & \text{otherwise,} \end{cases} \tag{2}$$

where $\mu \in (0, 1/2]$ is the so-called convergence parameter. When no Poisson event occurs for any of the edges incident to u, its opinion remains unchanged. Therefore, the opinions of two interacting agents shift towards each other by a relative amount μ ($\mu = 1/2$ means that the two agents meet halfway) if they meet, discuss and find that their opinions differ by less than a given threshold (in that case, we say that an opinion adjustment occurs). In the Fukushima-triggered panic buying, the state variable $X_t(u)$ represents the attitude of agent u concerning the rumor. At time t, $X_t(u) = 1$ represents that agent u completely believe the rumor, while $X_t(u) = 0$ indicates that she does not believe it at all. When two neighboring agents u, v with $e = \{u, v\}$ have opinions close enough to each other in terms of D_e (for example, they both tend to believe that the iodized salt is anti-radiation), they would compromise following the rules (1) and (2).

We give a couple of remarks here. Firstly, the above interacting particle system description entails that G has bounded degrees for technical reasons (see e.g. [35, p. 28]). Two examples studied below, finite graphs and $G = \mathbb{Z}$, meet this condition. Secondly, the model described above is basically *edge-centric* in the sense that each edge (potential interaction) $e \in E$ possesses its own measure D_e. A slightly different scheme can be called *vertex-centric* model, where A_v is an attribute of vertex v, and the actual threshold between agents u and v (if $e = \{u, v\} \in E$) can be expressed by some function $D_e = f(A_u, A_v)$. Frequently used vertex-centric heterogeneous opinion model is the relative agreement (RA) models [36, 37, 38], which are more effective to treat extremism propagation than BC models. Finally, for each edge $e \in E$, the threshold D_e is defined once and for all. We readily reproduce the classical homogeneous DW model (see e.g. [25, 26]) if D reduces to a degenerate random variable. Note that the homogeneous model is trivially edge-centric (as well as vertex-centric).

In the following, we consider the above opinion model on $\mathbb{Z}$. More specifically, we take $G = (V, E)$ with $V = \mathbb{Z}$ and $E = \{\{u, u + 1\} : u \in \mathbb{Z}\}$. For an edge $e = \{u, u + 1\} \in E$ we write $D_u := D_e$. Recall that the random variable D has the same distribution as D_u for any $u \in V$.

3 Critical confidence value in continuous opinion formation

In this section, we establish the following main result concerning the critical confidence threshold for our continuous opinion model.

Theorem 1. *Consider the above continuous opinion model on $\mathbb{Z}$ with convergence parameter $\mu \in (0, 1/2]$ and random confidence thresholds $\{D_u\}_{u\in\mathbb{Z}}$. Recall that agents u and $u + 1$ update their states following the rules*

$$X_t(u) = \begin{cases} X_{t-}(u) + \mu(X_{t-}(u+1) - X_{t-}(u)) & \text{if } |X_{t-}(u) - X_{t-}(u+1)| \le D_u; \\ X_{t-}(u) & \text{otherwise,} \end{cases}$$

and

$$X_t(u+1) = \begin{cases} X_{t-}(u+1) + \mu(X_{t-}(u) - X_{t-}(u+1)) & \text{if } |X_{t-}(u) - X_{t-}(u+1)| \le D_u; \\ X_{t-}(u+1) & \text{otherwise.} \end{cases}$$

Then 1/2 is the critical confidence threshold in terms of the expectation of D:

- *If* $\mathrm{E}(D) < 1/2$, *then with probability 1, the limiting value* $X_\infty(u) := \lim_{t\to\infty} X_t(u)$ *exists and* $\{|X_\infty(u) - X_\infty(u+1)|\} \in \{0\} \cup [D_u, 1]$ *for every* $u \in \mathbb{Z}$.
- *If* $\mathrm{E}(D) > 1/2$, *then with probability 1,* $X_\infty(u) = \lim_{t\to\infty} X_t(u) = 1/2$ *for every* $u \in \mathbb{Z}$.

A striking implication of Theorem 1 is that neither the critical confidence value of phase transition nor the limit opinion value relies on the distribution of confidence threshold except its mathematical expectation. When $\mathrm{E}(D) < 1/2$, the limiting configuration is piecewise constant interrupted by jumps of size at least D_u for edge $\{u, u+1\}$. When $\mathrm{E}(D) > 1/2$, the complete consensus (i.e., a single opinion cluster) is formed at 1/2, the average of initial opinions.

In the following we show Theorem 1 in two regimes $\mathrm{E}(D) < 1/2$ and $\mathrm{E}(D) > 1/2$, respectively, following the same lines drawn in [26, 27]. A key technique used is called the sharing a drink (SAD) scheme (see the Appendix). Due to the limitation of space, we only sketch the outline of the proof and refer the interested reader to [26] for more technical details.

3.1 Subcritical regime: $\mathrm{E}(D) < 1/2$

Given $\varepsilon > 0$ and the initial configuration $\{X_0(v)\}_{v\in\mathbb{Z}}$, a vertex $u \in \mathbb{Z}$ is said to be an *ε-flat point to the right* if for all $n \ge 0$,

$$\frac{1}{n+1}\sum_{v=u}^{u+n} X_0(v) \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

Likewise, $u \in \mathbb{Z}$ is said to be an *ε-flat point to the left* if for all $n \ge 0$,

$$\frac{1}{n+1}\sum_{v=u-n}^{u} X_0(v) \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right],$$

and *two-sidedly ε-flat* if for all $n, m \ge 0$,

$$\frac{1}{n+m+1}\sum_{v=u-n}^{u+m} X_0(v) \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

When u is ε-flat to the right, the Kolmogorov strong law of large numbers implies

$$\mathrm{P}\left(\lim_{n\to\infty}\frac{1}{n+1}\sum_{v=u}^{u+n}X_0(v)=\frac{1}{2}\right)=1.$$

By using the translation invariance of the configuration $\{X_0(v)\}_{v\in\mathbb{Z}}$, we can show that the following events happen with positive probability.

Lemma 1.[26] *For $\varepsilon>0$ and $u\in\mathbb{Z}$, $\mathrm{P}(u\ is\ \varepsilon\text{-}flat\ to\ the\ right)=\mathrm{P}(u\ is\ \varepsilon\text{-}flat\ to\ the\ left)$ >0.*

Lemma 2.[26] *For $\varepsilon>0$ and $u\in\mathbb{Z}$, $\mathrm{P}(u\ is\ two\text{-}sidedly\ \varepsilon\text{-}flat)>0$.*

Assume that $\mathrm{E}(D)<1/2$. Take $\delta=(1/2-\mathrm{E}(D))/3>0$. For $u\in\mathbb{Z}$, define the following events $\mathcal{A}(u)=\{|X_t(u)-X_t(u+1)|>D_u,\ \text{for all}\ t\geq 0\}$, $\mathcal{B}(u)=\{D_{u-1}\leq \mathrm{E}(D)+\delta$ and $D_u\leq \mathrm{E}(D)+\delta\}$, $\mathcal{C}_1(u)=\{u-1$ is δ-flat to the left$\}$, $\mathcal{C}_2(u)=\{X_0(u)>1-\delta\}$, and $\mathcal{C}_3(u)=\{u+1$ is δ-flat to the right$\}$, .

Proposition 1. *Under the assumption of Theorem 1, if $\mathrm{E}(D)<1/2$, then for any $u\in\mathbb{Z}$, $\mathrm{P}(\mathcal{A}(u))>0$.*

Proof. For any $u\in\mathbb{Z}$, define the event $\mathcal{C}(u)=\mathcal{C}_1(u)\cap\mathcal{C}_2(u)\cap\mathcal{C}_3(u)$. By using Lemma 1 and the independence, we obtain

$$\mathrm{P}(\mathcal{B}(u)\cap\mathcal{C}(u))=\mathrm{P}(\mathcal{B}(u))\,\mathrm{P}(\mathcal{C}_1(u))\,\mathrm{P}(\mathcal{C}_2(u))\,\mathrm{P}(\mathcal{C}_3(u))>0.$$

It suffices to show

$$\mathcal{B}(u)\cap\mathcal{C}(u)\subseteq\mathcal{A}(u). \tag{3}$$

To this end, suppose that $\mathcal{B}(u)$ and $\mathcal{C}(u)$ hold. Let $T<\infty$ be the first time that opinion adjustment happens across any of the two edges $\{u-1,u\}$ and $\{u,u+1\}$. Therefore, $X_t(u)=X_0(u)$ for any $t<T$. We will show that such a finite T does not exist at all. Indeed, on one hand, there must exist some $t_0<T$ such that either

(i) $X_{t_0}(u-1)>1-\delta-D_{u-1}\geq 1-\delta-\mathrm{E}(D)-\delta=\frac{1}{2}+\delta.$

or

(ii) $X_{t_0}(u+1)>1-\delta-D_u\geq 1-\delta-\mathrm{E}(D)-\delta=\frac{1}{2}+\delta.$

happens.

On the other hand, for any $t < T$ we obtain from Lemma A1 (by replacing 0 with $u+1$ due to translation invariance)

$$X_t(u+1) = \sum_{v \in \mathbb{Z}} Y_t(v) X_0(v),$$

and $Y_t(v) = 0$ for all $v \leq u$. By a monotonic property of the SAD process (see [26, Lemma 2.1]) we obtain

$$Y_t(u+1) \geq Y_t(u+2) \geq \cdots \geq Y_t(u+N) > 0 = Y_t(u+N+1) = \cdots$$

for some $1 \leq N < \infty$. Set $c_k = k(Y_t(u+k) - Y_t(u+k+1)) \geq 0$ for $k = 1, \cdots, N$. Following equations (19) and (20) in [26], we derive $\sum_{n=1}^{N} c_n = 1$ and

$$X_t(u+1) = \sum_{n=1}^{N} c_n \left(\frac{1}{n} \sum_{k=1}^{n} X_0(u+k) \right). \tag{4}$$

Since the event $\mathcal{C}_3(u)$ holds, it follows from (4) that

$$X_t(u+1) \in \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta \right]. \tag{5}$$

Similarly, $X_t(u-1) \in [1/2 - \delta, 1/2 + \delta]$. These contradict with the existence of a $t_0 < T$ satisfying (i) or (ii). Consequently, we obtain $T = \infty$.

Now given the events $\mathcal{C}_2(u)$, $\mathcal{B}(u)$ and using (5), we have for all $t \geq 0$,

$$|X_t(u) - X_t(u+1)| > 1 - \delta - \left(\frac{1}{2} + \delta \right) = \frac{1}{2} - 2\delta \geq D_u.$$

Thus, $\mathcal{A}(u)$ holds, and (3) is established. The proof is complete. □

Note that the above proof virtually says that, for all $u \in \mathbb{Z}$, $\mathcal{B}(u) \cap \mathcal{C}(u) \subseteq \mathcal{A}(u)$ and $\mathrm{P}(\mathcal{A}(u)) \geq \mathrm{P}(\mathcal{B}(u) \cap \mathcal{C}(u)) > 0$. By the ergodicity ([39, p. 340 Theorem (1.3)]) of the indicator processes $\{I_{\mathcal{A}(u)}\}_{u \in \mathbb{Z}}$ and $\{I_{\mathcal{B}(u) \cap \mathcal{C}(u)}\}_{u \in \mathbb{Z}}$, we can obtain the following corollary (c.f. [26, Lemma 5.2])

Corollary 1. *With probability 1, there are infinitely many vertices u to the left (and right) of 0 such that $\mathcal{A}(u)$ happens. The same thing holds for $\mathcal{B}(u) \cap \mathcal{C}(u)$.*

Proposition 2. *Under the assumption of Theorem 1, if $\mathrm{E}(D) < 1/2$, then with probability 1 the limiting value $X_\infty(u) = \lim_{t \to \infty} X_t(u)$ exists and $\{|X_\infty(u) - X_\infty(u+1)|\} \in \{0\} \cup [D_u, 1]$ for all $u \in \mathbb{Z}$.*

Proof. Given the initial opinion configuration $\{X_0(u)\}_{u \in \mathbb{Z}}$, let u_1 be a vertex such that $\mathcal{B}(u_1 - 1) \cap \mathcal{C}(u_1 - 1)$ happens, and let $u_2 = \min\{u > u_1 : \mathcal{B}(u) \cap \mathcal{C}(u) \text{ happens}\}$. It follows from (3) that the opinions in the interval $\{u_1, u_1+1, \cdots, u_2\}$ will not be affected by vertices outside and vice versa. By Corollary 1 we see that each $u \in \mathbb{Z}$ is located in some such interval. Therefore, it suffices to show the proposition for every $u \in \{u_1, u_1+1, \cdots, u_2\}$.

Following [26, Theorem 5.3], we define the energy of the interval $\{u_1, u_1+1, \cdots, u_2\}$ at time t as

$$W_t = \sum_{u \in \{u_1, u_1+1, \cdots, u_2\}} X_t(u)^2 \geq 0.$$

If two vertices u and $u+1$ in the interval exchange opinions at time t, W_t drops by an amount of $2\mu(1-\mu)|X_{t-}(u) - X_{t-}(u+1)|$. In addition, W_t is always decreasing with respect to time t. By employing the conditional version of the Borel-Cantelli lemma [39, p. 240, Corollary (3.2)], we can show [26, Theorem 5.3]

$$\lim_{t\to\infty} \max \left\{ |X_t(u) - X_t(u+1)| I_{\{|X_t(u)-X_t(u+1)| \leq D_u\}} : u \in \{u_1, u_1+1, \cdots, u_2 - 1\} \right\} = 0. \tag{6}$$

For any edge $\{u, u+1\}$ in the interval $\{u_1, u_1+1, \cdots, u_2\}$, a single opinion adjustment can only increase $|X_t(u) - X_t(u+1)|$ by at most μD_u. From (6) we can see that either $|X_t(u) - X_t(u+1)| > D_u$ for all sufficiently large t or $\lim_{t\to\infty} |X_t(u) - X_t(u+1)| = 0$. This together with the fact that the quantity $\sum_{u\in\{u_1,u_1+1,\cdots,u_2\}} X_t(u)$ remains unchanged over time implies [26, Theorem 5.3] the existence of the limit $\lim_{t\to\infty} X_t(u)$. □

3.2 Supercritical regime: $\mathrm{E}(D) > 1/2$

Recall that a vertex $u \in \mathbb{Z}$ is said to be two-sidedly ε-flat if for all $n, m \geq 0$,

$$\frac{1}{n+m+1} \sum_{v=u-n}^{u+m} X_0(v) \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

The next result tells us what values can be achieved for a two-sidedly flat vertex regardless of all future Poisson events.

Lemma 3.[26] *Given $\varepsilon > 0$ and the initial configuration $\{X_0(u)\}_{u\in\mathbb{Z}}$. If $u \in \mathbb{Z}$ is two-sidedly ε-flat, then $X_t(u) \in [1/2 - 6\varepsilon, 1/2 + 6\varepsilon]$ for all $t \geq 0$.*

Define the energy at vertex $u \in \mathbb{Z}$ as $W_t(u) := X_t(u)^2$. Define an auxiliary continuous-time step function $W_t^\dagger(u)$ by $W_0^\dagger(u) = 0$ with $W_t^\dagger(u)$ increasing by an amount of $2\mu(1-\mu)|X_{t-}(u) - X_{t-}(u+1)|^2$ if and only if opinion adjustment occurs on the edge $\{u, u+1\}$ at time t. We have the following

Lemma 4.[26] For any $u \in \mathbb{Z}$ and $t \geq 0$,

$$\mathrm{E}\left(W_t(u)\right) + \mathrm{E}\left(W_t^\dagger(u)\right) = \frac{1}{3}.$$

The next result holds for all $\mathrm{E}(D) \in (0,1)$.

Proposition 3. *With probability 1, for any $u \in \mathbb{Z}$, either $|X_t(u) - X_t(u+1)| > D_u$ for all sufficiently large t, or $\lim_{t\to\infty} |X_t(u) - X_t(u+1)| = 0$.*

Proof. Given $u \in \mathbb{Z}$ and a sufficiently small η with $D_u > \eta > 0$, we claim that, with probability 1,

$$|X_t(u) - X_t(u+1)| \in [0,\eta] \cup (D_u, 1] \tag{7}$$

for all large enough t.

Indeed, when a Poisson event occurs on any of the three edges $\{u-1,u\}$, $\{u,u+1\}$ or $\{u+1,u+2\}$, the next Poisson event will occur at $\{u,u+1\}$ with probability 1/3. If $|X_t(u) - X_t(u+1)| \in (\eta, D_u]$, then $W_t^\dagger(u)$ will increase at least $2\mu(1-\mu)\eta^2$ if a Poisson event occurs at $\{u,u+1\}$. Drawing on the conditional Borel-Cantelli lemma [39, p.240, Corollary(3.2)], this will happen infinitely often with probability 1. Hence, $\lim_{t\to\infty} W_t^\dagger(u) = \infty$, which is in contradiction to Lemma 4. Therefore, (7) is established.

What remains to show is that for small enough $\eta > 0$, $|X_t(u) - X_t(u+1)|$ cannot jump back and forth between $[0,\eta]$ and $(D_u, 1]$ infinitely often. This is because a single Poisson event can not increase $|X_t(u) - X_t(u+1)|$ by more than μD_u, which for small enough η, is always less than the span of the gap $(\eta, D_u]$ that needs to be bridged. □

Proposition 4. *Under the assumptions of Theorem 1, if $\mathrm{E}(D) > 1/2$, then with probability 1, $X_\infty(u) := \lim_{t\to\infty} X_t(u) = 1/2$ for every $u \in \mathbb{Z}$.*

Proof. Take an $\varepsilon > 0$ satisfying $\mathrm{E}(D) > 1/2 + 7\varepsilon$. We first show that with probability 1,

$$\lim_{t\to\infty} |X_t(u) - X_t(u+1)| = 0 \tag{8}$$

for any $u \in \mathbb{Z}$. Thanks to Proposition 3, we only need to show that for each u,

$$\mathrm{P}(|X_t(u) - X_t(u+1)| > D_u \text{ for all large enough } t) = 0. \tag{9}$$

Suppose, on the contrary, that the probability in (9) is strictly positive. Then the event in (9) happens for infinitely many u on $\mathbb{Z}^-$ and $\mathbb{Z}^+$ with probability 1 by using the ergodicity theorem (see e.g. [39, p. 340, Theorem (1.3)]). We next show that the limit

$$X_\infty(u) := \lim_{t\to\infty} X_t(u) \tag{10}$$

exists for any $u \in \mathbb{Z}$.

Fix $u \in \mathbb{Z}$. Note that we can always pick two vertices $v_1, v_2 \subseteq \mathbb{Z}$ with $v_1 \leq u < v_2$ such that $\{v_1, v_1+1, \cdots, v_2\}$ are the vertices locating between two edges $\{v_1-1, v_1\}$ and $\{v_2, v_2+1\}$ which never exchange opinions for any $t \geq T$ with some $T > 0$. To show (10) we consider two cases: (a) No edge $\{v, v+1\}$ in $\{v_1, v_1+1, \cdots, v_2\}$ gets stuck with the event $\{|X_t(v) - X_t(v+1)| > D_v\}$ happening for any $t \geq T$. (b) Some edge $\{v, v+1\}$ in $\{v_1, v_1+1, \cdots, v_2\}$ gets stuck with this event at some time $T' \geq T$. If (a) happens, we obtain $\lim_{t\to\infty} |X_t(v) - X_t(v+1)| = 0$ for all v in this interval in view of Proposition 3. Since $\sum_{v=v_1}^{v_2} X_t(v)$ is invariant over time, $X_t(v_1), \cdots, X_t(v_2)$ must all converge to the average value $\frac{1}{v_2-v_1+1}\sum_{v=v_1}^{v_2} X_T(v)$. If (b) happens, u will still belong to some subinterval $\{v_1', v_1'+1, \cdots, v_2'\} \subseteq \{v_1, v_1+1, \cdots, v_2\}$ such that $\lim_{t\to\infty} |X_t(v) - X_t(v+1)| = 0$ for all v in this subinterval and that no opinion adjustment will occur on $\{v_1'-1, v_1'\}$ and $\{v_2', v_2'+1\}$ from some time $T'' \geq T'$ onwards. Here we allow that $v_1' = v_2'$. Reasoning as above, we know that the opinion for any vertex in the subinterval must converge to the average value $\frac{1}{v_2'-v_1'+1}\sum_{v=v_1'}^{v_2'} X_{T''}(v)$. Thus, (10) holds true.

It follows from Lemma 2 and ergodicity that with probability 1 there exists a vertex w which is two-sidedly ε-flat. Employing Lemma 3 and (10) we obtain $X_\infty(w) \in [1/2 - 6\varepsilon, 1/2+6\varepsilon]$. By Proposition 3, we obtain either $|X_\infty(w) - X_\infty(w+1)| > D_w$ or $X_\infty(w) = X_\infty(w+1)$. However, the first option leads to

(i) $X_\infty(w+1) > \frac{1}{2} - 6\varepsilon + D_w$,

or

(ii) $X_\infty(w+1) < \frac{1}{2} + 6\varepsilon - D_w$.

Since $\mathrm{P}(D_w \geq \mathrm{E}(D) - \varepsilon) > 0$, it follows from (i) that the event $X_\infty(w+1) > 1/2 - 6\varepsilon + \mathrm{E}(D) - \varepsilon > 1$ happens with strictly positive probability. This clearly is impossible. Likewise, (ii) gives rise to the event $X_\infty(w+1) < 0$ happening with strictly positive probability, which is impossible either. Hence, we obtain $X_\infty(w) = X_\infty(w+1)$. Iteratively, we obtain $X_\infty(w) = X_\infty(u)$ for all $u \in \mathbb{Z}$. This, however, contradicts the assumption that the event in (9) holds with positive probability. Therefore, the equality (9) holds true and (8) is proved.

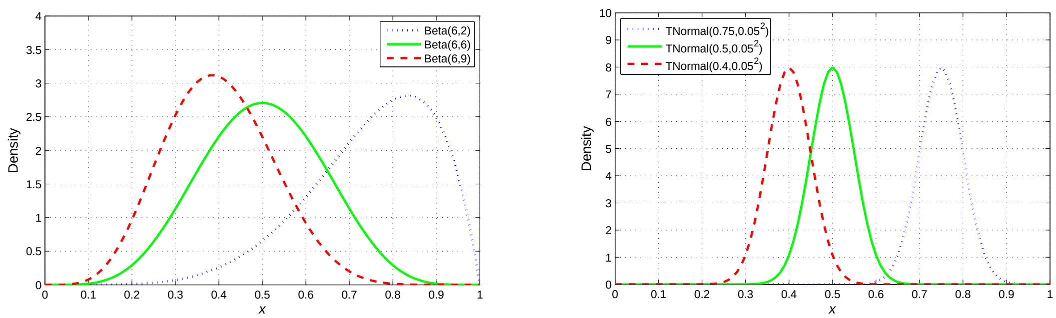
Next, for the vertex w chosen above, we obtain by Lemma 3 that $X_t(w) \in [1/2 - 6\varepsilon, 1/2 + 6\varepsilon]$ for all $t \geq 0$. For any $u \in \mathbb{Z}$, we obtain with probability 1 that $X_t(u) \in [1/2-7\varepsilon, 1/2+7\varepsilon]$ for large enough t by applying (8), because there are only finitely many edges between u and w. We get the result as desired by taking $\varepsilon \to 0$. □

Combining Proposition 2 and Proposition 4 we finish the proof of Theorem 1.

4 Simulation results

In this section, we carry out extensive agent-based simulations on rings/circles as well as BA networks [34] of different sizes to illustrate the availability and enhance our understanding of the analytical results.

We deal with three types of confidence threshold D, which are modeled by continuous as well as discrete random variables taking values in $[0, 1]$. Specifically, we consider the following distributions (their density curves are plotted in Figure 1) [27]:



(a) $D \sim \mathrm{Beta}(\alpha, \beta)$ (b) $D \sim \mathrm{TNormal}(\hat{\mu}, \hat{\sigma}^2)$

Figure 1: Probability density functions of confidence threshold D studied in the simulations. (a) beta distributions $\mathrm{Beta}(\alpha, \beta)$ with $(\alpha, \beta) = (6, 2)$ (blue dotted line), $(\alpha, \beta) = (6, 6)$ (green solid line), and $(\alpha, \beta) = (6, 9)$ (red dashed line); (b) truncated normal distributions $\mathrm{TNormal}(\hat{\mu}, \hat{\sigma}^2)$ with $(\hat{\mu}, \hat{\sigma}) = (0.75, 0.05)$ (blue dotted line), $(\hat{\mu}, \hat{\sigma}) = (0.5, 0.05)$ (green solid line), and $(\hat{\mu}, \hat{\sigma}) = (0.4, 0.05)$ (red dashed line).

- $D \sim \mathrm{Beta}(\alpha, \beta)$: beta distribution on the interval $[0, 1]$ with parameters $\alpha > 0$ and $\beta > 0$. Its probability density function is $f_{\mathrm{Beta}(\alpha,\beta)}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} I_{\{0 \leq x \leq 1\}}$, where B is the beta function. The expectation is $\mathrm{E}(D) = \frac{\alpha}{\alpha+\beta}$.

We will fix $\alpha = 6$ and vary the value of β, representing unimodal but (relatively) flatly distributed confidence level; see Figure 1(a).

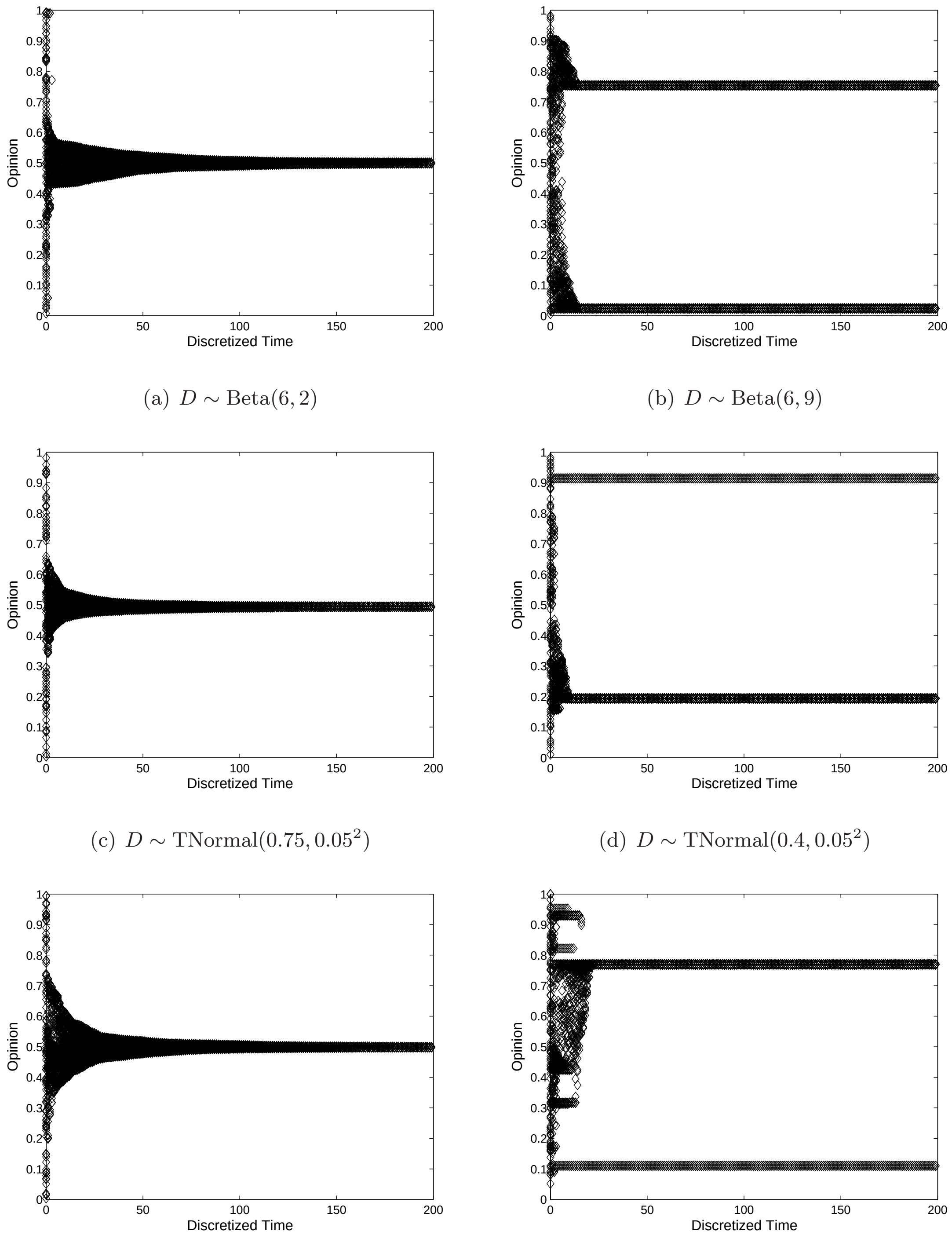
- $D \sim \text{TNormal}(\hat{\mu}, \hat{\sigma}^2)$: truncated normal distribution on $[0,1]$. The probability density function is $f_{\text{TNormal}(\hat{\mu},\hat{\sigma}^2)}(x) = \frac{1}{\hat{\sigma} Z}\phi(\frac{x-\hat{\mu}}{\hat{\sigma}})I_{\{0\le x\le 1\}}$, where $Z = \Phi(\frac{1-\hat{\mu}}{\hat{\sigma}}) - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})$, with ϕ and Φ being the probability density function and cumulative distribution function of the standard normal distribution, respectively. Its expectation can be calculated as $\text{E}(D) = \hat{\mu} + \frac{\hat{\sigma}}{Z}\left(\phi(\frac{-\hat{\mu}}{\hat{\sigma}}) - \phi(\frac{1-\hat{\mu}}{\hat{\sigma}})\right)$.

  We will fix $\hat{\sigma} = 0.05$ and vary the value of $\hat{\mu}$, representing unimodal and sharply peaked confidence level; see Figure 1(b).

- $D \sim \text{TwoPoint}(a, b, p)$: two-point distribution focusing on a and b, which takes value a with probability p and takes value b with probability $1 - p$. The expectation is $\text{E}(D) = ap + b(1 - p)$.

  We will fix $a = 0.3$, $b = 0.8$, and vary the probability $p \in (0, 1)$. Although this probability distribution is essentially discrete, an inspection of the proof shows that Theorem 1 still holds under this distribution of D, which represents bimodal confidence level.

To begin with, we perform simulation to display the time evolution of opinions among a population of $n = 1000$ agents distributed on a ring. Six random confidence thresholds D are treated, namely, Beta$(6, 2)$ (Figure 2(a)), TNormal$(0.75, 0.05^2)$ (Figure 2(c)), and TwoPoint$(0.3, 0.8, 0.1)$ (Figure 2(e)) all with approximately the same $\text{E}(D) = 0.75$; Beta$(6, 9)$ (Figure 2(b)), TNormal$(0.4, 0.05^2)$ (Figure 2(d)), and TwoPoint$(0.3, 0.8, 0.8)$ (Figure 2(f)) all with approximately the same $\text{E}(D) = 0.4$. The opinion dynamics is performed by monitoring 1000 independent Poisson processes, each of which determines the meeting time of a pair of agents as described in the model. For better illustration, we plot in Figure 2 the opinion behavior of a randomly selected subset of 100 agents out of 1000, and discretize the time axis by compressing 50000 times of Poisson jumps into one time unit.



(a) $D \sim \text{Beta}(6, 2)$

(b) $D \sim \text{Beta}(6, 9)$

(c) $D \sim \text{TNormal}(0.75, 0.05^2)$

(d) $D \sim \text{TNormal}(0.4, 0.05^2)$

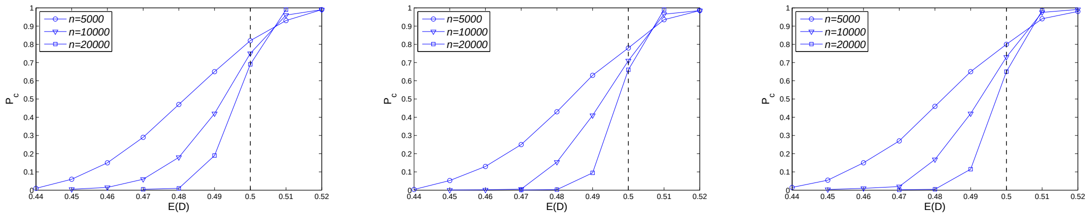
(e) $D \sim \text{TwoPoint}(0.3, 0.8, 0.1)$

(f) $D \sim \text{TwoPoint}(0.3, 0.8, 0.8)$

Figure 2: Results of opinion dynamics with $\mu = 0.5$ on a circle consisting of $n = 1000$ agents. Only a subset of 100 agents out of 1000 is displayed. Each opinion is represented by a hollow diamond. Each plot is the sketch of one realization of the simulation. In the left column: $\text{E}(D) \approx 0.75$; In the right column: $\text{E}(D) \approx 0.4$.

We draw the following observation from Figure 2. Firstly, the opinions converge to 1/2, the average of the initial opinions, when $\mathrm{E}(D) > 1/2$ (see Figure 2(a), (c) and (e)); while they diverge when $\mathrm{E}(D) < 1/2$ (see Figure 2(b), (d) and (f)). These agree well with theoretical predictions of Theorem 1. Next, comparing Figure 2(a), (c) and (e) we see that consensus occurs over time, but the rate of consensus is controlled by the distribution of D to some extent: Opinions converge clearly faster for D obeying a truncated normal distribution than a beta or two-point distribution. For example, in Figure 2(c) the consensus is reached around $t = 50$, while in Figure 2(a) and (e) the consensus is reached around $t = 90$. We performed several tests by keeping the same $\mathrm{E}(D) > 1/2$ for these three distributions, and they confirm this phenomenon. We interpret it as follows. Given a convergence parameter μ, the convergence rate is determined by the small values of D, reminiscent of the well-known Wooden Bucket Theory. An agent u with a smaller D_u or D_{u-1} has less chance to exchange opinion with its neighbors, and thus dramatically impedes the convergence. When D follows a sharp distribution like the truncated normal distribution illustrated in Figure 1(b), there are extremely few agents that have small confidence thresholds. Therefore, the consensus forms rapidly. On the contrary, there exist some agents that have small confidence thresholds (e.g. $D_u < 1/2$) for D with beta and two-point distributions, and relatively slow consensus is expected. An interesting implication behind could be that the existence of a handful of closed-minded individuals in a community A might make agreement difficult as opposed to another community B which has even the same average openness of mind (i.e., the same $\mathrm{E}(D)$) as community A. In the Fukushima-triggered panic buying case, this implies that a group of nearly homogenous agents (i.e., D with a sharply peaked unimodal distribution) could reach consensus faster than a group of heterogeneous ones. This sheds light on the opinion spreading pattern and is instrumental in fighting rumors for the government and health officials.

Next, we investigate the critical value of the opinion phase transition on rings for the above three types of confidence thresholds. We fix $\mu = 0.5$ and consider three different population sizes, namely, $n = 5000, 10000$, and 20000. For a given population n and a distribution of confidence threshold D, we run the opinion model algorithm on 500 samples. The algorithm proceeds until no agent changes its opinion by more than 10^{-4} for 10^6 times of consecutive Poisson jumps. We qualify this regime as quasi steady state. Once the system reaches the quasi steady state, we examine whether all agents belong to the same cluster or not. Define P_c as the fraction of samples which attain a complete consensus, i.e., the final opinion configuration is a single cluster. In Figure 3, we plot P_c as a function of expected confidence threshold $\mathrm{E}(D)$ for different distributions of D and values of n.



(a) $D \sim \mathrm{Beta}(6, \beta)$ (b) $D \sim \mathrm{TNormal}(\hat{\mu}, 0.05^2)$ (c) $D \sim \mathrm{TwoPoint}(0.3, 0.8, p)$

Figure 3: Fraction of samples with complete opinion consensus as a function of $\mathrm{E}(D)$ in a quasi-steady-state regime. Three different numbers of individuals located on rings are considered: $n = 5000$ (circles), 10000 (lower triangles), and 20000 (squares). We fix $\mu = 0.5$ in the simulation. (a) $D \sim \mathrm{Beta}(6, \beta)$ with $\beta \in [5.54, 7.64]$; (b) $D \sim \mathrm{TNormal}(\hat{\mu}, 0.05^2)$ with $\hat{\mu} \in [0.44, 0.52]$; (c) $D \sim \mathrm{TwoPoint}(0.3, 0.8, p)$ with $p \in [0.56, 0.72]$. The threshold $\mathrm{E}(D) = 1/2$ is indicated by a vertical dashed line in each plot.

These results show sigmoidal variations of the probability P_c with respect to the expectation of confidence threshold $\mathrm{E}(D)$. For all distributions of D considered, the population is able to reach consensus with probability 1 if $\mathrm{E}(D)$ is large. On the other hand, no consensus emerges at a small $\mathrm{E}(D)$. On examining the growth of P_c against different population size n, we find that the swift transition from a disordered phase—corresponding to $P_c = 0$—to an ordered one—corresponding to $P_c = 1$—happens at the critical value $\mathrm{E}(D) = 1/2$ as n grows as expected. Taking a closer look at Figure 3, we further find that the transition in Figure 3(b) is sharper than those in Figure 3(a) and Figure 3(c). By comparing these with the study in [17], where even sharper transition was displayed for deterministic confidence threshold, we are therefore led to conclude that the sharper the distribution of D is, the sharper the transition of consensus formation turns. In other words, a community of individuals with homogeneous confidence levels will have a sharper phase transition of opinion agreement than a community of individuals with heterogeneous confidence levels. In the economic literature, *homophily* refers to the tendency of individuals to associate disproportionately with others having similar traits [31]. It is one of the most popular and robust trends of the way in which people relate to each other. A group of agents with homogeneous confidence levels is attributed to a high level of homophily. Our finding reveals that the sharpness of phase transition at $\mathrm{E}(D) = 1/2$ depends on homophily in ways that are comparable to the recent studies in social economy [32, 33], where homophily is shown to have intricate impact on the speed of some learning and updating dynamics.

In the final part of this section, we move our battlefield from one dimensional lattices to a heterogeneous network, the BA scale-free network [34], where the probability of a vertex having degree k is proportional to k^{-3} (see Figure 4) and the network size $n = 1000$. We fix $\mu = 0.5$ as before. Given the distribution of confidence threshold D, we perform the continuous opinion model algorithm on this network until the system enters a quasi steady state. Figure 5(a) shows the numbers of opinion clusters in different settings of α and β when D obeys beta distribution $\mathrm{Beta}(\alpha, \beta)$; Figure 5(b) shows the numbers of opinion clusters in different settings of a and b when D obeys two-point distribution $\mathrm{TwoPoint}(a, b, p)$ with fixed $p = 0.5$.

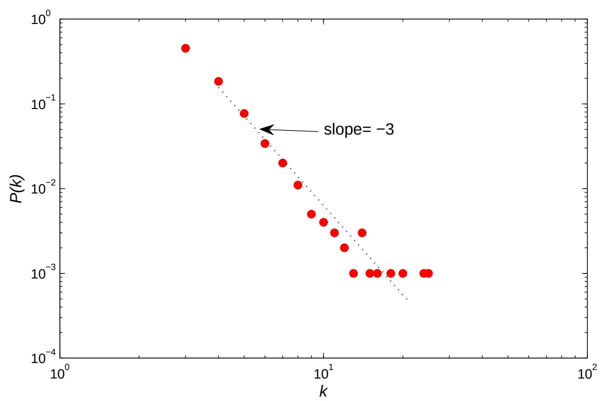
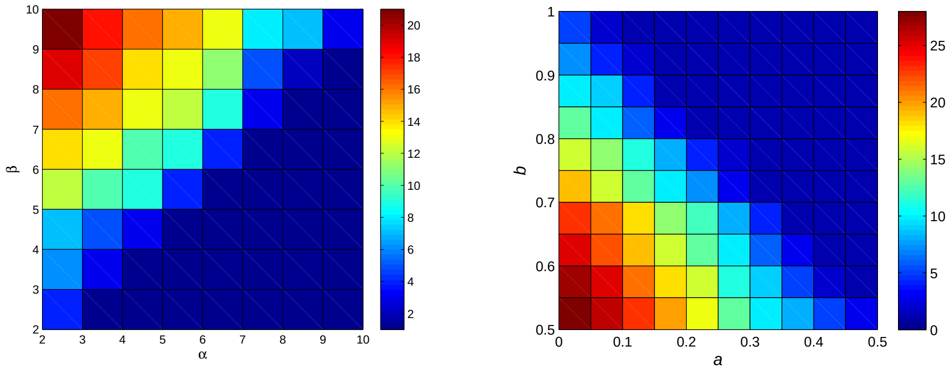


Figure 4: The degree distribution $P(k)$ (i.e., the fraction of vertices having degree k) of a BA scale-free network used in the simulation.

Note that when $D \sim \mathrm{Beta}(\alpha, \beta)$, the critical confidence threshold (if Theorem 1 holds for scale-free networks) corresponds to $\mathrm{E}(D) = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$, i.e., $\alpha = \beta$. This agrees with Figure 5(a), in which the consensus is seen to emerge when $\alpha > \beta$. Likewise, the critical confidence threshold for the case $D \sim \mathrm{TwoPoint}(a, b, 0.5)$ would be $\mathrm{E}(D) = \frac{a+b}{2} = \frac{1}{2}$, i.e., $a + b = 1$. Figure 5(b) shows that the opinions converge to a single cluster when $a + b > 1$, going some way to supporting a universality of the critical value $\mathrm{E}(D) = 1/2$ stated in



(a) $D \sim \text{Beta}(\alpha, \beta)$ (b) $D \sim \text{TwoPoint}(a, b, 0.5)$

Figure 5: Display of the average number (coded by colors) of opinion clusters in a quasi-steady-state regime. (a) D follows $\text{Beta}(\alpha, \beta)$ distribution with α and β varying between 2 and 10; (b) D follows $\text{TwoPoint}(a, b, p)$ distribution with fixed $p = 0.5$, a varying between 0 and 0.5, and b varying between 0.5 and 1. 10 replications are done for each tested couple (α, β) in (a) and (a, b) in (b), respectively.

Theorem 1. Although we do not have a direct proof yet, an intuitive argument towards the universality can be as follows. Since the opinion dynamics is symmetric, the distribution of the opinion clusters is also symmetric around the central opinion 1/2. Consider two extreme opinion clusters close to 0 and 1. The opinion close to 0 can exchange with opinions in $[0, D]$. The center of the peak $p_0 \approx D/2$ due to the symmetric dynamics. Likewise, the other opinion peak $p_1 \approx 1 - D/2$. The distance between these two clusters is about $1 - D < D$, or $1 - \text{E}(D) < \text{E}(D)$ on average if $\text{E}(D) > 1/2$. Therefore, when $\text{E}(D) > 1/2$ all opinions will be ultimately attracted by the major cluster centered at 1/2, irrespective of the structure of the underlying social network.

5 Conclusions

We have presented a continuous opinion model with random confidence threshold D, which is preferable for a realistic community when heterogeneous confidence levels are involved. The results reported in this paper allow us to formulate the following important conclusions:

(i) When the underlying network is modeled as $\mathbb{Z}$, the critical confidence threshold is identified to be $\text{E}(D) = 1/2$—with probability one, all opinions converge to the average of the initial opinions, $1/2$, once $\mathrm{E}(D) > 1/2$. The specific distribution of D plays no part in the critical phenomenon. This result is further extended to the case of scale-free networks, which is common in real-world complex systems.

(ii) For $\mathrm{E}(D) > 1/2$, the rate of reaching consensus is closely related to the distribution of D—opinions converge faster with sharply distributed D than with broadly distributed or bimodal D. This implies that a handful of closed-minded agents could prominently delay the progress of reaching agreement in a community even the confidence threshold on average, namely $\mathrm{E}(D)$, is not low.

(iii) The phase transition at the critical value $\mathrm{E}(D) = 1/2$—from non-consensus to consensus—becomes sharper when the distribution of D is sharper. Hence, a community of individuals with homogeneous confidence levels will have a more rapid transition of opinion agreement as compared to a diverse community with heterogeneous confidence levels in line with the concept of homophily in social sciences and economics. This implies that it is possible to expedite the consensus process by partitioning a heterogeneous community into several more homogeneous subcommunities.

For the future research, it is intriguing to explore one particular problem: how can the social networks influence the rate of convergence and the number of final opinion clusters? Some widely used networks such as small-world networks, high-dimensional lattices as well as real social networks should be deeply investigated. The critique of these results in a sociological perspective might also be a challenge which is out of the province of this paper.

Moreover, several extensions of the model are desirable. We may consider, for example, the vertex-centric opinion dynamics such as the RA models [36] as discussed before. In this paper, we adopted a prompt opinion reaction mechanism. In reality, the opinion exchange should be of slowness or retardation for an individual. Therefore, one may consider other opinion updating mechanisms taking into account time delay. Besides, repulsive opinion interaction and multiple possible choices based on the Social Judgment Theory are also of practical interest [40].

Acknowledgments

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Appendix

In the appendix, we briefly review the sharing a drink (SAD) scheme proposed in [26], which turns out to be closely related with the DW model on $\mathbb{Z}$.

The SAD scheme, denoted by $\{Y_i(u)\}_{u\in\mathbb{Z}}$, is a deterministic process and runs in discrete time. It can be defined iteratively as follows. Set

$$Y_0(u) = \begin{cases} 1 & \text{for } u = 0; \\ 0 & \text{for } u \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{11}$$

For a given sequence $u_1, u_2, \cdots \in \mathbb{Z}$ and $\mu \in (0, 1/2]$, we obtain the configuration $\{Y_i(u)\}_{u\in\mathbb{Z}}$ for $i \geq 1$ by letting

$$Y_i(u) = \begin{cases} Y_{i-1}(u) + \mu(Y_{i-1}(u+1) - Y_{i-1}(u)) & \text{for } u = u_i; \\ Y_{i-1}(u) + \mu(Y_{i-1}(u-1) - Y_{i-1}(u)) & \text{for } u = u_i + 1; \\ Y_{i-1}(u) & \text{for } u \in \mathbb{Z}\backslash\{u_i, u_i+1\}. \end{cases} \tag{12}$$

This procedure can be vividly depicted as a liquid exchanging process on $\mathbb{Z}$. A glass is put at each site $u \in \mathbb{Z}$. At $i = 0$ only the glass located at the origin is full (indicated by 1) while all others are empty (indicated by 0). At each subsequent step i, we pick two adjacent glasses at u_i and $u_i + 1$, and pouring liquids from the glass with higher level to that with lower level by a relative amount μ.

Fix time $t > 0$ and consider our opinion model on $\mathbb{Z}$. Note that there exists a finite interval $[u_\alpha, u_\beta] \subseteq \mathbb{Z}$ containing 0 such that the Poisson events on the two boundary edges $\{u_\alpha - 1, u_\alpha\}$ and $\{u_\beta, u_\beta + 1\}$ have not happened up to time t [26]. Let N be the number of opinion adjustments occur in $[u_\alpha, u_\beta]$ up to time t. The times of these adjustments are arranged in the chronological order as

$$\tau_{N+1} := 0 < \tau_N < \tau_{N-1} < \cdots < \tau_1 \leq t,$$

where we set $\tau_{N+1} := 0$ for convenience. For $i = 1, \cdots, N$, we set u_i be the left endpoint of the edge $\{u_i, u_i + 1\}$ for which u_i and u_{i+1} adjust opinions at time τ_i. Given the sequence

$u_1, \cdots, u_N$ (in exactly this order) and $\mu \in (0, 1/2]$, we obtain a SAD process $\{Y_i(u)\}_{u\in\mathbb{Z}}$ as defined by (11) and (12). The following lemma is a key ingredient towards our main result.

Lemma A1.[26] *For* $i = 0, 1, \cdots, N$,

$$X_t(0) = \sum_{u\in\mathbb{Z}} Y_i(u) X_{\tau_{i+1}}(u).$$

In particular, $X_t(0) = \sum_{u\in\mathbb{Z}} Y_N(u) X_0(u) := \sum_{u\in\mathbb{Z}} Y_t(u) X_0(u)$.

In other words, the opinion at the origin can be expressed as a linear combination of initial opinions over sites of $\mathbb{Z}$, with coefficients given by the SAD process. Lemma A1 can be proved by using straightforward induction over i in the same way as Lemma 3.1 [26].

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