

# Fixed-time group consensus for multi-agent systems with nonlinear dynamics and uncertainties

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**Abstract:** In this paper, we study fixed-time group consensus problem in networks of dynamic agents with intrinsic nonlinear dynamics and bounded uncertainties. Three types of distributed control protocols are proposed to achieve fixed-time group consensus when the subgroups are connected and have inter-group common influence. By using Lyapunov theory, algebraic graph theory, and fixed-time stability, some conditions are derived to select the controller gains to ensure the convergence in a prescribed time regardless of the initial conditions. Numerical examples are worked out to illustrate the effectiveness of our theoretical results.

## 1 Introduction

The past decade or so has witnessed the prolific progress in studying consensus problems of multi-agent systems from various contexts [1–3]. In general, the main objective of consensus problems is to make the states of a group of agents converge to some common value by designing appropriate distributed protocols. For this purpose, a broad range of consensus protocols have been proposed; see the survey papers [4, 5] and references therein.

Most of the previous works are concerned with complete consensus, i.e., the protocols therein cause all agents to converge to the same consistent state. However, in practical scenarios, there may be multiple consistent states as agents are often divided into several disjoint subgroups to cope with unanticipated situations or carry out different cooperative tasks. Examples include social learning under different environments [6], obstacle avoidance of animal herds [7], heterogeneous robots sorting [8], and team hunting of predators. As a generalization to complete consensus protocols, group (or cluster) consensus protocols [9–12] have been proposed to solve these issues, where the states of multiple agents in each subgroup converge to an individual consistent state asymptotically when information exchanges exist not only among agents within the same subgroup but among those in different subgroups. For instance, under the inter-group balance condition, group consensus problems are solved in [9] in terms of linear matrix inequalities for networks with switching topologies by introducing the double-tree-form transformations. Based on the nonnegative matrix analysis, two cluster criteria are established in [10] to deal with the group consensus of discrete-time multi-agent systems. A novel combinatorial necessary and sufficient condition for group consensus is provided in [11]. In addition to these leaderless protocols, virtual leaders and pinning control techniques have been leveraged in [13–15] to help build the desired group pattern. However, consensus in the case of leaderless strategies is more challenging since the agreed trajectories in this situation is unknown to any agents and cannot prescribed in advance.

In the study of consensus problems, convergence rate plays a significant role reflecting the effectiveness of a consensus protocol. Compared to the usual asymptotic consensus algorithms, which mean that the consensus is only achieved as time tends to infinity, finite-time controllers enjoy many attractive properties such as faster convergence rate, higher tracking accuracy, better disturbance rejection, and more robustness to uncertainties [16]. As such, finite-time consensus problems of multi-agent systems have been investigated intensively for first-order [17–19], higher-order [20], and inherent nonlinear or uncertain dynamics [21–23]. To optimize the convergence rate, the authors in [24] propose a finite-time switching protocol covering both continuous control and discontinuous control. It is worthy of noting that, for the aforementioned works, the

settling time depends on the initial conditions of the agents and grows unboundedly with the increase of initial conditions. Thus, these control laws cannot guarantee a prescribed convergence time since the knowledge of initial conditions is usually not available in advance in distributed systems.

To overcome this shortcoming, some new results based upon the fixed-time stability theory [25] have been reported recently, which allow an upper-bounded settling time independent of the initial conditions of the agents. In the leaderless case, fixed-time consensus protocols are designed in [26–29] for multi-agent systems with integrator-type dynamics. For example, nonlinear finite-time consensus protocols are introduced in [29] for multi-agent systems with nonlinear dynamics and uncertain disturbances over fixed undirected communication topology. Leader-follower fixed-time consensus problems with bounded disturbances have been solved in [30] for first-order multi-agent systems. The results have been extended recently to higher-order systems [31, 32] and multiple leaders [33]. Furthermore, under the inter-group balance condition (c.f. Remark 2), leader-follower fixed-time group synchronization for complex networks has been investigated in [34, 35].

Motivated by the above work, we in this paper consider the fixed-time group consensus problems for multi-agent systems with unknown inherent nonlinear dynamics and bounded uncertain disturbances. Group consensus has been a hot topic in recent years, which means that the network can achieve multiple consensus values as time tends to infinity. Fixed-time group consensus further requires that the settling time is bounded by the same constant for any initial values. We mention that only a finite settling time does not ensure fixed-time group consensus since it may well depend on the initial conditions. The contribution of this paper is highlighted as follows. First, compared with the previous consensus results in [25–29], a generalization of the fixed-time consensus protocol to group consensus is proposed by splitting the group of agents into multiple subgroups. We not only present the settling time regardless of the initial conditions, but address the robustness against intrinsic nonlinear dynamics as well as uncertain disturbances. Second, the ultimate convergence trajectories can be expressed as the average positions of the agents in each subgroup instead of following predetermined trajectories of some leaders. Hence, the methods in [33–35] are no longer applicable here. Finally, explicit estimations of the settling time are provided for our protocols which are distributed in the sense that only local information of each agent is needed (c.f. Remark 6).

The rest of the paper is organized as follows. Section 2 provides some preliminaries and formulates the group consensus problem. Section 3 is devoted to the analysis of the fixed-time group protocols. Some numerical examples are presented in Section 4. Finally, the conclusion is drawn in Section 5.

## 2 Preliminaries and problem statement

We start with some notations that will be used later. Let  $\mathbb{R}$  and  $\mathbb{R}_+$  represent the set of real and positive real numbers, respectively. Let  $M^T$  be the transpose of a matrix  $M$ . For a symmetric matrix  $M \in \mathbb{R}^{N \times N}$ , its second smallest eigenvalue is denoted by  $\lambda_2(M)$ . The size of a set  $S$  is denoted by  $|S|$ . For a vector  $x = (x_1, \dots, x_N)^T \in \mathbb{R}^N$  and  $a \geq 0$ , we define  $|x|^a = (\text{sgn}(x_1)|x_1|^a, \dots, \text{sgn}(x_N)|x_N|^a)^T$ , where  $\text{sgn}(\cdot)$  is the signum function. Analogously, for a matrix  $M = (m_{ij})$ , we have  $|M|^a = (\text{sgn}(m_{ij})|m_{ij}|^a)$ . The upper right-hand Dini derivative of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  at the point  $t \in \mathbb{R}$  is defined by  $D^*f(t) := \limsup_{h \rightarrow 0^+} (f(t+h) - f(t))/h$ .

For  $p > 0$ , the  $p$ -norm  $\|\cdot\|_p$  is defined as  $\|x\|_p = (\sum_{i=1}^N |x_i|^p)^{1/p}$  for a vector  $x \in \mathbb{R}^N$ . The following lemma connecting different norms is very instrumental in tackling the fixed-time consensus problems, a proof of which can be found in [36].

**Lemma 1.** *Let  $x \in \mathbb{R}^N$  and  $q > p > 0$ . Then*

$$\|x\|_q \leq \|x\|_p \leq N^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

### 2.1 Graph theory

The communication topology of a multi-agent system can often be described by a graph [37]. An undirected graph,  $G = (V, E)$ , consists of a node set  $V = \{1, 2, \dots, N\}$  representing  $N$  agents and an edge set  $E \subseteq V \times V$  describing the information exchange among them. Let  $A = (a_{ij}) \in \mathbb{R}^{N \times N}$  be the associated weighted adjacency matrix of the graph, where  $a_{ij} = a_{ji} \neq 0$  if  $(i, j) \in E$  and  $a_{ij} = 0$  otherwise. For an undirected graph  $G$ , we have  $A^T = A$ . Conversely, given a symmetric matrix  $A$ , we refer to  $G(A)$  as the corresponding undirected graph with the weighted adjacency  $A$  following the same rule.

To investigate the group consensus, a *grouping*  $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_K\}$  of the graph  $G$  is defined by dividing its node set into disjoint subgroups  $\{\mathcal{G}_k\}_{k=1}^K$ . In other words,  $\mathcal{G}$  satisfies  $\cup_{k=1}^K \mathcal{G}_k = V$  and  $\mathcal{G}_k \cap \mathcal{G}_{k'} = \emptyset$  for  $k \neq k'$ . To fix the notation, we write  $\mathcal{G}_1 = \{1, \dots, r_1\}$ ,  $\mathcal{G}_2 = \{r_1 + 1, \dots, r_2\}$ ,  $\dots$ ,  $\mathcal{G}_K = \{r_{K-1} + 1, \dots, N\}$ . Let  $r_0 = 0$  and  $r_K = N$ . We assume that  $a_{ij} \geq 0$  if  $i, j \in \mathcal{G}_k$  for some  $k$ . This means that the interactions between agents in the same subgroup are cooperative. Naturally,  $\mathcal{G}_k$  ( $1 \leq k \leq K$ ) inherit the structure of  $G$  in the sense of induced subgraphs [37]. A graph is connected if there exists a path connecting any pair of distinct nodes in the graph. We make the following assumption.

**Assumption 1.** Each subgroup  $\mathcal{G}_k$  for  $1 \leq k \leq K$  is connected.

This assumption ensures that the information of agents can be exchanged with each other in every subgroup. According to the given grouping, the adjacency matrix  $A$  can be decomposed as

$$A = \begin{pmatrix} A_1 & A_{12} & \dots & A_{1K} \\ A_{21} & A_2 & \dots & A_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ A_{K1} & A_{K2} & \dots & A_K \end{pmatrix} \in \mathbb{R}^{N \times N}, \quad (1)$$

where  $A_k \in \mathbb{R}^{(r_k - r_{k-1}) \times (r_k - r_{k-1})}$  represents the adjacency matrix of  $\mathcal{G}_k$  and  $A_{kk'}^T = A_{k'k}$  for  $1 \leq k, k' \leq K$  and  $k \neq k'$ . The Laplacian matrix  $L = (l_{ij}) \in \mathbb{R}^{N \times N}$  induced by the graph  $G$  is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \sum_{j=1, j \neq i}^N a_{ij}, & j = i. \end{cases}$$

Clearly, we have  $x^T L x = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (x_j - x_i)^2$  for  $x = (x_1, \dots, x_N)^T \in \mathbb{R}^N$ . Hence,  $L$  is positive semi-definite, and it has a zero eigenvalue corresponding to a right eigenvector  $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^N$  by definition. A well-known property of the Laplacian matrix is stated as follows.

**Lemma 2.** ([38]) *Let  $L$  be the Laplacian matrix of graph  $G$  on  $N$  nodes.  $L$  has a simple zero eigenvalue if and only if  $G$  is connected.*

Moreover, the algebraic connectivity, i.e.,  $\lambda_2(L)$ , of a connected graph  $G$  satisfies  $\lambda_2(L) = \min_{\|x\|_2 \neq 0, \mathbf{1}^T x = 0} \frac{x^T L x}{\|x\|_2^2} > 0$ .

### 2.2 Fixed-time stability

Consider the general differential equation

$$\dot{x}(t) = g(t, x(t)), \quad x(0) = x_0, \quad (2)$$

where  $x \in \mathbb{R}^N$  and  $g: \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a nonlinear function. If  $g$  is discontinuous with respect to  $x$ , the solutions of (2) are understood in the sense of Filippov [39]. Suppose that the origin is an equilibrium point of (2).

**Definition 1.** ([16]) The origin of system (2) is said to be globally finite-time stable if it is globally asymptotically stable and there is a function  $T: \mathbb{R}^N \rightarrow \mathbb{R}_+ \cup \{0\}$ , called settling time function, such that for any  $x_0 \in \mathbb{R}^N$ , the solution  $x(t, x_0)$  of system (2) satisfies  $x(t, x_0) \in \mathbb{R}^N$  for  $t \in [0, T(x_0))$  and  $\lim_{t \rightarrow T(x_0)} x(t, x_0) = 0$ .

**Definition 2.** ([25]) The origin of system (2) is said to be globally fixed-time stable if it is globally finite-time stable and the settling-time function  $T(x_0)$  is bounded; i.e., there is some  $T_{\max} > 0$  satisfying  $T(x_0) \leq T_{\max}$  for any  $x_0 \in \mathbb{R}^N$ .

For example, the origin of the simple scalar system  $\dot{x} = -x^{1/3}$  is globally finite-time stable with  $T(x_0) = \frac{3}{2} \sqrt[3]{|x_0|^2}$ . The origin of  $\dot{x} = -|x|^{1/3} - |x|^2$  is globally fixed-time stable because its settling time function  $T(x_0) \leq \pi$  for any  $x_0 \in \mathbb{R}$ .

**Lemma 3.** ([26]) *If there exists a continuous radially unbounded function  $V: \mathbb{R}^N \rightarrow \mathbb{R}_+ \cup \{0\}$  such that*

- (1)  $V(x(t)) = 0$  if and only if  $x(t) = 0$ ;
- (2) any solution  $x(t)$  satisfies the inequality

$$D^*V(x(t)) \leq -aV^p(x) - bV^q(x),$$

with  $a, b > 0$ ,  $0 < p < 1$ , and  $q > 1$ ,

then the origin of the system (2) is globally fixed-time stable and the following estimate holds:

$$T(x_0) \leq T_{\max} = \frac{1}{a(1-p)} + \frac{1}{b(q-1)}, \quad \forall x_0 \in \mathbb{R}^N.$$

This lemma presents a good estimate of the settling time independent of the initial conditions, which will be used to analyze the fixed-time group consensus protocols.

### 2.3 Problem formulation

Now we are in the position to formulate our fixed-time group consensus problem. Consider the following multi-agent system with  $N$  mobile agents governed by

$$\dot{x}_i(t) = f(t, x_i(t)) + u_i(t) + d_i(t, x_i(t)), \quad i \in V, \quad (3)$$

where  $x_i \in \mathbb{R}$  is the state of agent  $i$ ,  $u_i \in \mathbb{R}$  is the control input of agent  $i$  to be designed,  $f$  is a nonlinear function, and  $d_i$  is the uncertain disturbance. The disturbances  $\{d_i\}_{i \in V}$  are assumed to be continuous and bounded by a known constant  $\bar{d}$ , i.e.,  $|d_i(t, x_i(t))| \leq \bar{d}$  for all  $i \in V$ . The function  $f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  represents the intrinsic dynamics of an agent, which is continuous in  $t$ . Since  $f$  in general is a nonlinear function, we assume that there exist positive constants  $r$  and  $\ell$  such that

$$|f(t, x_1) - f(t, x_2)| \leq \ell |x_1 - x_2|^r \quad (4)$$

holds for all  $x_1, x_2 \in \mathbb{R}$  and  $t \geq 0$ .

**Remark 1.** First, it is worth noting that the solutions of (3) exist in the sense of Filippov according to [40] with the control laws  $u_i$  to be designed later. Second, the presented condition (4) indicates

the Hölder continuity with exponent  $r > 0$ . The system includes Lipschitz ( $r = 1$ ) and other nonlinear models ( $r \neq 1$ ), which may appear, for instance, in mechanical models with certain viscous friction force [30]. The condition (4) is more general than most of the existing works concerning consensus with nonlinear inherent dynamics [21, 34]. Note that it is not possible to cancel the effect of  $f$  by the control law since  $f$  is uncertain.

The objective of this paper is to design appropriate control protocols  $u_i$  based on available local information such that agents in the same subgroup converge to a common value in a prescribed time  $T_{\max}$ , that is,  $x_i(t) = x_j(t)$  for all  $i, j \in \mathcal{G}_k$ ,  $1 \leq k \leq K$ , and  $t \geq T_{\max}$ .

**Assumption 2.** In the weighted adjacency matrix  $A$  of  $G$ , we assume that  $\sum_{j \in \mathcal{G}_{k'}} a_{ij} = \mu_{k,k'}$  for all  $i \in \mathcal{G}_k$  and  $k \neq k'$ .

**Remark 2.** Recall that the interaction between different subgroups is allowed to be cooperative (i.e., with non-negative weights) or competitive (i.e., with non-positive weights). Assumption 2 is called the inter-group common influence condition (with the special case of  $\mu_{k,k'} \equiv 0$  referring to as the inter-group balance condition, implying a balance of influence between an agent in a subgroup and all agents in any of the other subgroups). The inter-group balance assumption is widely made in most of the existing literature, see, e.g. [9, 10, 12–15, 34, 35], to guarantee group consensus.

### 3 Analysis of fixed-time group consensus

In this section, we solve the fixed-time group consensus problem for (3) by examining three novel distributed control protocols. Denote  $x(t) = (x_1(t), \dots, x_N(t))^T$ . For each  $i \in \mathcal{G}_k$ ,  $1 \leq k \leq K$ , we consider

$$u_i(t) = \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) + \psi_i(x), \quad (5)$$

where

$$\begin{aligned} \text{(I)} \quad & \varphi_{ij}(x_j - x_i) = \alpha |a_{ij}(x_j - x_i)|^p + \beta |a_{ij}(x_j - x_i)|^q + \\ & \gamma |a_{ij}(x_j - x_i)|^r + \xi a_{ij} \text{sgn}(x_j - x_i) \text{ and } \\ & \psi_i(x) = \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^p + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^q \\ & + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^r + \text{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right); \text{ or} \\ \text{(II)} \quad & \varphi_{ij}(x_j - x_i) = \alpha |a_{ij}(x_j - x_i)|^p + \beta |a_{ij}(x_j - x_i)|^q + \\ & \xi a_{ij} \text{sgn}(x_j - x_i) \text{ and } \psi_i(x) = \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^p \\ & + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^q + \text{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right); \text{ or} \\ \text{(III)} \quad & \varphi_{ij}(x_j - x_i) = \beta |a_{ij}(x_j - x_i)|^q + \xi a_{ij} \text{sgn}(x_j - x_i) \text{ and} \\ & \psi_i(x) = \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right]^q \\ & + \text{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j \right) \end{aligned}$$

for  $0 < p < 1$ ,  $q > 1$  and the parameters  $\alpha, \beta, \gamma, \xi$  will be designed later. Clearly, the above three types of controllers turn simpler in the order of (I), (II), and (III). As we will see below, the design of control parameters becomes more conservative if the controller is simpler. For example, the first two terms of  $\varphi_{ij}$  in (I), the most complicated controller here, take care of the fixed-time consensus, while the third term controls the intrinsic nonlinear dynamics, and the last term deals with the unknown disturbances. By doing so, we are hopefully to obtain a relatively accurate estimation for the settling time. See Remark 6 and Section 4 for further comments on the comparison of these protocols.

#### 3.1 Group consensus for system (3) with (5) and (I)

We first consider the group consensus scheme with (5) and (I). For  $1 \leq k \leq K$ , let  $\bar{x}_k(t) = \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} x_j(t)$  be the average position within each subgroup. We refer to  $e_i(t) = x_i(t) - \bar{x}_k(t)$  as the error of agent  $i$  in  $\mathcal{G}_k$ . It follows from (3) and (5) that  $\dot{x}_i(t) = f(t, x_i(t)) + \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) + \psi_i(x) + d_i(t, x_i(t))$ . Since  $\varphi_{ij}(x_j - x_i) = -\varphi_{ji}(x_i - x_j)$ , we obtain  $\sum_{i, j \in \mathcal{G}_k} \varphi_{ij}(x_j -$

$x_i) = 0$  and hence

$$\begin{aligned} \dot{e}_i(t) &= \dot{x}_i(t) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} \dot{x}_j(t) \\ &= f(t, x_i(t)) + \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) + \psi_i(x) + d_i(t, x_i(t)) \\ &\quad - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} f(t, x_j(t)) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} \psi_j(x) \\ &\quad - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} d_j(t, x_j(t)) \end{aligned} \quad (6)$$

for  $i \in \mathcal{G}_k$ . Let  $e(t) = (e|_{\mathcal{G}_1}^T, \dots, e|_{\mathcal{G}_K}^T)^T$  with  $e|_{\mathcal{G}_1} = (e_1(t), \dots, e_{r_1}(t))^T$ ,  $e|_{\mathcal{G}_2} = (e_{r_1+1}(t), \dots, e_{r_2}(t))^T$ ,  $\dots$ , and  $e|_{\mathcal{G}_K} = (e_{r_{K-1}+1}(t), \dots, e_N(t))^T$ .

**Theorem 1.** Under Assumption 1 and Assumption 2 with  $\mu_{k,k'} \equiv 0$ , the multi-agent system (3) with protocol (5) and (I) satisfying  $\alpha, \beta \in \mathbb{R}_+$ ,

$$\begin{aligned} \gamma &= \frac{\ell \cdot \max \left\{ \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{r-1}, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\}}{2^{\frac{r-1}{2}} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,r}) \right)^{\frac{r+1}{2}}}, \\ \xi &= (\bar{d} + 1) \sqrt{\frac{2K \max_{1 \leq k \leq K} |\mathcal{G}_k|}{\min_{1 \leq k \leq K} \lambda_2(L_{k,s})}} \end{aligned}$$

and  $0 < p < 1$ ,  $q > 1$ , achieves fixed-time group consensus with

$$T_{\max}^1 = \frac{2}{\bar{\alpha}(1-p)} + \frac{2}{\bar{\beta}(q-1)},$$

where  $\bar{\alpha} = 2^p \alpha \left( \min_k \lambda_2(L_{k,p}) \right)^{(p+1)/2}$ ,  $\bar{\beta} = 2^q \beta \left( \sqrt{K} \max_k |\mathcal{G}_k| \right)^{1-q} \left( \min_k \lambda_2(L_{k,q}) \right)^{(q+1)/2}$ . Here,  $L_{k,p}$ ,  $L_{k,q}$ , and  $L_{k,r}$  are the Laplacian matrices of the graphs  $G \left( [A]^{2p/(p+1)} \right)$ ,  $G \left( [A]^{2q/(q+1)} \right)$ , and  $G \left( [A]^{2r/(r+1)} \right)$ , respectively.  $L_{k,s}$  is the Laplacian matrix of the graph  $G \left( [A]^2 \right)$ .

**Proof.** Consider the Lyapunov function  $V(t) = \frac{1}{2} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i^2(t) := \sum_{k=1}^K V_k(t)$  for the system (3). Here,  $V_k(t) = \frac{1}{2} e|_{\mathcal{G}_k}^T e|_{\mathcal{G}_k}$ . The derivative of  $V$  satisfies

$$\begin{aligned} \dot{V}(t) &= \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) \\ &\quad + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( f(t, x_i(t)) - f(t, \bar{x}_k(t)) + f(t, \bar{x}_k(t)) \right. \\ &\quad \left. - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} f(t, x_j(t)) \right) \\ &\quad + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( d_i(t, x_i(t)) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} d_j(t, x_j(t)) \right) \\ &\quad + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( \psi_i(x) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} \psi_j(x) \right). \end{aligned}$$

By noting that  $\sum_{i \in \mathcal{G}_k} e_i(t) = 0$  for each  $k$ , we obtain  $\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( f(t, \bar{x}_k(t)) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} f(t, x_j(t)) \right) = 0$ ,  $\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( d_i(t, x_i(t)) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} d_j(t, x_j(t)) \right) = 0$ ,  $\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( \psi_i(x) - \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} \psi_j(x) \right) = 0$ .

$e_i(t) \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} d_j(t, x_j(t)) = 0$ , and  $\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \frac{1}{|\mathcal{G}_k|} \sum_{j \in \mathcal{G}_k} \psi_j(x) = 0$ . Therefore,

$$\begin{aligned} \dot{V}(t) &= \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) \\ &\quad + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) (f(t, x_i(t)) - f(t, \bar{x}_k(t))) \\ &\quad + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) d_i(t, x_i(t)) + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \psi_i(x) \\ &\leq \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \sum_{j \in \mathcal{G}_k} \varphi_{ij}(x_j - x_i) + \ell \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|^{1+r} \\ &\quad + \bar{d} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \psi_i(x) \\ &= -\frac{1}{2} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} (e_j(t) - e_i(t)) \varphi_{ij}(e_j - e_i) \\ &\quad + \ell \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|^{1+r} \\ &\quad + \bar{d} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \psi_i(e), \end{aligned} \quad (7)$$

where we used the relations  $x_j - x_i = e_j - e_i$  for  $i, j \in \mathcal{G}_k$  and  $\psi_i(x) = \psi_i(e)$  by Assumption 2.

Define  $\bar{a} := \max_{i \in \mathcal{G}_k, j \in \mathcal{G}_{k'}, k \neq k'} |a_{ij}|$ . Employing the protocol (I) and the fact that  $\sum_{i \in \mathcal{G}_k} e_i(t) = 0$  for each  $k$ , the last term on the right-hand side of (7) can be estimated by

$$\begin{aligned} &\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \psi_i(e) \\ &= \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} e_i(t) \left( \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} e_j \right|^p + \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} e_j \right|^q \right. \\ &\quad \left. + \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} e_j \right|^r + \operatorname{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} e_j \right) \right) \\ &\leq \bar{a}^p \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} e_j \right|^p \\ &\quad + \bar{a}^q \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} e_j \right|^q \\ &\quad + \bar{a}^r \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| \left| \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} e_j \right|^r + \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| \\ &= \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|. \end{aligned}$$

Hence, using (7) and (I), one has

$$\begin{aligned} \dot{V}(t) &\leq -\frac{1}{2} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} (e_j - e_i) \left( \alpha [a_{ij}(e_j - e_i)]^p \right. \\ &\quad \left. + \beta [a_{ij}(e_j - e_i)]^q + \gamma [a_{ij}(e_j - e_i)]^r \right) \\ &\quad - \frac{1}{2} \xi \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij} (e_j - e_i) \operatorname{sgn}(e_j - e_i) \\ &\quad + \ell \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|^{1+r} + (\bar{d} + 1) \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)| \\ &= -\frac{1}{2} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} \left( \alpha a_{ij}^p |e_j - e_i|^{p+1} + \beta a_{ij}^q |e_j - e_i|^{q+1} \right. \\ &\quad \left. + \gamma a_{ij}^r |e_j - e_i|^{r+1} \right) \\ &\quad - \frac{1}{2} \xi \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij} |e_j - e_i| + \ell \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|^{1+r} \\ &\quad + (\bar{d} + 1) \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|. \end{aligned} \quad (8)$$

It follows from Lemma 1 that

$$\begin{aligned} &\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i|^{1+r} \\ &\leq \begin{cases} \sum_{k=1}^K |\mathcal{G}_k|^{\frac{1-r}{2}} \left( \sum_{i \in \mathcal{G}_k} |e_i|^2 \right)^{\frac{1+r}{2}}, & 0 < r \leq 1, \\ \sum_{k=1}^K \left( \sum_{i \in \mathcal{G}_k} |e_i|^2 \right)^{\frac{1+r}{2}}, & 1 \leq r, \end{cases} \end{aligned} \quad (9)$$

Similarly,

$$\begin{aligned} &\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^r |e_j - e_i|^{r+1} \geq \\ &\begin{cases} \sum_{k=1}^K \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2r}{r+1}} (e_j - e_i)^2 \right)^{\frac{r+1}{2}}, & 0 < r \leq 1, \\ \sum_{k=1}^K |\mathcal{G}_k|^{1-r} \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2r}{r+1}} (e_j - e_i)^2 \right)^{\frac{r+1}{2}}, & 1 \leq r, \end{cases} \end{aligned} \quad (10)$$

Consider the following two cases:

(a)  $0 < r < 1$ . Substituting (9) and (10) into (8) and exploiting Lemma 1 again gives

$$\begin{aligned} \dot{V}(t) \leq & -\frac{1}{2}\alpha \sum_{k=1}^K \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2p}{p+1}} (e_j - e_i)^2 \right)^{\frac{p+1}{2}} \\ & -\frac{1}{2}\beta \sum_{k=1}^K |\mathcal{G}_k|^{1-q} \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2q}{q+1}} (e_j - e_i)^2 \right)^{\frac{q+1}{2}} \\ & -\frac{1}{2}\gamma \sum_{k=1}^K \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2r}{r+1}} (e_j - e_i)^2 \right)^{\frac{r+1}{2}} \\ & -\frac{1}{2}\xi \sum_{k=1}^K \left( \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^2 (e_j - e_i)^2 \right)^{\frac{1}{2}} \\ & + (\bar{d} + 1) \sum_{k=1}^K |\mathcal{G}_k|^{\frac{1}{2}} \left( \sum_{i \in \mathcal{G}_k} e_i^2 \right)^{\frac{1}{2}} \\ & + \ell \sum_{k=1}^K |\mathcal{G}_k|^{\frac{1-r}{2}} \left( \sum_{i \in \mathcal{G}_k} e_i^2 \right)^{\frac{r+1}{2}}. \end{aligned}$$

It follows from Assumption 1 and Lemma 2 that for each  $k$ ,  $\sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2p}{p+1}} (e_j - e_i)^2 = 2e|_{\mathcal{G}_k}^T L_{k,p} e|_{\mathcal{G}_k} \geq 4\lambda_2(L_{k,p})\mathbf{V}_k$  holds. Analogously, we have  $\sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2q}{q+1}} (e_j - e_i)^2 \geq 4\lambda_2(L_{k,q})\mathbf{V}_k$ ,  $\sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^{\frac{2r}{r+1}} (e_j - e_i)^2 \geq 4\lambda_2(L_{k,r})\mathbf{V}_k$ , and  $\sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij}^2 (e_j - e_i)^2 \geq 4\lambda_2(L_{k,s})\mathbf{V}_k$ . Consequently,

$$\begin{aligned} \dot{V}(t) \leq & -2^p \alpha \sum_{k=1}^K (\lambda_2(L_{k,p})\mathbf{V}_k)^{\frac{p+1}{2}} \\ & -2^q \beta \sum_{k=1}^K |\mathcal{G}_k|^{1-q} (\lambda_2(L_{k,q})\mathbf{V}_k)^{\frac{q+1}{2}} \\ & -2^r \gamma \sum_{k=1}^K (\lambda_2(L_{k,r})\mathbf{V}_k)^{\frac{r+1}{2}} - \xi \sum_{k=1}^K (\lambda_2(L_{k,s})\mathbf{V}_k)^{\frac{1}{2}} \\ & + \sqrt{2}(\bar{d} + 1) \sum_{k=1}^K |\mathcal{G}_k|^{\frac{1}{2}} \mathbf{V}_k^{\frac{1}{2}} + 2^{\frac{r+1}{2}} \ell \sum_{k=1}^K |\mathcal{G}_k|^{\frac{1-r}{2}} \mathbf{V}_k^{\frac{r+1}{2}}. \end{aligned}$$

Next, we estimate each term on the right-hand side of the above inequality. By repeatedly using Lemma 1, we have the following estimates:

$$\sum_{k=1}^K (\lambda_2(L_{k,p})\mathbf{V}_k)^{(p+1)/2} \geq (\min_k \lambda_2(L_{k,p}))^{(p+1)/2} \mathbf{V}^{(p+1)/2},$$

$$\begin{aligned} \sum_{k=1}^K |\mathcal{G}_k|^{1-q} (\lambda_2(L_{k,q})\mathbf{V}_k)^{(q+1)/2} & \geq (\max_k |\mathcal{G}_k|)^{1-q} \\ & \cdot (\min_k \lambda_2(L_{k,q}))^{\frac{q+1}{2}} K^{(1-q)/2} \mathbf{V}^{(q+1)/2}, \end{aligned}$$

$$\sum_{k=1}^K (\lambda_2(L_{k,r})\mathbf{V}_k)^{(r+1)/2} \geq (\min_k \lambda_2(L_{k,r}))^{(r+1)/2} \mathbf{V}^{(r+1)/2},$$

$$\sum_{k=1}^K (\lambda_2(L_{k,s})\mathbf{V}_k)^{1/2} \geq (\min_k \lambda_2(L_{k,s}))^{1/2} \mathbf{V}^{1/2},$$

$$\sum_{k=1}^K |\mathcal{G}_k|^{1/2} \mathbf{V}_k^{1/2} \leq (K \max_k |\mathcal{G}_k|)^{\frac{1}{2}} \mathbf{V}^{1/2},$$

and

$$\sum_{k=1}^K |\mathcal{G}_k|^{(1-r)/2} \mathbf{V}_k^{(r+1)/2} \leq (K \max_k |\mathcal{G}_k|)^{(1-r)/2} \mathbf{V}^{(r+1)/2}.$$

Therefore,

$$\begin{aligned} \dot{V}(t) \leq & -2^p \alpha \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \mathbf{V}^{\frac{p+1}{2}} \\ & -2^q \beta \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathbf{V}^{\frac{q+1}{2}} - 2^r \gamma \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,r}) \right)^{\frac{r+1}{2}} \mathbf{V}^{\frac{r+1}{2}} \\ & - \left( \xi \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} \right. \\ & \left. - \sqrt{2}(\bar{d} + 1) \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1}{2}} \right) \mathbf{V}^{\frac{1}{2}} \\ & + 2^{\frac{r+1}{2}} \ell \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \mathbf{V}^{\frac{r+1}{2}}. \end{aligned} \quad (11)$$

(b)  $1 \leq r$ . We can similarly derive, in parallel with (11), that

$$\begin{aligned} \dot{V}(t) \leq & -2^p \alpha \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \mathbf{V}^{\frac{p+1}{2}} \\ & -2^q \beta \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathbf{V}^{\frac{q+1}{2}} - 2^r \gamma \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-r} \\ & \cdot \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,r}) \right)^{\frac{r+1}{2}} \mathbf{V}^{\frac{r+1}{2}} \\ & - \left( \xi \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} - \sqrt{2}(\bar{d} + 1) \right. \\ & \left. \cdot \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1}{2}} \right) \mathbf{V}^{\frac{1}{2}} + 2^{\frac{r+1}{2}} \ell \mathbf{V}^{\frac{r+1}{2}}. \end{aligned} \quad (12)$$

Combining (11) and (12), for all  $r > 0$ , it follows from the chosen values of  $\gamma$  and  $\xi$  in the statement of Theorem 1, we obtain

$$\begin{aligned} \dot{V}(t) \leq & -2^p \alpha \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \mathbf{V}^{\frac{p+1}{2}} \\ & -2^q \beta \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathbf{V}^{\frac{q+1}{2}} \\ & = -\bar{\alpha} \mathbf{V}^{\frac{p+1}{2}} - \bar{\beta} \mathbf{V}^{\frac{q+1}{2}}, \end{aligned}$$

where  $\bar{\alpha} = 2^p \alpha (\min_k \lambda_2(L_{k,p}))^{(p+1)/2}$  and  $\bar{\beta} = 2^q \beta \cdot (\sqrt{K} \max_k |\mathcal{G}_k|)^{1-q} (\min_k \lambda_2(L_{k,q}))^{(q+1)/2}$ . Then from

Lemma 3, the origin of the system (6) is fixed-time stable with the settling time bounded by  $T_{\max}^1$ . The proof is complete.  $\square$

**Remark 3.** The settling time does not rely on the initial conditions. If a predetermined convergence time is required, one may easily tune  $\alpha, \beta > 0, 0 < p < 1$ , and  $q > 1$  to solve the group consensus problem. The terms  $\gamma|a_{ij}(x_j - x_i)|^r$  and  $\xi a_{ij} \text{sgn}(x_j - x_i)$  in  $\varphi_{ij}$  in (I) are meant to, respectively, control the nonlinear function and the unknown disturbances, while the first two terms in  $\varphi_{ij}$  guarantee the fixed-time consensus. It is also worth noting that, when  $K = 1$ , namely, the graph  $G$  is a connected graph with nonnegative weights, our result is essentially consistent with that in [29], which addresses complete consensus problems.

**Remark 4.** From Theorem 1, we can choose  $\gamma = 0$  if the nonlinear function  $f(t, x)$  is constant in  $x$  (or simply equal to zero as in, e.g., [26, 28]). Interestingly, we will see in Section 3.2 below that this is feasible for our general  $f$  as long as the control parameters  $\alpha$  and  $\beta$  are made more restrictive so that they contribute to not only the fixed-time stability but also the control of  $f$ .

In the case of general inter-group common influence  $\mu_{k,k'}$  in Assumption 2, we redefine the function  $\psi_i(x)$  in (I) as

$$\begin{aligned} \psi_i(x) = & \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^p \\ & + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^q \\ & + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^r \\ & + \text{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right). \end{aligned}$$

With the above modification, the following corollary can be established using similar arguments.

**Corollary 1.** *The result of Theorem 1 holds verbatim with the above updated definition of  $\psi_i(x)$  under full generality of Assumption 2.*

### 3.2 Group consensus for system (3) with (5) and (II)

In this section, we consider the group consensus scheme with (5) and (II). As in the above derivation, we will focus on the error dynamics (6). The main result reads as follows.

**Theorem 2.** *Under Assumption 1 and Assumption 2 with  $\mu_{k,k'} \equiv 0$ , the multi-agent system (3) with protocol (5) and (II) satisfying*

$$\begin{aligned} \alpha = \hat{\alpha} + & \frac{\ell \cdot \max \left\{ 1, (K \max_{1 \leq k \leq K} |\mathcal{G}_k|)^{\frac{1-r}{2}} \right\}}{2^{p-r} \frac{r+1}{2} (\min_{1 \leq k \leq K} \lambda_2(L_{k,p}))^{\frac{p+1}{2}}}, \\ \beta = & \hat{\beta} \\ & + \frac{\ell \cdot \max \left\{ 1, (K \max_{1 \leq k \leq K} |\mathcal{G}_k|)^{\frac{1-r}{2}} \right\}}{2^{q-r} \frac{r+1}{2} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} (\min_{1 \leq k \leq K} \lambda_2(L_{k,q}))^{\frac{q+1}{2}}}, \\ \xi = & (\bar{d} + 1) \sqrt{\frac{2K \max_{1 \leq k \leq K} |\mathcal{G}_k|}{\min_{1 \leq k \leq K} \lambda_2(L_{k,s})}} \end{aligned}$$

and  $p \leq r \leq q$ ,  $\hat{\alpha}, \hat{\beta} \in \mathbb{R}_+$ , achieves fixed-time group consensus with

$$T_{\max}^2 = \frac{2}{\bar{\alpha}(1-p)} + \frac{2}{\bar{\beta}(q-1)},$$

where  $\bar{\alpha} = 2^p \hat{\alpha} (\min_k \lambda_2(L_{k,p}))^{(p+1)/2}$ ,  $\bar{\beta} = 2^q \hat{\beta} \cdot \left( \sqrt{K} \max_k |\mathcal{G}_k| \right)^{1-q} (\min_k \lambda_2(L_{k,q}))^{(q+1)/2}$ . Here,  $L_{k,p}$  and

$L_{k,q}$  are the Laplacian matrices of the graphs  $G \left( [A]^{2p/(p+1)} \right)$  and  $G \left( [A]^{2q/(q+1)} \right)$ , respectively.  $L_{k,s}$  is the Laplacian matrix of the graph  $G \left( [A]^2 \right)$ .

**Proof.** Following the same proof of Theorem 1, we can obtain the following estimation in parallel with (8):

$$\begin{aligned} \dot{V}(t) \leq & -\frac{1}{2} \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} \left( \alpha a_{ij}^p |e_j - e_i|^{p+1} + \beta a_{ij}^q |e_j - e_i|^{q+1} \right) \\ & - \frac{1}{2} \xi \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} \sum_{j \in \mathcal{G}_k} a_{ij} |e_j - e_i| + \ell \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|^{1+r} \\ & + (\bar{d} + 1) \sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i(t)|. \end{aligned} \quad (13)$$

Note that the inequality (9) can be rewritten as

$$\sum_{k=1}^K \sum_{i \in \mathcal{G}_k} |e_i|^{1+r} \leq \sum_{k=1}^K \max \left\{ 1, |\mathcal{G}_k|^{\frac{1-r}{2}} \right\} \left( \sum_{i \in \mathcal{G}_k} |e_i|^2 \right)^{\frac{1+r}{2}}. \quad (14)$$

By using (14) and combining the calculations leading to (11) and (12), we obtain for all  $r > 0$  that

$$\begin{aligned} \dot{V}(t) \leq & -2^p \alpha \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \sqrt{\frac{p+1}{2}} \\ & - 2^q \beta \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \sqrt{\frac{q+1}{2}} - \left( \xi \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} - \sqrt{2}(\bar{d} + 1) \right) \\ & \cdot \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1}{2}} \sqrt{\frac{1}{2}} \\ & + 2 \frac{r+1}{2} \ell \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\} \sqrt{\frac{r+1}{2}} \\ = & -2^p \hat{\alpha} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \sqrt{\frac{p+1}{2}} \\ & - 2^q \hat{\beta} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \sqrt{\frac{q+1}{2}} + 2 \frac{r+1}{2} \ell \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\} \\ & \cdot \left( \sqrt{\frac{r+1}{2}} - \sqrt{\frac{p+1}{2}} - \sqrt{\frac{q+1}{2}} \right). \end{aligned}$$

Since  $p \leq r \leq q$ , we find that

$$\begin{aligned} \dot{V}(t) \leq & -2^p \hat{\alpha} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,p}) \right)^{\frac{p+1}{2}} \sqrt{\frac{p+1}{2}} \\ & - 2^q \hat{\beta} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \sqrt{\frac{q+1}{2}} \\ = & -\bar{\alpha} \sqrt{\frac{p+1}{2}} - \bar{\beta} \sqrt{\frac{q+1}{2}}, \end{aligned}$$

where  $\bar{\alpha} = 2^p \hat{\alpha} (\min_k \lambda_2(L_{k,p}))^{(p+1)/2}$  and  $\bar{\beta} = 2^q \hat{\beta} \cdot \left( \sqrt{K} \max_k |\mathcal{G}_k| \right)^{1-q} (\min_k \lambda_2(L_{k,q}))^{(q+1)/2}$ . Again from

Lemma 3, the origin of the error system (6) is fixed-time stable with the settling time bounded by  $T_{\max}^2$ . The proof is complete.  $\square$

**Remark 5.** As in Theorem 1, we can conveniently tune  $\alpha, \beta > 0$ ,  $0 < p < 1$  and  $q > 1$  to meet any prescribed convergence time regardless of the initial conditions. However, in Theorem 2 we additionally require  $p \leq r \leq q$ , which can be satisfied when  $r$  is given. Moreover, we observe that the control parameters  $\alpha$  and  $\beta$  are more restrictive in Theorem 2, since they are also used to control the nonlinear dynamics in (II). A simpler controller comes at the price of more conservatively designed parameters.

In the case of general inter-group common influence  $\mu_{k,k'}$  in Assumption 2, we similarly redefine the function  $\psi_i(x)$  in (II) as

$$\begin{aligned} \psi_i(x) = & \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^p \\ & + \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^q \\ & + \operatorname{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right). \end{aligned}$$

**Corollary 2.** The result of Theorem 2 holds verbatim with the above updated definition of  $\psi_i(x)$  under full generality of Assumption 2.

### 3.3 Group consensus for system (3) with (5) and (III)

The fixed-time group consensus can be achieved with an even simpler protocol (5) with (III).

**Theorem 3.** Under Assumption 1 and Assumption 2 with  $\mu_{k,k'} \equiv 0$ , the multi-agent system (3) with protocol (5) and (III) satisfying

$$\begin{aligned} \beta = & \hat{\beta} \\ & + \frac{\ell \cdot \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\}}{2^{q-r} \frac{r+1}{2} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}}}, \\ \xi = & \hat{\alpha} + (\bar{d} + 1) \sqrt{\frac{2K \max_{1 \leq k \leq K} |\mathcal{G}_k|}{\min_{1 \leq k \leq K} \lambda_2(L_{k,s})}} \\ & + \frac{2^{\frac{r+1}{2}} \ell \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\}}{\left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}}}, \end{aligned}$$

and  $r \leq q$ ,  $\hat{\alpha}, \hat{\beta} \in \mathbb{R}_+$ , achieves fixed-time group consensus with

$$T_{\max}^3 = \frac{2}{\bar{\alpha}} + \frac{2}{\bar{\beta}(q-1)},$$

where  $\bar{\alpha} = \hat{\alpha} \left( \min_k \lambda_2(L_{k,s}) \right)^{1/2}$ ,  $\bar{\beta} = 2^q \hat{\beta} \left( \sqrt{K} \max_k |\mathcal{G}_k| \right)^{1-q} \cdot \left( \min_k \lambda_2(L_{k,q}) \right)^{(q+1)/2}$ . Here,  $L_{k,q}$  and  $L_{k,s}$  are the Laplacian matrices of the graphs  $G \left( [A]^{2q/(q+1)} \right)$  and  $G \left( [A]^2 \right)$ , respectively.

**Proof.** Similar to the proofs of Theorems 1 and 2, for all  $r > 0$ , we derive

$$\begin{aligned} \dot{V}(t) \leq & -2^q \beta \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathcal{V}^{\frac{q+1}{2}} - \left( \xi \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} - \sqrt{2}(\bar{d} + 1) \right) \\ & \cdot \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1}{2}} \mathcal{V}^{\frac{1}{2}} \\ & + 2^{\frac{r+1}{2}} \ell \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\} \mathcal{V}^{\frac{r+1}{2}} \\ = & -2^q \hat{\beta} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathcal{V}^{\frac{q+1}{2}} - \hat{\alpha} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} \mathcal{V}^{\frac{1}{2}} \\ & + 2^{\frac{r+1}{2}} \ell \max \left\{ 1, \left( K \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{\frac{1-r}{2}} \right\} \\ & \cdot \left( \mathcal{V}^{\frac{r+1}{2}} - \mathcal{V}^{\frac{1}{2}} - \mathcal{V}^{\frac{q+1}{2}} \right). \end{aligned}$$

Since  $0 < r \leq q$ , we find that

$$\begin{aligned} \dot{V}(t) \leq & -2^q \hat{\beta} \left( \sqrt{K} \max_{1 \leq k \leq K} |\mathcal{G}_k| \right)^{1-q} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,q}) \right)^{\frac{q+1}{2}} \\ & \cdot \mathcal{V}^{\frac{q+1}{2}} - \hat{\alpha} \left( \min_{1 \leq k \leq K} \lambda_2(L_{k,s}) \right)^{\frac{1}{2}} \mathcal{V}^{\frac{1}{2}} \\ = & -\bar{\alpha} \mathcal{V}^{\frac{1}{2}} - \bar{\beta} \mathcal{V}^{\frac{q+1}{2}}, \end{aligned}$$

where  $\bar{\alpha} = \hat{\alpha} \left( \min_k \lambda_2(L_{k,s}) \right)^{1/2}$  and  $\bar{\beta} = 2^q \hat{\beta} \cdot \left( \sqrt{K} \max_k |\mathcal{G}_k| \right)^{1-q} \left( \min_k \lambda_2(L_{k,q}) \right)^{(q+1)/2}$ . Again from Lemma 3, the origin of the error system (6) is fixed-time stable with the settling time bounded by  $T_{\max}^3$ . We complete the proof.  $\square$

**Remark 6.** We can conveniently tune  $\beta > 0$ ,  $0 < p < 1$  and  $q > 1$  to meet any prescribed convergence time regardless of the initial conditions. Here, the control parameter  $\xi$  is more conservative than those in Theorems 1 and 2 as one would expect. In fact, the term  $\xi a_{ij} \operatorname{sgn}(x_j - x_i)$  in  $\varphi_{ij}$  in (III) vitally contributes to the control of nonlinear term, uncertain disturbances, and the fixed-time consensus. From (I) to (II) and (III), the design of controller becomes simpler and consequently the control parameters  $\alpha, \beta, \gamma, \xi$  are more conservative since they are multi-tasked, leading to worse estimation of settling time as is shown in Section 4. However, it is noteworthy that the result of Theorem 3 outperforms those of Theorems 1 and 2 in the sense that protocol (5) with (III) is essentially fully distributed since  $\hat{\beta}, \hat{\alpha} \in \mathbb{R}_+$  are arbitrary whereas the global information such as eigenvalues is only used in the estimation of  $T_{\max}^3$ .

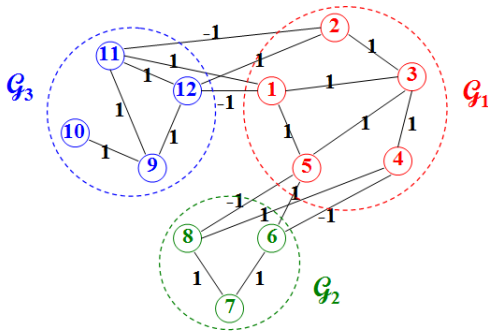
**Remark 7.** Note that the upper bound of settling time for the leader-follower fixed-time group consensus protocol presented in [34] increases with the coupling strengths among agents from different subgroups. Such coupling strengths can be characterized by the quantity,  $\bar{a}$ , defined in the proof of Theorem 1. Remarkably, for our protocols (5) with (I), (II), and (III),  $T_{\max}^i$  for  $i = 1, 2, 3$  do not rely on these inter-group coupling strengths, providing desirable flexibility for some realistic physical systems, where inter-group coupling strengths could be large. Finally, we mention that the above analyses remain valid essentially for multidimensional dynamics by a straightforward application of Kronecker operations.

In the case of general inter-group common influence  $\mu_{k,k'}$  in Assumption 2, we similarly redefine the function  $\psi_i(x)$  in (III) as

$$\psi_i(x) = \left[ \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right]^q + \text{sgn} \left( \sum_{k' \neq k} \sum_{j \in \mathcal{G}_{k'}} a_{ij} x_j - \sum_{k' \neq k} \mu_{k,k'} \bar{x}_{k'} \right).$$

**Corollary 3.** The result of Theorem 3 holds verbatim with the above updated definition of  $\psi_i(x)$  under full generality of Assumption 2.

#### 4 Numerical examples



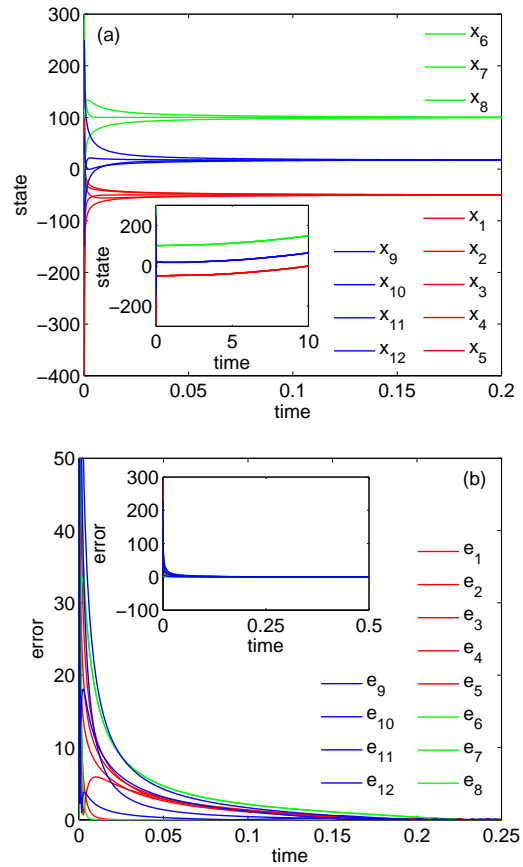
**Fig. 1** Communication topology containing  $N = 12$  nodes with the grouping  $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$  satisfying the inter-group balance condition.

**Example 1.** Consider an undirected graph  $G$  of  $N = 12$  nodes as shown in Fig. 1. The network is divided into  $K = 3$  subgroups with  $\mathcal{G}_1 = \{1, 2, 3, 4, 5\}$ ,  $\mathcal{G}_2 = \{6, 7, 8\}$ , and  $\mathcal{G}_3 = \{9, 10, 11, 12\}$ , which satisfy Assumptions 1 and 2 (with  $\mu_{k,k'} \equiv 0$ ). Based on the decomposition in (1), the communication relationship of agents is fully represented by the matrices:

$$\text{ces: } A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, A_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \text{ and } A_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We assume  $f(t, x_i(t)) = t + \sin(x_i(t))$  and  $d_i(t, x_i(t)) = \cos(x_i(t))$  for  $i = 1, \dots, 12$ . Hence, we have  $\ell = r = \bar{d} = 1$ . The initial condition is chosen as  $x(0) = (200, 100, -400, 50, -200, -100, 300, 100, -150, 250, 70, -100)^T$ . The state evolution of system (3) with protocol (5) and (I) is shown in Fig. 2(a). The corresponding error dynamics is shown in Fig. 2(b). The control parameters are taken as  $\alpha = 2.5$ ,  $\beta = 5$ ,  $\gamma = 1$ ,  $\xi = 10.95$  by calculation as per Theorem 1. Similarly, the behavior of system (3) using (5) with (II) is shown in Fig. 3 with the same parameters as per Theorem 2. In Fig. 4, we show the behavior of system (3) using (5) with (III), where  $\xi = 14.04$  following Theorem 3 (with other parameters remain the same).

The theoretical upper-bounds are calculated as  $T_{\max}^1 \approx 2.00$ ,  $T_{\max}^2 \approx 9.07$ , and  $T_{\max}^3 \approx 8.30$ , respectively. From Figs. 2, 3, and 4, it can be seen that the state trajectories in each subgroup converge as expected and the settling time in all these three situations is around, say,  $t = 0.25$ , much less than the estimated



**Fig. 2** Group consensus with protocol (5) and (I). (a) State evolution with zoom-out view shown in the inset. (b) Error evolution with zoom-out view shown in the inset.

upper-bounds. More estimates of the settling time under these protocols for different values of parameter  $\beta$  are shown in Tab. 1. It is also worth noting that although the state trajectories of the agents under these three protocols are very similar, the estimate  $T_{\max}^1$  is obviously smaller than  $T_{\max}^2$  and  $T_{\max}^3$ . We contend that the relatively accurate prediction for (I) is attributed to the refined controller design, where four parameters  $\alpha, \beta, \gamma, \xi$  play an important role in governing the group consensus in a detailed manner.

In view of the conservativeness of the theoretical estimates, a more practical convergence time can be obtained by simulating the closed-loop system with large initial conditions. The settling time will approach a finite limit as the initial conditions grow. Further simulations show that a more practical settling time upper bound can be determined as around  $t = 0.3$ .

We mention that the errors may increase in some time segments as shown in Figs. 2(b), 3(b), and 4(b). This is because they measure the difference between the status of each agent and the mean value of the subgroup to which the very agent belongs according to our definition. Here, for example, we take  $e_1 = |x_1 - (\sum_{i=1}^5 x_i)/5|$ .

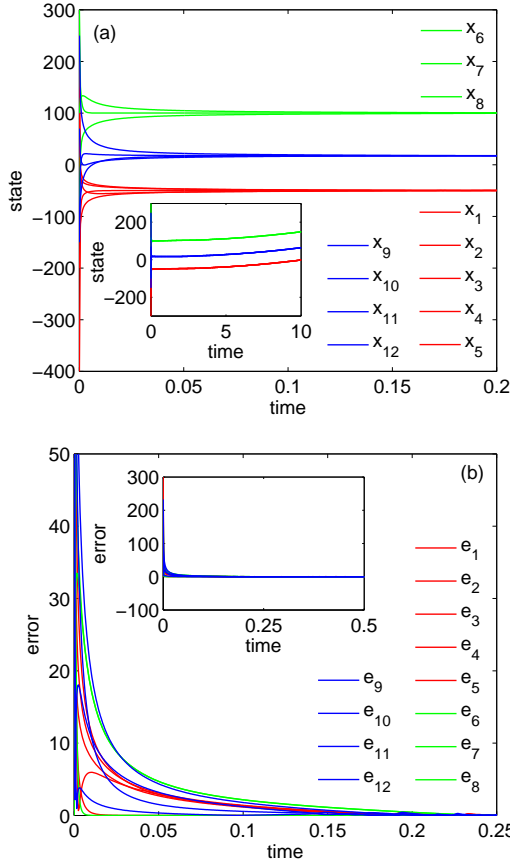
**Example 2.** In this example, we consider a graph  $G$  of  $N = 8$  nodes illustrated in Fig. 5. The network is divided into  $K = 3$  subgroups with  $\mathcal{G}_1 = \{1, 2\}$ ,  $\mathcal{G}_2 = \{3, 4, 5\}$ , and  $\mathcal{G}_3 = \{6, 7, 8\}$ , which satisfy Assumptions 1 and 2 with  $\mu_{1,2} = \mu_{2,1} = -1$ ,  $\mu_{2,3} = \mu_{3,2} = 1$ , and  $\mu_{1,3} = \mu_{3,1} = 0$ . Based on the decomposition in (1), the communication relationship of agents is described by the matrices:  $A_1 =$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, A_{21} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \\ 0 & 0 \end{pmatrix}, A_{31} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } A_{32} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

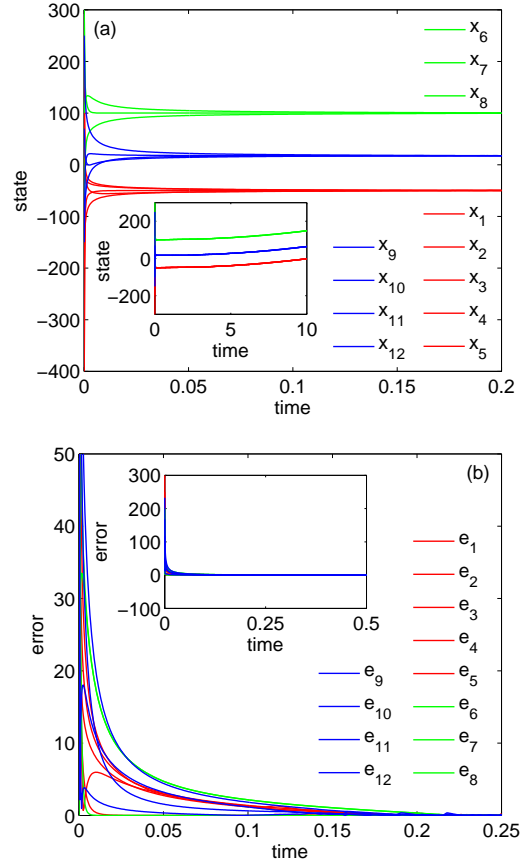


	$\beta = 5$	$\beta = 6$	$\beta = 7$	$\beta = 8$	$\beta = 9$
$T_{\max}^1$	2.00	1.85	1.75	1.67	1.61
$T_{\max}^2$	9.07	5.20	4.23	3.78	3.53
$T_{\max}^3$	8.30	4.43	3.46	3.02	2.77

**Table 1** List of estimated settling time  $T_{\max}$  in Theorems 1, 2, 3 for different  $\beta$ . Here,  $\alpha = 2.5$ ,  $\gamma = 1$ ,  $p = 0.5$ ,  $q = 2$ ,  $\gamma = 1$ ,  $\xi = 10.95$  for (I), (II) and  $\xi = 14.04$  for (III).



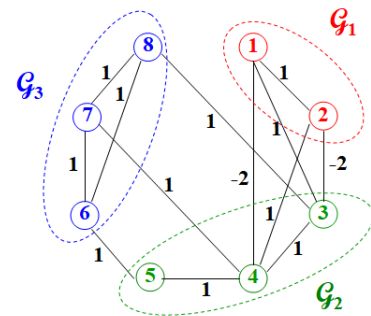
**Fig. 3** Group consensus with protocol (5) and (II). (a) State evolution with zoom-out view shown in the inset. (b) Error evolution with zoom-out view shown in the inset.



**Fig. 4** Group consensus with protocol (5) and (III). (a) State evolution with zoom-out view shown in the inset. (b) Error evolution with zoom-out view shown in the inset.

We assume  $f(t, x_i(t)) = t + x_i(t)$  and  $d_i(t, x_i(t)) = \cos(x_i(t))$  for  $i = 1, \dots, 8$ . Hence, we have  $\ell = r = \bar{d} = 1$ . The initial condition is chosen as  $x(0) = (100, -300, 100, -200, -50, -150, 150, 50)^T$ . The state evolution of system (3) with protocol (5), (I), (II), and (III) is displayed in the main panels of Fig. 6. The error behaviors have also been shown in the respective insets. Note that the modified definitions of  $\psi_i$  are in use according to Corollaries 1, 2, and 3, respectively. The control parameters are taken as  $\alpha = 2$ ,  $\beta = 4$ ,  $\gamma = 1$ ,  $\xi = 8.49$  for (I) and (II), and  $\xi = 11.07$  for (III).

Similarly as in Example 1, we observe that all state trajectories in every subgroup converge as one would expect and the settling time in all these three cases is about  $t = 0.3$ . The settling time is less than our theoretical upper-bounds, which are  $T_{\max}^1 \approx 2.06$ ,  $T_{\max}^2 \approx 6.68$ , and  $T_{\max}^3 \approx 5.27$ , respectively. In Tab. 2 we show more estimates of the settling time under these protocols for different values of parameters  $p$  and  $q$ . In the considered range of parameters, we observe that  $T_{\max}^i$  ( $i = 1, 2$ ) increase with both  $p$  and  $q$ . As such, we are likely to obtain more accurate upper-bounds for protocols (I) and (II) by choosing small  $p$  and  $q$  since the practical settling time is often much less than our predictions (as we have observed above). Note that the protocol (III) is not compared here as it is independent of  $p$  and relies on a different mechanism of  $\xi$ .



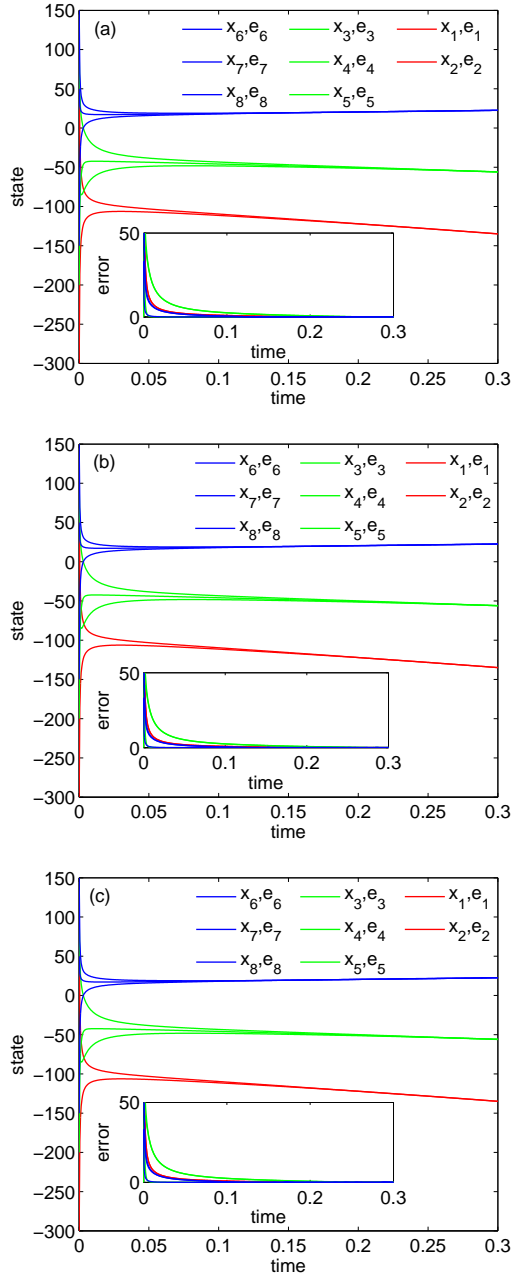
**Fig. 5** Communication topology containing  $N = 8$  nodes with the grouping  $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$  satisfying the inter-group common influence condition.

## 5 Conclusion

This paper discusses fixed-time group consensus problems for multi-agent systems with inherent nonlinear dynamics and bounded uncertainties. Three novel distributed control strategies are proposed to solve the fixed-time group consensus problem. Some numerical simulations are presented to show the correctness of our obtained

	$p = 0.5$	$p = 0.7$	$p = 0.9$
$q = 2$	(1.67, 2.77)	(2.31, 3.02)	(5.62, 6.12)
$q = 2.5$	(1.69, 2.89)	(2.33, 3.15)	(5.64, 6.25)
$q = 3$	(2.76, 3.45)	(3.40, 3.71)	(6.71, 6.81)

**Table 2** List of estimated settling time pair  $(T_{\max}^1, T_{\max}^2)$  in Theorems 1 and 2 for different combination of  $p$  and  $q$ . Here, we take  $\alpha = 2$  and  $\beta = 10$ .



**Fig. 6** Group consensus with protocols (5) and (I) in panel (a); protocols (5) and (II) in panel (b); protocols (5) and (III) in panel (c). The insets show the corresponding error evolution.

theoretical results. For future work, there are some challenging problems to be addressed. For instance, fixed-time group consensus over directed networks will be studied as directed network is more general in the real world [41]. It would be also interesting to see if the current analysis can be generalized to accommodate networks with adaptive couplings and communication delays.

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