

Effective Rate Analysis in Weibull Fading Channels

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Abstract—Recently the theory of effective rate has attracted much attention, since it can take the delay aspect into account when performing channel capacity analysis. Weibull fading model is a flexible and effective model for describing fading channels in both indoor and outdoor environments. This letter derives the exact analytical expressions of effective rate under independent but not necessarily identical Weibull fading channels, which can be used in the system analysis and design in real-time communication scenarios. The exact analytical results as well as the closed-form approximation under high signal-to-noise ratio (SNR) and low-SNR situations are given for reducing computational complexity.

Index Terms—Weibull fading, fading channels, effective capacity, effective rate.

I. INTRODUCTION

Many emerging applications, such as the voice over IP and most smart grid applications, are real-time applications. We need to consider not only the throughput but also the delay performance as quality of service (QoS) metrics. The delay performance would vary with time due to channels' temporal fading feature. However, traditional Shannon's ergodic capacity cannot account for the delay aspect. In order to solve this issue, the effective rate (or effective capacity) theory is proposed by Wu and Negi [1]. Similar to the traditional channel capacity theory, effective rate analysis will rely on both communication mechanism design and system performance analysis involving the delay as one of QoS metrics. Hence it has attracted much attention of effective rate analysis on single or independent and identical (i.i.d) fading channels, such as Nakagami- m , Rician and generalized-K [1], $\eta-\mu$, $\alpha-\mu$ [2] and $\kappa-\mu$ shadowed channels [3]. But in practical systems, the fading parameters associated to different channels may not be identical, particularly in the situations where the antennas are separated in space, such as the distributed antenna system (DAS). We refer these channels as independent and not necessarily identical (i.n.i.d) fading channels. To the authors' best knowledge, only the effective rate under i.n.i.d $\kappa-\mu$ fading channels has been studied [4].

The performance analysis over fading channels plays an important role in communication systems. IEEE Vehicular Technology Society Committee on Radio Propagation has recommended Nakagami- m and Weibull models for theoretical studies of fading channels [5]. The effective rate under Nakagami- m fading channels has been well studied [1], but the effective rate under Weibull fading model has not been

researched yet. Weibull fading is an important fading model, since it is flexible in describing the fading severity of the wireless channel. Measurements show that distributions of the small scale fading in many scenarios follow Weibull distributions, such as fixed-to-walk and mobile-to-mobile channels [6].

This letter derives exact analytical expressions of effective rate under i.n.i.d Weibull fading channels, as well as closed-form approximated expressions in high signal-to-noise ratio (SNR) and low-SNR regimes. These approximations give a deeper insight of the parameters influence on the systems' performance and have a lower computational complexity. Simulation results are also presented to verify the analysis.

II. SYSTEM MODEL

Consider a multiple input single output (MISO) channel model, where there are N transmit antennas separated in space and only one receive antenna. It is assumed that the channels are flat-fading and the channels between different antennas are i.n.i.d channels. Then the channel's input-output relation can be expressed as $r = \mathbf{h}\mathbf{x} + n_0$, where $\mathbf{h} \in \mathbb{C}^{1 \times N}$ denotes the MISO channel vector and \mathbf{x} is the transmit vector. n_0 represents the complex additive white Gaussian noise with zero mean and variance N_0 . In addition, uniform power allocation across the antennas is assumed.

A. Effective Rate

Effective rate is the maximum channel rate that a fading channel can support under a statistical QoS requirement described by QoS exponent θ . The effective rate of the service process can be written as [1]

$$\alpha(\theta) = -\frac{1}{\theta T} \ln \mathbb{E} \{ e^{-\theta TC} \}, \quad \theta \neq 0, \quad (1)$$

where C represents the system's throughput during a single time block and T denotes the duration of a time block. $\mathbb{E}\{\cdot\}$ is the expectation operator. The QoS exponent θ is given by $\theta = -\lim_{x \rightarrow \infty} \ln Pr[L > x]/x$, where L is the equilibrium queue-length of the buffer present at the transmitter. The QoS exponent θ has to satisfy the constraint $\theta \geq \theta_0$, where θ_0 is the minimum required QoS exponent. Larger θ_0 corresponds to a tighter delay constraint, while $\theta_0 \rightarrow 0$ implies no delay constraints.

Suppose that the transmitter sends uncorrelated circularly symmetric zero-mean complex Gaussian signals, then the effective rate can be written as [1]

$$R(\rho, \theta) = -\frac{1}{A} \log_2 \left(\mathbb{E} \left\{ \left(1 + \frac{\rho}{N} \mathbf{h} \mathbf{h}^\dagger \right)^{-A} \right\} \right), \quad (2)$$

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where $A = \theta TB / \ln 2$, with B denoting the bandwidth of the system and ρ is the average SNR. The symbol $(\cdot)^\dagger$ represents the Hermitian transpose.

B. The Weibull Distribution

Weibull fading occurs when the signal recovered from the wireless channel is composed of clusters of a multipath wave, each of which propagates in non-homogeneous environment, such that they possess similar delay times and with the delay-time spreads of different clusters and their phases are independent [7]. Assume that $|h_n|$ are i.n.i.d Weibull random variables (RVs) and denote $Y_n = |h_n|^2$. Then Y_n is also distributed as a Weibull random variable. For a single Weibull fading channel, the probability density function (pdf) of Y_n becomes [6]

$$f_{Y_n}(y) = \frac{\beta_n}{\omega_n} \left(\frac{y}{\omega_n} \right)^{\beta_n - 1} e^{-\left(\frac{y}{\omega_n}\right)^{\beta_n}}, \quad (3)$$

where β_n and ω_n are the shape parameter and scale parameter corresponding to that fading channel, respectively.

III. EFFECTIVE RATE ANALYSIS

A. Exact Analysis

Let $Y = \sum_{n=0}^{N-1} Y_n$ denote the sum of N i.n.i.d Weibull RVs, then the pdf of Y can be given by [7]

$$f_Y(y) = \sum_{l=0}^{\infty} \sum_{k=0}^l (-1)^k \binom{l}{k} a_l \frac{y^{N+k-1}}{\Gamma(N+k)\Omega^{N+k}} e^{-y/\Omega}, \quad (4)$$

where

$$a_l = \sum_{k_0 + \dots + k_{N-1} = l} \prod_{n=0}^{N-1} \sum_{k=0}^{k_n} \frac{(-1)^k}{\Omega^k k!} \binom{k_n}{k} \Gamma\left(1 + \frac{k}{\beta_n}\right) \omega_n^{\frac{k}{\beta_n}}, \quad (5)$$

$$\Omega = \frac{2}{N} \sum_{n=0}^{N-1} \Gamma\left(1 + \frac{1}{\beta_n}\right) \omega_n^{\frac{1}{\beta_n}}, \quad (6)$$

where $\binom{l}{k} = \frac{\Gamma(l+1)}{\Gamma(k+1)\Gamma(l-k+1)}$, and $\sum_{k_0 + \dots + k_{N-1} = l}$ denotes the summation over all the possible non-negative integers k_0, \dots, k_{N-1} satisfying the condition of $k_0 + \dots + k_{N-1} = l$. $\Gamma(t)$ is Gamma function given by $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx$.

Theorem 1. *The exact analytical expression of effective rate $R(\rho, \theta)$ for MISO i.n.i.d Weibull fading channels can be given by*

$$R(\rho, \theta) = \log_2 \frac{\Omega \rho}{N} - \frac{1}{A} \log_2 \left\{ \sum_{l=0}^{\infty} \frac{a_l \Gamma(A+l)}{\Gamma(A)} \times U\left(A+l; A+1-N; \frac{N}{\Omega \rho}\right) \right\}, \quad (7)$$

where $U(\cdot)$ is Tricomi hypergeometric function [8, eq. (13.2.5)].

Proof. See Appendix A. \square

Let $b_l = \Gamma(A+l) U(A+l; A+1-N; N/(\Omega \rho)) / \Gamma(A)$, it can be proved that b_l is positive, monotonically decreasing and bounded by b_0 . As a_l converges [9], the convergence of (7)

follows the Abel's criterion. Tricomi U function has already been a build-in function in numerical softwares such as Matlab and Maple. Also as highlighted in [9] and verified in the simulation in Section IV, small l (e.g $l=50$) shows a good approximation to the exact value.

Corollary 1. *For i.i.d Rayleigh situations, if let $\beta_n=1$, $\omega_n=\omega$, the effective rate expression (7) simplifies to*

$$R(\rho, \theta) = \log_2 \frac{\omega \rho}{N} - \frac{1}{A} \log_2 U\left(A; A+1-N; \frac{N}{\omega \rho}\right). \quad (8)$$

Proof. Substitute $\beta_n=1$, $\omega_n=\omega$ to (7) and use the binomial coefficient identities [8, eq. (26.3.3-4)], the desired results follows. \square

Remark: This result coincides with [10, eq. (11)], verifying the accuracy of our derivations.

B. Approximation Analysis

Although the given expression in (7) is exact, the effects of the parameters, such as the delay exponent and antenna numbers, are implicit. In addition, the exact analytical expression involves infinite sum, which is not very favourable in application. Hence in this part, we provide approximated closed forms of the effective rate in both high-SNR and low-SNR regimes. These approximations are easy to compute and give deeper insight of the parameters' influence on the systems' performance, as shown below.

Theorem 2. *If assume the joint pdf $f_Y(y)$ and the associated moment generating function $M_Y(s)$ satisfy the following assumption as [11]*

$$\begin{cases} \lim_{y \rightarrow 0} f_Y(y) = py^t \\ \lim_{s \rightarrow +\infty} M_Y(s) = q|s|^{-d} \end{cases} \quad (9)$$

where $p = q/\Gamma(d)$, $t = d-1$ and $M_Y(s) = \mathbb{E}\{e^{-sy}\}$, then for MISO i.n.i.d Weibull fading channels in high-SNR situations, the effective rate can be approximated by

$$R(\rho, \theta) \approx -\frac{1}{A} \log_2 \frac{(N\bar{\omega})^{N\bar{\beta}} \prod_{n=0}^{N-1} \omega_n^{-\beta_n}}{\Gamma(N\bar{\beta}+1)} F(A, N\bar{\beta}; N\bar{\beta}+1; -\rho\bar{\omega}). \quad (10)$$

where $\bar{\beta} = \frac{1}{N} \sum_{n=0}^{N-1} \beta_n$ and $\bar{\omega} = \frac{1}{N} \sum_{n=0}^{N-1} \omega_n$ are the average shape parameter and scale parameter, and $F(\cdot)$ is the hypergeometric function [8, eq.(15.1.1)].

Proof. See Appendix B. \square

It is much easier to evaluate the closed-form (10) compared to (7), where hypergeometric function can be easily calculated since it is also a build-in function in numerical softwares such as Matlab and Maple. It can be seen that the effective rate in high SNR is mainly related to the averaged shape parameter and scale parameter, which means the effect of fading on the communication system has been alleviated. In many communication networks, such as cellular networks, systems often operate in low-SNR situations [12]. Hence it is also beneficial to derive a more effective approximation in

the low-SNR regime. We note that in the low-SNR regime, analysis based on SNR would lead to misleading conclusions as explained in [12]. The capacity analysis should be better expressed as a function of transmitted normalized energy per information bit E_b/N_0 rather than SNR. Hence following the same method in [13], the effective rate in the low-SNR regime can be approximated as

$$R\left(\frac{E_b}{N_0}, \theta\right) \approx S_0 \log_2\left(\frac{\frac{E_b}{N_0}}{\frac{E_b}{N_0}_{\min}}\right), \quad (11)$$

where S_0 denotes the capacity slope in bits/s/Hz/(3dB) and $E_b/N_{0\min}$ is the minimum energy per information bit required to reliably convey any positive rate, which is given by [12]

$$\begin{cases} \frac{E_b}{N_{0\min}} = \lim_{\rho \rightarrow 0} \frac{\rho}{R(\rho, \theta)} = \frac{1}{R'(0, \theta)} \\ S_0 = -\frac{2[R'(0, \theta)]^2 \ln 2}{R''(0, \theta)} \end{cases} \quad (12)$$

where $R'(0, \theta)$ and $R''(0, \theta)$ denote the first and second order partial derivatives with respect to ρ and evaluated at $\rho=0$.

Theorem 3. *The effective rate for MISO i.n.i.d Weibull fading channels in low-SNR regimes can be approximated by (11), with the metrics given by*

$$\frac{E_b}{N_{0\min}} = \frac{2 \ln 2}{\Omega} = \frac{N \ln 2}{\sum_{n=0}^{N-1} \Gamma\left(1 + \frac{1}{\beta_n}\right) \omega_n^{\frac{1}{\beta_n}}}, \quad (13)$$

$$S_0 = \frac{2}{1 + (A+1) \frac{\sum_{n=0}^{N-1} \omega_n^{\frac{2}{\beta_n}} \left\{ \Gamma\left(1 + \frac{2}{\beta_n}\right) - \left[\Gamma\left(1 + \frac{1}{\beta_n}\right) \right]^2 \right\}}{\left[\sum_{n=0}^{N-1} \Gamma\left(1 + \frac{1}{\beta_n}\right) \omega_n^{\frac{1}{\beta_n}} \right]^2}}. \quad (14)$$

Proof. See Appendix C. \square

We note that the capacity slope S_0 has a reverse relationship with the parameter $A = \theta TB / \ln 2$. Hence with the same increase of E_b/N_0 , the transmission with more relaxed QoS requirement will gain more improvement for the upper bound of the supported rate. It can be inferred that, the increase of the bandwidth or the QoS requirement will result in a lower effective rate, under the same fading channel conditions and E_b/N_0 . Also the low-SNR approximation is closed-form, which is easy to compute.

IV. NUMERICAL RESULTS

In this section, numerical simulations are used to verify the proposed theoretical analysis in Section III. As β_n describes the fading severity of the channel while ω_n describes the average power of fading, the parameters under different numbers of transmit antennas are assumed and listed in Table I [9]. Without loss of generality, $B=1$ and $T=1$ is assumed [4].

As QoS exponent θ increases, the delay constraints become more stringent, which is corresponding to the increase of A in this case. It is illustrated in Fig. 1 that analytical results agree with the simulation results. Also under the same channel fading scenarios, the overall effective rate decreases with the increase of the delay constraints.

TABLE I
PARAMETERS FOR DIFFERENT I.N.I.D WEIBULL FADING SCENARIOS

Scenario	Parameters
$N=1$	$\omega_1=1, \beta_1=1$
$N=2$	$\omega_1=1, \beta_1=1, \omega_2=1, \beta_2=2$
$N=3$	$\omega_1=1, \beta_1=1, \omega_2=1, \beta_2=2, \omega_3=1, \beta_3=3$

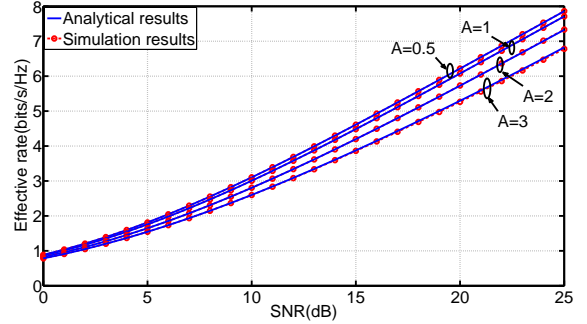


Fig. 1. Analytical and simulated effective rate versus the average SNR of i.n.i.d Weibull fading channels with $N=2$.

High-SNR approximation results and the simulation results are compared in Fig. 2 under the same QoS requirement. It can be seen that the high-SNR approximation results agree with the simulation results under different i.n.i.d scenarios. When the average SNR is relative high, in this case above 10dB, the high-SNR approximation results have a good agreement with the simulation results.

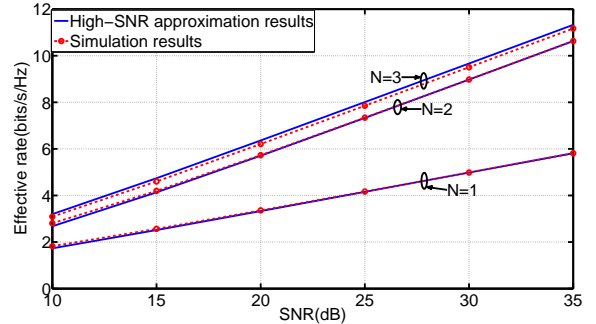


Fig. 2. High-SNR approximated and simulated effective rate versus the average SNR of i.n.i.d Weibull fading channels with different N , $A=2$.

In low-SNR regime, it can be observed in Fig. 3 that the proposed low-SNR approximation results agree with the simulation results in different i.n.i.d scenarios under the same system conditions. From both Fig. 2 and Fig. 3, it can be inferred that with the increase of the antenna numbers, the effective rate of the system also increases. As the shape parameter β_n describes the fading severity of the channel, under the same QoS requirements, the channels with larger severity of the fading contributes less to the total effective rate. It should also be noticed that the simulated channel follows Rayleigh fading when $N = 1$ and the simulation results coincide with the results in [4].

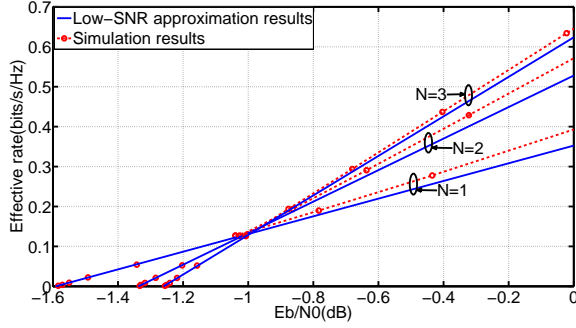


Fig. 3. Low-SNR approximated and simulated effective rate versus E_b/N_0 of i.n.i.d Weibull fading channels with different N , $A=1$.

V. CONCLUSION

In this letter, we derived new analytical expressions for the effective rate under i.n.i.d Weibull fading channels as well as the closed-form approximations in high-SNR and low-SNR situations. The convergence of our proposed expressions was proved. Numerical results showed that the simulations agreed with the analysis and the approximations. It was shown that tighter QoS requirement resulted in lower effective rate. Also the channel with more severity of fading contributed less to the overall effective rate. These results extended and complemented previous analysis on other channel fading models.

APPENDIX

A. Proof of Theorem 1

By substituting (4) into (2) and using the identity of $\int_0^\infty (1+ax)^{-v} x^{q-1} e^{-px} dx = \Gamma(q) U(q; q+1-v; p/a) / a^q$ [8, eq. (13.4.4)] as well as the Kummer's transform $U(a; b; x) = x^{1-b} U(a-b+1; 2-b; x)$ [8, eq. (13.2.40)], $R(\rho, \theta)$ can be simplified to

$$R(\rho, \theta) = -\frac{1}{A} \log_2 \sum_{l=0}^{\infty} \sum_{k=0}^l (-1)^k \binom{l}{k} a_l U\left(A; A+1-N-k; \frac{N}{\Omega\rho}\right) \quad (15)$$

Then expand the Tricomi U function with [8, eq. (13.4.4)] and use the binomial coefficient identity of $\sum_{k=0}^l \binom{l}{k} x^k = (1+x)^l$ [8, eq. (26.3.3)], (7) follows after some simplifications.

B. Proof of Theorem 2

First we truncate the integral involved in (2) by $Q = \sum_{n=0}^{N-1} \omega_n$, which captures the major part of the pdf of the sum of Weibull random variables [9]. Then the truncation error for the integration is $E(Q) = \int_Q^\infty (1 + \frac{\rho y}{N})^{-A} f(y) dy$. It can be proved that $(1 + \frac{\rho y}{N})^{-A}$ is monotonically decreasing with respect to ρ , then $E(Q)$ is upper bounded by $(1 + \frac{\rho Q}{N})^{-A}$. Hence we have $\lim_{\rho \rightarrow 0} E(Q) = 0$. Then use the relation $M_Y(s) = \prod_{n=0}^{N-1} M_{Y_n}(s)$ and the help of [9, eq. (19)] and [14], we can get the expansion as $\lim_{|s| \rightarrow \infty} M_Y(s) = \prod_{n=0}^{N-1} \omega_n^{-\beta_n} s^{-N\beta}$. Using the assumptions in (9) and [8, eq. (3.194.1)], (10) can be achieved.

C. Proof of Theorem 3

When $\rho \rightarrow 0$, the Tricomi function has the Poincare-Type expansions as $U(a; b; z) \approx z^{-a} \sum_{n=0}^{\infty} (a)_n (a-b+1)_n (-z)^{-n} / n!$ [8], where $(a)_n$ is the Pochhammer operator given by $(a)_n = a(a+1) \cdots (a+n-1)$ and $(a)_0 = 1$. By taking $\rho \rightarrow 0$ and follow the rules of limitation, we get $R'(0, \theta) = \frac{\Omega}{2 \ln 2}$ and $R''(0, \theta) = -\frac{\Omega^2}{4 \ln 2} - \frac{A+1}{N^2 \ln 2} \sum_{n=0}^{N-1} [\mathbb{E}(Y_n^2) - (\mathbb{E}Y_n)^2]$. Substitute these results into (12), (13) and (14) can be obtained.

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