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Research article

# On comparison between the distance energies of a connected graph

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ABSTRACT

Let $G$ be a simple connected graph of order $n$ having Wiener index $W(G)$. The distance, distance Laplacian and the distance signless Laplacian energies of $G$ are respectively defined as

$$DE(G)=\sum_{i=1}^{n}|\upsilon_i^D|,\ DLE(G)=\sum_{i=1}^{n}|\upsilon_i^L-\overline{Tr}| \text{ and } DSLE(G)=\sum_{i=1}^{n}|\upsilon_i^Q-\overline{Tr}|,$$

where $\upsilon_i^D,\upsilon_i^L$ and $\upsilon_i^Q, 1\le i\le n$ are respectively the distance, distance Laplacian and the distance signless Laplacian eigenvalues of $G$ and $\overline{Tr}=\frac{2W(G)}{n}$ is the average transmission degree. In this paper, we will study the relation between $DE(G)$, $DLE(G)$ and $DSLE(G)$. We obtain some necessary conditions for the inequalities $DLE(G)\ge DSLE(G), DLE(G)\le DSLE(G), DLE(G)\ge DE(G)$ and $DSLE(G)\ge DE(G)$ to hold. We will show for graphs with one positive distance eigenvalue the inequality $DSLE(G)\ge DE(G)$ always holds. Further, we will show for the complete bipartite graphs the inequality $DLE(G)\ge DSLE(G)\ge DE(G)$ holds. We end this paper by computational results on graphs of order at most 6.

## 1. Introduction

Only connected (undirected, finite and simple) graphs are considered here in this article. A graph is represented by $G=(V(G),E(G))$, with $V(G)=\{w_1,w_2,\dots,w_n\}$ as its vertex set and its edge set is denoted by $E(G)$. The number $n=|V(G)|$ is the *order* of $G$ and $m=|E(G)|$ is the *size*. The *neighborhood* $N(w)$ of $w\in V(G)$ is a set of vertices adjacent to it.

In a graph $G$, the *distance* $d_{wx}$ among 2 vertices $w,x\in V(G)$ is taken as the length of a smallest path connecting $w$ with $x$. The maximum distance among any two vertices is the *diameter* of $G$. The *distance matrix* $D(G)$ of $G$ defined as $D(G)=(d_{wx})_{w,x\in V(G)}$. The *transmission* $Tr_G(w)$ of $w\in V(G)$ is taken as the sum of the distances from $w$ to remaining vertices in $G$, so, $Tr_G(w)=\sum_{v\in V(G)} d_{wv}$. A graph is $k$-*transmission regular* if $Tr_G(w)=k$, for all $w\in V(G)$. The Wiener index $W(G)$ (also called transmission number) of $G$ is the sum of distances between all unordered pairs of vertices in $G$. Clearly, $W(G)=\frac{1}{2}\sum_{w\in V(G)} Tr_G(w)$. For any vertex $w_i\in V(G)$, the transmission (transmission degree) $Tr_G(w_i)$ is represented by $Tr_i$ and the transmission degree sequence is given as: $\{Tr_1,Tr_2,\dots,Tr_n\}$.

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The diagonal matrix of vertex transmissions of $G$ is denoted by $Tr(G) = diag(Tr_1, \ldots, Tr_n)$. The Laplacian and the signless Laplacian of the distance matrix of a graph $G$ are respectively defined as $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$, respectively. They are known as the *distance Laplacian matrix* and the *distance signless Laplacian matrix* of $G$, respectively.

Gutman in 1978 introduced the concept of energy of $G$ [14], whose origin of credit goes back to theoretical/mathematical chemistry. Let $\curlywedge_1 \geq \curlywedge_2 \geq \cdots \geq \curlywedge_n$ be the eigenvalues of adjacency matrix $A(G)$. The *energy* (trace norm of a symmetric matrix) of $G$ is given as

$$\mathbb{E}(G) = \sum_{i=1}^{n} | \curlywedge_i |.$$

Detailed results and recent survey of $\mathbb{E}(G)$ can be seen in [17].

This spectral-based invariant is much studied in theoretical chemistry/mathematical literature. The energy (trace norm) of $G$ (symmetric real matrices) is a natural concept in mathematics and theoretical (chemistry/computer science) in terms of general matrices and particularly those coming from graph structure, see [2,11,19,21].

Let $\upsilon_1^D \geq \upsilon_2^D \geq \ldots \geq \upsilon_n^D$, $\upsilon_1^L \geq \upsilon_2^L \geq \ldots \geq \upsilon_n^L$ and $\upsilon_1^Q \geq \upsilon_2^Q \geq \ldots \geq \upsilon_n^Q$ be the distance, $D^L$ and $D^Q$ eigenvalues of $G$, respectively. The distance energy (see [16]) of $G$ is defined as

$$DE(G) = \sum_{i=1}^{n} |\upsilon_i^D|.$$

Various interesting papers on distance energy along with recent work can be found in [1,6,7,26] and the articles therein.

The concept of distance energy $DE(G)$ of $G$ was elaborated to $D^L$ matrix in [27] and $D^Q$ matrix in [6]. The $DLE(G)$ (distance Laplacian energy) of $G$ is given as

$$DLE(G) = \sum_{i=1}^{n} \left| \upsilon_i^L - \frac{2W(G)}{n} \right|.$$

Let $\zeta$ be the largest positive integer so that $\upsilon_\zeta^L \geq \frac{2W(G)}{n}$. Using $\sum_{i=1}^{n} \upsilon_i^L = 2W(G)$, it is shown in [6], that

$$DLE(G) = 2\left( \sum_{i=1}^{\zeta} \upsilon_i^L(G) - \frac{2\zeta W(G)}{n} \right) = 2 \max_{1 \leq j \leq n} \left( \sum_{i=1}^{j} \upsilon_i^L(G) - \frac{2jW(G)}{n} \right),$$

for more work on $DLE(G)$, we refer to [6,7,9,10,13,23].

The distance signless Laplacian energy $DSLE(G)$ of $G$ is defined as

$$DSLE(G) = \sum_{i=1}^{n} |\upsilon_i^Q - \frac{2W(G)}{n}|.$$

Let $\zeta^+$ be the number of distance signless Laplacian eigenvalues of $G$ which are larger than or equal to $\frac{2W(G)}{n}$, then

$$DSLE(G) = 2\left( \sum_{i=1}^{\zeta^+} \upsilon_i^Q(G) - \frac{2\zeta^+ W(G)}{n} \right) = 2 \max_{1 \leq j \leq n} \left( \sum_{i=1}^{j} \upsilon_i^Q(G) - \frac{2jW(G)}{n} \right).$$

For more work on $DSLE(G)$, we refer to [6,7,22].

It is clear that $DE(G)$, $DLE(G)$ and $DSLE(G)$ of $G$ is the trace norm of the matrices $D(G), D^L(G) - \frac{2W(G)}{n} I_n$ and $D^Q(G) - \frac{2W(G)}{n} I_n$, respectively. We recall that the trace norm of a $n \times n$ complex square matrix $M$ is defined as $\|M\|_* = \sum_{i=1}^{n} \zeta_i(M)$, where $\zeta_1(M) \geq \zeta_2(M) \geq \cdots \geq \zeta_n(M)$ are the singular values of $M$ (the square roots of the eigenvalues of $MM^*$ or $M^*M$, with $M^*$ being the complex conjugate of $M$). Therefore, the study of these spectral graph invariants is not only interesting and important from spectral graph theory point of view, but also important from Matrix Theory point of view.

One of the interesting open question in spectral graph theory is to find the extremal graphs with respect to a spectral graph invariant for a given graph matrix among all or among a class of graphs of order $n$. A lot of papers can be found in this direction. Another problem of interest is to compare two different spectral graph invariants for a given graph matrix among all or among a class of graphs of order $n$. Again this problem has been considered for various spectral graph invariants with respect to adjacency, Laplacian and the signless Laplacian matrices. The problem of comparing the spectral graph invariants: the distance energy $DE(G)$, the distance Laplacian energy $DLE(G)$ and the distance signless Laplacian energy $DSLE(G)$ of a connected graph $G$ was considered in [6] and the following relations were observed (see also Table 1).

(1) There are non-transmission regular graphs for which the inequality $DSLE(G) \geq DLE(G)$ holds and the non-transmission regular graphs for which the reverse inequality $DSLE(G) \leq DLE(G)$ holds.

(2) There are non-transmission regular graphs for which the inequality $DLE(G) \geq DE(G)$ holds and the non-transmission regular graphs for which the reverse inequality $DLE(G) \leq DE(G)$ holds.

Based on these observations the following problems can be put forward, see also [6].

**Problem 1.** Characterize all graphs for which $DSLE(G) > DE(G)$, $DSLE(G) < DE(G)$ and $DSLE(G) = DE(G)$.

**Problem 2.** Characterize all graphs for which $DLE(G) > DE(G)$, $DLE(G) < DE(G)$ and $DLE(G) = DE(G)$.

**Problem 3.** Characterize all graphs for which $DSLE(G) > DLE(G)$, $DSLE(G) < DLE(G)$ and $DSLE(G) = DLE(G)$.

Clearly, solving Problems (1), (2) and (3) for all graphs of order $n$ is difficult. Therefore, giving a construction of a family of graphs for any of these problems is always interesting and appreciable. This paper is a contribution in this direction.

The paper is organized as follows. In Section 2, we obtain some results about the distance Laplacian eigenvalues. We will show for the complete bipartite graph the inequality $DLE(G) \geq DSLE(G) \geq DE(G)$ holds. Further, we obtain some necessary conditions for the inequalities $DLE(G) \geq DSLE(G)$, $DLE(G) \leq DSLE(G)$, $DLE(G) \geq DE(G)$ and $DSLE(G) \geq DE(G)$ to hold. We will show for graphs with one positive distance eigenvalue the inequality $DSLE(G) \geq DE(G)$ always holds. We end this paper with some computational results on graphs of order at most 6, in Section 3.

## 2. Relation between distance and distance (signless) Laplacian energies

In the present section, we first obtain some results about the distance Laplacian eigenvalues. We will establish some relations between the distance signless Laplacian $DSLE(G)$, the distance Laplacian energy $DLE(G)$ and the distance energy $DE(G)$ of a graph $G$. We will show for graphs with one positive distance eigenvalue the inequality $DSLE(G) \geq DE(G)$ always holds.

The following result gives an inequality (upper bound) for the second smallest distance Laplacian eigenvalue in terms of vertex transmission degrees of a graph $G$.

**Theorem 2.1.** *Let $G$ be a graph of order $n \geq 3$ having transmission degrees $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Then*

$$\upsilon_{n-1}^L \leq \min_{w_i w_j \in E(G)} \left\{ \frac{Tr_i + Tr_j}{2} \right\} + 1. \tag{2.1}$$

*Equality occurs in (2.1) if and only if (shortly iff) $Tr_i = Tr_j$ and $d(w_i, w_k) = d(w_j, w_k)$, for all $w_k \in V(G) - \{w_i, w_j\}$. In particular, if $G \cong K_2 \vee H$, where $H$ is a graph of order $n-2$, then equality holds in (2.1).*

**Proof.** Let $X = (x_1, x_2, \ldots, x_n)^T \neq \mathbf{0}$ be a vector in $\mathbb{R}^n$ with sum of entries equal to zero, then by Rayleigh-Ritz Theorem, we have

$$\upsilon_{n-1}^L \leq \frac{X^T D^L X}{X^T X}, \tag{2.2}$$

where $\mathbb{R}^n$ is the real vector space of dimension $n$ and $X^T$ is the transpose of $X$. Let $w_i$ and $w_j$ be two adjacent vertices in $G$, then $d(w_i, w_j) = 1$. Taking $x_i = 1, x_j = -1$ and $x_k = 0$, for $k \neq i, j$ in (2.2), we get

$$\upsilon_{n-1}^L \leq \frac{Tr_i + Tr_j}{2} + 1,$$

with this Inequality (2.1) follows. Suppose that equality holds in (2.1), then equality occurs in Rayleigh-Ritz Theorem, giving that $X = (0, 0, \ldots, 1, \ldots, 0, -1, 0, \ldots, 0)^T$, where $x_i = 1$ and $x_j = -1$, is an eigenvector of the matrix $D^L(G)$ corresponding to the eigenvalue $\upsilon_{n-1}^L$. For the vertex $w_i$, it follows from the equation $D^L X = \upsilon_{n-1}^L X$ that $\upsilon_{n-1}^L(1) = Tr_i(1) - (-1 + 0 + \cdots + 0)$. This gives that $\upsilon_{n-1}^L = Tr_i + 1$. Similarly, for the vertex $w_j$, we get $\upsilon_{n-1}^L = Tr_j + 1$. These two equations together give that $Tr_i = Tr_j$. Let $w_k$ be a vertex different from $w_i$ and $w_j$. For this vertex, it follows from the equation $D^L X = \upsilon_{n-1}^L X$ that $\upsilon_{n-1}^L(0) = Tr_k(0) - (d(w_i, w_k) - d(w_j, w_k) + 0 + \cdots + 0)$. This gives that $d(w_i, w_k) = d(w_j, w_k)$. Thus, it follows that equality occurs in (2.1) iff $w_i$ and $w_j$ are adjacent with $Tr_i = Tr_j$ and $d(w_i, w_k) = d(w_j, w_k)$, for all $w_k \in V(G) - \{w_i, w_j\}$. For the graph $G = K_2 \vee H$, let $w_1$ and $w_2$ be the vertices of $K_2$ and $w_3, \ldots, w_n$ be the vertices of $H$. It is clear that $Tr_1 = n - 1 = Tr_2$ and $Tr_k \geq n - 1$, for all $k = 3, 4, \ldots, n$. Since complement of the graph $K_2 \vee H$ is disconnected, therefore it follows that $\upsilon_{n-1}^L = n$ and also we note that $\min_{w_i w_j \in E(G)} \left\{ \frac{Tr_i + Tr_j}{2} \right\} + 1 = \frac{2n-2}{2} + 1 = n$. This completes the proof. □

In the following result, a lower bound inequality is given for $DLE(G)$ in terms of order $n$, $W(G)$ and the transmission degrees of $G$.

**Theorem 2.2.** *Let $G$ be a graph of order $n \geq 3$ with Wiener index $W(G)$. Let $\zeta(G)$ be the number of distance Laplacian eigenvalues of $G$ which are greater than or equal to $\frac{2W(G)}{n}$, then*

$$DLE(G) \geq \frac{8W(G)}{n} - \min_{w_i w_j \in E(G)} \Big\{ Tr_i + Tr_j + 2 \Big\}, \tag{2.3}$$

*equality occurs iff* $\zeta(G) = n-2$ *and* $Tr_i = Tr_j$ *with* $d(w_i, w_k) = d(w_j, w_k)$*, for all* $w_k \in V(G) - \{w_i, w_j\}$.

**Proof.** Let $G$ be a graph (connected) of order $n \geq 3$ with $D^L$ eigenvalues $\upsilon_1^L(G) \geq \upsilon_2^L(G) \geq \cdots \geq \upsilon_{n-1}^L(G) \geq \upsilon_n^L(G) = 0$. Let $\zeta$ be the number of $D^L$ eigenvalues of $G$ which are larger than or equal to $\frac{2W(G)}{n}$. With $\sum_{i=1}^{n-1} \upsilon_i^L(G) = 2W(G)$ and the inequality (2.1), we have

$$\begin{aligned} DLE(G) &= 2 \max_{1\leq j \leq n-1} \left( \sum_{i=1}^{j} \upsilon_i^L(G) - \frac{2jW(G)}{n} \right) \\ &\geq 2 \left( \sum_{i=1}^{n-2} \upsilon_i^L(G) - \frac{2(n-2)W(G)}{n} \right) \\ &= 2 \left( 2W(G) - \upsilon_{n-1}^L(G) - \frac{2(n-2)W(G)}{n} \right) = \frac{8W(G)}{n} - 2\upsilon_{n-1}^L(G) \\ &\geq \frac{8W(G)}{n} - 2 \min_{w_i w_j \in E(G)} \left\{ \frac{Tr_i + Tr_j}{2} \right\} - 2. \end{aligned}$$

Equality holds in (2.3) iff equality occurs in

$$\max_{1\leq j \leq n-1} \left( \sum_{i=1}^{j} \upsilon_i^L(G) - \frac{2jW(G)}{n} \right) = \left( \sum_{i=1}^{n-2} \upsilon_i^L(G) - \frac{2(n-2)W(G)}{n} \right) \tag{2.4}$$

and equality holds in $\upsilon_{n-1}^L(G) \leq \min_{w_i w_j \in E(G)} \left\{ \frac{Tr_i + Tr_j}{2} \right\} + 1$. Equality holds in (2.4) iff $\zeta = n-2$ and by Theorem 2.1 equality holds in (2.1) iff $Tr_i = Tr_j$ and $d(w_i, w_k) = d(w_j, w_k)$, for all $w_k \in V(G) - \{w_i, w_j\}$. This finishes the proof. □

The following inequality (upper bound) for the second smallest $D^L$ eigenvalue $\upsilon_{n-1}^L(G)$ in terms of order $n$ and the minimum transmission degree $Tr_{\min}$ of $G$ was obtained in [6].

$$\upsilon_{n-1}^L(G) \leq \frac{n}{n-1} Tr_{\min}, \tag{2.5}$$

with equality iff $G$ contains a vertex of transmission degree $n-1$.

Using this upper bound for $\upsilon_{n-1}^L(G)$ together with the definition of $D^L$ energy, the following lower bound is obtained in [12] for $DLE(G)$ of $G$.

$$DLE(G) \geq \frac{8W(G)}{n} - \frac{2nTr_{\min}}{n-1}, \tag{2.6}$$

equality occurs iff $\zeta(G) = n-2$ and $G$ has a vertex of transmission degree $n-1$.

It is easy to verify that the upper bound given by (2.1) for the second smallest $D^L$ eigenvalue $\upsilon_{n-1}^L(G)$ is better than the bound given by (2.5), for all $G$ with $2Tr_{\min} \geq (n-1)\Big(Tr_j - Tr_{\min} + 2\Big)$, where $Tr_j = \min\{Tr_k : w_k w_n \in E(G)\}$, $w_n$ is the vertex with minimum transmission $Tr_{\min}$. In particular, for connected graphs with $Tr_j = Tr_{\min}$, the inequality $2Tr_{\min} \geq (n-1)\Big(Tr_j - Tr_{\min} + 2\Big)$ always holds. Clearly, for connected graphs $G$ which satisfy the inequality $2Tr_{\min} \geq (n-1)\Big(Tr_j - Tr_{\min} + 2\Big)$, the lower bound given by (2.3) is better than the lower bound given by (2.6).

The next result shows that the $D^L$ energy of the complete bipartite graph is always greater the corresponding $D^Q$ energy.

**Theorem 2.3.** *For* $K_{a,b}$ *with* $a \leq b$ *and* $a \geq 5$*, we have*

$$DLE(K_{a,b}) \geq DSLE(K_{a,b}),$$

*with equality iff* $a = b$.

**Proof.** Consider the complete bipartite graph $K_{a,b}$ with $a \leq b$ and $a \geq 5$. The $D^L$ spectrum of $K_{a,b}$ is $\{(2n-a)^{[b-1]}, (2n-b)^{[a-1]}, n, 0\}$ and $2W(K_{a,b}) = 2n^2 - 2n - 2ab$. We first suppose that $5 \leq a < b$. Let $\zeta$ be the number of $D^L$ eigenvalues of $K_{a,b}$ which are greater than or equal to $\frac{2W(K_{a,b})}{n}$, then using definition of $D^L$ energy, we have

$$DLE(K_{a,b}) = 2\Big( S_\zeta^L(K_{a,b}) - \frac{2\zeta W(K_{a,b})}{n} \Big), \tag{2.7}$$

where $S_k^L(G)$ is the sum of the $k$ largest distance Laplacian eigenvalues of $G$. We first compute $\zeta$ for the graph $K_{a,b}$. Since $a<b$, therefore we always have $2n-a>2n-b$. Also $2n-a>n$ holds as $n>a$, likewise $n>b$ implies $2n-a$ is $D^L$ spectral radius of $G$ and so we always have $2n-a\geq\frac{2W(G)}{n}$. For the eigenvalue $n$, we have $n<\frac{2W(K_{a,b})}{n}$ giving that $n^2-2n-2ab>0$, which gives

$$2a^2-2an+n^2-2n>0,\quad \text{as}\quad a+b=n. \tag{2.8}$$

Consider a polynomial $f(a)=2a^2-2an+n^2-2n$, for $1\leq a<n$, then the discriminant of $f(a)$ is $d=4n(4-n)<0$, for all $n\geq 5$. This shows that (2.8) always holds and so we always have $n<\frac{2W(K_{a,b})}{n}$, for $n\geq 5$. For the eigenvalue $2n-b$, we have $2n-b\geq\frac{2W(K_{a,b})}{n}$ implying that $2ab\geq n(b-2)$, which further gives that $\frac{a+2-\sqrt{a^2+12a+4}}{2}\leq b\leq\frac{a+2+\sqrt{a^2+12a+4}}{2}$. Since $\frac{a+2-\sqrt{a^2+12a+4}}{2}<a$ and $a+3\leq\frac{a+2+\sqrt{a^2+12a+4}}{2}\leq a+4$, it follows that $2n-b\geq\frac{2W(G)}{n}$ for all $a<b\leq a+3$ and $2n-b<\frac{2W(G)}{n}$ for all $b\geq a+4$. Thus, it follows that if $a<b\leq a+3$, then $\zeta=n-2$ and if $b\geq a+4$, then $\zeta=b-1$. With this it follows from (2.7) that

$$DLE(K_{a,b})=2\Big(S_{n-2}^L(K_{a,b})-\frac{2(n-2)W(K_{a,b})}{n}\Big)=2\Big(3n-4-\frac{4ab}{n}\Big),$$

for all $a<b\leq a+3$ and

$$DLE(K_{a,b})=2\Big(S_{b-1}^L(K_{a,b})-\frac{2(b-1)W(K_{a,b})}{n}\Big)=2(b-1)\Big(\frac{2ab}{n}-a+2\Big),$$

for all $b\geq a+4$. If $a=b$, then the $D^L$ spectrum of $K_{a,b}$ is $\{3a^{[2a-2]},2a,0\}$ and $\frac{2W(K_{a,b})}{n}=3a-2$. It is easy to see that $\zeta=2a-2$ and so using (2.7), we have

$$DLE(K_{a,b})=2\Big(S_{2a-2}^L(K_{a,b})-\frac{2(2a-2)W(K_{a,b})}{n}\Big)=8(a-1)=DSLE(K_{a,b}),$$

for $a=b$, as the graph $K_{a,b}$ is regular in this case and for regular graphs $G$, we have $DLE(G)=DSLE(G)$.

The $D^Q$ spectrum of the graph $K_{a,b}$ is

$$\left\{\frac{5n-8+\sqrt{9(b-a)^2+4ab}}{2},\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2},2n-a-4^{[b-1]},2n-b-4^{[a-1]}\right\}.$$

Let $\zeta^+$ be the number of $D^Q$ eigenvalues of the graph $K_{a,b}$, which are greater than or equal to $\frac{2W(K_{a,b})}{n}$, then using definition of $D^Q$ energy, we have

$$DSLE(K_{a,b})=2\Big(S_{\zeta^+}^Q(K_{a,b})-\frac{2\zeta^+W(K_{a,b})}{n}\Big),$$

$S_k^Q(G)$ is the sum of the $k$ largest distance signless Laplacian eigenvalues of $G$. We first compute $\zeta^+$ for the graph $K_{a,b}$. Let us suppose that $5\leq a<b$. It is easy to see that $\frac{5n-8+\sqrt{9(b-a)^2+4ab}}{2}$ is the $D^Q$ spectral radius of $K_{a,b}$. For $a<b$, we always have $2n-a-4>2n-b-4$ and $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}>2n-b-4$. From this it follows that we have two choices for the second largest distance signless Laplacian. In fact, either $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}$ or $2n-a-4$ is the second largest $D^Q$ eigenvalue of $K_{a,b}$. We have $2n-a-4\geq\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}$ giving that $b\geq\frac{5a}{2}$. This shows that for $b<\frac{5a}{2}$, the eigenvalue $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}$ is the second largest eigenvalue and for $b\geq\frac{5a}{2}$, the eigenvalue $2n-a-4$ is the second largest eigenvalue of the graph $K_{a,b}$. Since, $\frac{5n-8+\sqrt{9(b-a)^2+4ab}}{2}$ is the distance signless Laplacian spectral radius so we always have $\frac{5n-8+\sqrt{9(b-a)^2+4ab}}{2}\geq\frac{2W(K_{a,b})}{n}$. For the eigenvalue $2n-b-4$, it is easy to see that $2n-b-4<\frac{2W(K_{a,b})}{n}$. For the eigenvalue $2n-a-4$, we have $2n-a-4\geq\frac{2W(K_{a,b})}{n}=\frac{2n^2-2n-2ab}{n}$ giving that $b\geq\frac{a(a+2)}{a-2}$. This shows that $2n-a-4\geq\frac{2W(K_{a,b})}{n}$, for all $b\geq\frac{a(a+2)}{a-2}$ and $2n-a-4<\frac{2W(K_{a,b})}{n}$, for all $a<b<\frac{a(a+2)}{a-2}$. Further for the eigenvalue $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}$, we have $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2}<\frac{2W(K_{a,b})}{n}=\frac{2n^2-2n-2ab}{n}$ giving that

$$n^4-(5a-1)n^3+(3a^2+4a-2)n^2+4a^2n(a-1)-2a^4>0. \tag{2.9}$$

Consider the function $f(n)=n^4-(5a-1)n^3+(3a^2+4a-2)n^2+4a^2n(a-1)-2a^4$, for $n>2a$. Since for $n>2a$, we have $4a^2n(a-1)-2a^4>0$, also $n^4-(5a-1)n^3+(3a^2+4a-2)n^2\geq 0$ gives that $n^2-(5a-1)n+(3a^2+4a-2)\geq 0$, which further gives that $n\geq\frac{5a-1+\sqrt{13a^2-26a+9}}{2}$. It is easy to see that $\frac{5a-1+\sqrt{13a^2-26a+9}}{2}<\Big(\frac{5+\sqrt{13}}{2}\Big)a-\Big(\frac{1+\sqrt{13}}{2}\Big)\approx 4.3027a-2.3027$. From this it follows that the inequality (2.9) always holds for $n\geq 4.31a-2$ (that is, for $b\geq 3.31a-2$). Now, for $2a<n<4.31a-2$, we have $f(n)=ng(n)-2a^4$, where $g(n)=n^3-(5a-1)n^2+(3a^2+4a-2)n+4a^2(a-1)$. For the function $g(n)$ the first derivative is $g'(n)=3n^2-2(5a-1)n+3a^2+4a-2$ giving that $g(n)$ is decreasing for $n\in(2a,\gamma]$ and increasing for $n\in[\gamma,3.31a-2)$, where $\gamma=\frac{5a-1+\sqrt{16a^2-22a+7}}{3}$. We have $g(2a)=8a^3-4a^2(5a-1)+2a(3a^2+4a-2)+4a^2(a-1)=-2a(a^2-4a+2)<0$, for all $a\geq 2$. Also, $g(4a-3)=-5a^2+19a-12<0$, for all

$a \geq 3$. From this it follows that $g(n) < 0$, for all $2a < n \leq 4a-3$, giving that $f(n) < 0$, for all $2a < n \leq 4a-3$, which in turn gives that the inequality (2.9) does not hold for $2a < n \leq 4a-3$. For $4a-2 \leq n < 4.31a-2$, it can be seen that $f(n)$ is positive for some values of $n$ and negative for some values. From this discussion, it follows that $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2} < \frac{2W(K_{a,b})}{n}$, for all $n \geq 4.31a-2$, $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2} \geq \frac{2W(K_{a,b})}{n}$, for all $2a < n \leq 4a-3$ and $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2} < \frac{2W(K_{a,b})}{n}$ or $\frac{5n-8-\sqrt{9(b-a)^2+4ab}}{2} \geq \frac{2W(K_{a,b})}{n}$, for $4a-2 \leq n < 4.31a-2$. Thus, it follows that for $a < b < \frac{a(a+2)}{a-2}$, we have $\zeta^+ = 2$; for $\frac{a(a+2)}{a-2} \leq b < \frac{5a}{2}$, we have $\zeta^+ = b+1$; for $\frac{5a}{2} \leq b \leq 3a-3$, we have $\zeta^+ = b+1$; for $3a-2 \leq b < 3.31a-2$, we have $\zeta^+ = b$ or $b+1$; and for $b \geq 3.31a-2$, we have $\zeta = b$. With this it follows from (2.7) that if $a < b < \frac{a(a+2)}{a-2}$, then

$$DSLE(K_{a,b}) = 2\Big(S_2^Q(K_{a,b}) - \frac{4W(K_{a,b})}{n}\Big) = 2\Big(n-4+\frac{4ab}{n}\Big).$$

If $\frac{a(a+2)}{a-2} \leq b \leq 3a-3$, then

$$DSLE(K_{a,b}) = 2\Big(S_{b+1}^Q(K_{a,b}) - \frac{2(b+1)W(K_{a,b})}{n}\Big) = 2\Big(2(a-1)-b(a+1)+\frac{2ab(b+1)}{n}\Big).$$

If $3a-2 \leq b < 3.31a-2$, then

$$DSLE(K_{a,b}) = 2\Big(S_{b+1}^Q(K_{a,b}) - \frac{2(b+1)W(K_{a,b})}{n}\Big) = 2\Big(2(a-1)-b(a+1)+\frac{2ab(b+1)}{n}\Big).$$

or

$$DSLE(K_{a,b}) = 2\Big(S_b^Q(K_{a,b}) - \frac{2bW(K_{a,b})}{n}\Big) = n+2a-2b(a+2)+\sqrt{9(a-b)^2+4ab}+\frac{4ab^2}{n}.$$

If $b \geq 3.31a-2$, then

$$DSLE(K_{a,b}) = 2\Big(S_b^Q(K_{a,b}) - \frac{2bW(K_{a,b})}{n}\Big) = n+2a-2b(a+2)+\sqrt{9(a-b)^2+4ab}+\frac{4ab^2}{n}.$$

Now, for $a < b \leq a+3$, we have $\zeta = n-2$, $\zeta^+ = 2$ and so $DLE(K_{a,b}) = 2\Big(3n-4-\frac{4ab}{n}\Big) > 2\Big(n-4+\frac{4ab}{n}\Big) = DSLE(K_{a,b})$, giving that $(a-b)^2 > 0$, which is always true. This shows that the result holds in this case. For $a+3 < b < \frac{a(a+2)}{a-2}$, we have $\zeta = b-1$, $\zeta^+ = 2$ and so $DLE(K_{a,b}) = 2(b-1)\Big(\frac{2ab}{n}-a+2\Big) > 2\Big(n-4+\frac{4ab}{n}\Big) = DSLE(K_{a,b})$, giving that

$$\frac{ab}{n}\Big(b-a-6\Big)+b+2 > 0. \tag{2.10}$$

Since $a+3 < b < \frac{a(a+2)}{a-2}$ and $\frac{a(a+2)}{a-2} \leq a+6$, for $a \geq 6$. It follows that the possible values of $b$ in this interval are $b = a+4$ or $a+5$. It is easy to see that inequality (2.10) holds for these values of $b$. For $a = 5$, the inequality (2.10) becomes $6b(b-8)+10 > 0$, which is always true as $b > a+3 \geq 9$. This shows that $DLE(K_{a,b}) > DSLE(K_{a,b})$ holds in this case as well. For $\frac{a(a+2)}{a-2} \leq b \leq 3a-3$, we have $\zeta = b-1$, $\zeta^+ = b+1$ and so $DLE(K_{a,b}) = 2(b-1)\Big(\frac{2ab}{n}-a+2\Big) > 2\Big(n+a-2-b(a+2)+\frac{2ab(b+1)}{n}\Big) = DSLE(K_{a,b})$, giving that $b^2-a^2+2b(b-a) > 0$, which is always true as $b > a$. For $3a-2 \leq b < 3.31a-2$, we have $\zeta = b-1$, $\zeta^+ = b$ or $b+1$. If $\zeta^+ = b+1$, then using above case the result follows. If $\zeta^+ = b$, then $DLE(K_{a,b}) = 2(b-1)\Big(\frac{2ab}{n}-a+2\Big) > n-2ab-2(b-a)+\sqrt{9(a-b)^2+4ab}+\frac{4ab^2}{n}$, giving that

$$5b-a-4 > \sqrt{9(a-b)^2+4ab}+\frac{4ab}{n}. \tag{2.11}$$

Since $\sqrt{9(a-b)^2+4ab} < 3b-a$, for all $a < b$, it follows from inequality (2.11) that $5b-a-4 > 3b-a+\frac{4ab}{n}$, which implies that $b(b-a-1)-a > 0$, which is always true as $b > 3a-2$. This shows that the result is true in this case as well. Lastly, for $b \geq 3.31a-2$, we have $\zeta = b-1$, $\zeta^+ = b$ and so by previous case the result follows in this case as well. This finishes the proof. □

The next Lemma by Fan [8] is very important related to sum of eigenvalue inequalities.

**Lemma 2.4.** *Let $L$, $M$ and $N$ be square matrices of order $n$ such that $N = L+M$. Then*

$$\sum_{i=1}^{n} s_i(N) \leq \sum_{i=1}^{n} s_i(L) + \sum_{i=1}^{n} s_i(M),$$

*where $s_i$ denote the singular values of a matrix. Furthermore, the equality holds iff there exists a matrix $P$ $(PP^T - I)$ such that $PL$ and $PM$ are both positive semi-definite matrices.*

We have $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$. Subtracting the corresponding sides, we obtain $D^Q(G) - D^L(G) = 2D(G)$ which we can rewrite as

$$\left(D^Q(G)-\frac{2W(G)}{n}I_n\right)-\left(D^L(G)-\frac{2W(G)}{n}I_n\right)=2D(G).$$

Taking $Z=2D(G)$, $X=D^Q(G)-\frac{2W(G)}{n}I_n$ and $Y=D^L(G)-\frac{2W(G)}{n}I_n$ in Lemma 2.4, we obtain

$$DSLE(G)+DLE(G)\geq 2DE(G). \tag{2.12}$$

From (2.12) it is clear that if $G$ is a connected graph with $DSLE(G)\leq DLE(G)$, then for such a graph $G$, we have $DLE(G)\geq DE(G)$, while as for a connected graph $G$ with $DLE(G)\leq DSLE(G)$, we have $DSLE(G)\geq DE(G)$. Therefore, we have the following observation.

**Theorem 2.5.** *Let $G$ be a graph of order $n$.*

1. *If $DSLE(G)\leq DLE(G)$, then $DLE(G)\geq DE(G)$.*
2. *If $DLE(G)\leq DSLE(G)$, then $DSLE(G)\geq DE(G)$.*

The next consequence follows from Theorems 2.3 and 2.5.

**Corollary 2.6.** *For $K_{a,b}$ with $a+b=n$, $b\geq a$ and $a\geq 5$, we have $DLE(K_{a,b})\geq DE(K_{a,b})$.*

The next lemma by W. So is given in [25].

**Lemma 2.7.** *Let $L$ and $M$ be Hermitian matrices of order $n$ such that $Z=L+M$. Then*

$$\curlywedge_k(Z)\leq \curlywedge_j(L)+\curlywedge_{k-j+1}(M),\quad n\geq k\geq j\geq 1,$$
$$\curlywedge_k(Z)\geq \curlywedge_j(L)+\curlywedge_{k-j+n}(M),\quad n\geq j\geq k\geq 1,$$

*where $\curlywedge_i(M)$ is the $i$-th largest eigenvalue of matrix $M$. In either of the above eigenvalue inequalities, the equality occurs iff there exists a unit vector that is an eigenvector of each of the three eigenvalues involved.*

The following result gives that the distance signless Laplacian energy of a graph with one positive distance eigenvalue is always greater than the corresponding distance energy.

**Theorem 2.8.** *Let $G$ be a graph of order $n\geq 3$ with Wiener index $W(G)$. Let $n_+$ the number of positive distance eigenvalues of $G$. If $\sum_{i=1}^{n_+}\left(\upsilon_i^Q(G)-\frac{4W(G)}{n}\right)\geq 0$, then $DSLE(G)\geq DE(G)$. In particular, for graphs with one positive distance eigenvalue, we always have $DSLE(G)\geq DE(G)$.*

**Proof.** Let $\upsilon_1^D\geq \upsilon_2^D\geq\cdots\geq \upsilon_n^D$ be the distance eigenvalues of $G$ and let $n_+$ be the number of positive distance eigenvalues of $G$. Then, using the fact that $\upsilon_1^D+\upsilon_2^D+\cdots+\upsilon_n^D=0$, it follows that

$$DE(G)=\sum_{i=1}^{n}|\upsilon_i^D|=2\sum_{i=1}^{n_+}\upsilon_i^D.$$

Let $\zeta^+$ be the number of distance signless Laplacian eigenvalues which are greater than $\frac{2W(G)}{n}$, then as shown in [6] the $D^Q$ energy can be expressed as

$$\begin{aligned}DSLE(G)&=\sum_{i=1}^{n}\left|\upsilon_i^Q-\frac{2W(G)}{n}\right|=2\sum_{i=1}^{\zeta^+}\left(\upsilon_i^Q-\frac{2W(G)}{n}\right)\\&=2\max_{1\leq k\leq n}\left\{\sum_{i=1}^{k}\upsilon_i^Q-\frac{2kW(G)}{n}\right\}.\end{aligned} \tag{2.13}$$

Since $D^Q(G)=2D(G)+D^L(G)$, it follows from the second part of Lemma 2.7 by taking $Z=D^Q(G), L=2D(G), M=D^L(G), k=i$ and $j=i$ that $\curlywedge_i(D^Q(G))\geq \curlywedge_i(2D(G))+\curlywedge_n(D^L(G))$. That is,

$$\upsilon_i^Q\geq 2\upsilon_i^D,\quad \text{for}\quad \text{all } i=1,2,\dots,n, \tag{2.14}$$

as the smallest eigenvalue of $D^L(G)$ is zero. Since $n_+\in\{1,2,\dots,n-1\}$, it follows from equation (2.13) and inequality (2.14) that

$$DSLE(G)=2\max_{1\leq k\leq n}\left\{\sum_{i=1}^{k}\upsilon_i^Q-\frac{2kW(G)}{n}\right\}$$

$$\geq 2\Big(\sum_{i=1}^{n_+}\upsilon_i^Q - \frac{2n_+W(G)}{n}\Big) = \sum_{i=1}^{n_+}\upsilon_i^Q + \sum_{i=1}^{n_+}\upsilon_i^Q - \frac{4n_+W(G)}{n}$$
$$\geq 2\sum_{i=1}^{n_+}\upsilon_i^D + \sum_{i=1}^{n_+}\left(\upsilon_i^Q - \frac{4W(G)}{n}\right) = DE(G) + \sum_{i=1}^{n_+}\left(\upsilon_i^Q - \frac{4W(G)}{n}\right).$$

It is now clear that if $\sum_{i=1}^{n_+}\left(\upsilon_i^Q - \frac{4W(G)}{n}\right) \geq 0$, then $DSLE(G) \geq DE(G)$. If $G$ has one positive $D$ eigenvalue then $n_+ = 1$ and so $\sum_{i=1}^{n_+}\left(\upsilon_i^Q - \frac{4W(G)}{n}\right) = \upsilon_i^Q - \frac{4W(G)}{n}$. Since, it is well known that $\upsilon_1^Q \geq \frac{4W(G)}{n}$, for any connected graph $G$, it follows that $\upsilon_i^Q - \frac{4W(G)}{n} \geq 0$ always holds in this case. The proof is done. □

The wheel graph $W_n$ with $n$ vertices is a graph that has a cycle of length $n-1$ and a vertex $w_0$ not in the cycle so that $w_0$ is adjacent to other vertices. A block of a graph $G$ is a maximal subgraph (connected) of $G$ with no cut-vertex. A graph $G$ is called a clique tree if all the blocks of $G$ are cliques. A cactus is a graph whose all blocks on three or more vertices forms a cycle. Alternatively, a connected graph where two cycles have at most one vertex in common is known as cactus. A complete subgraph of a graph is said to be clique of the graph.

We recall that there exists many classes of graphs with one positive $D$ eigenvalue, mostly notable graphs ((see [3], [20] and the references therein)) are as:

(1). Trees with one positive $D$ eigenvalue.
(2). Graphs $K_n$ and the complement of $K_n \setminus \{e\}$ have one positive $D$ eigenvalue.
(3). The odd order cycle has one positive $D$ eigenvalue.
(4). For a unicyclic graph $G$ on $2t+1+p$ vertices with cycle length $2t+1$ and for a unicyclic graph on $2t+p$ vertices with cycle length $2t$, the number of positive $D$ eigenvalue is 1.
(5). The number of positive $D$ eigenvalues of cactus, the linear hexagonal chains $L_n$ with $n$ hexagons, $H_n$ the $n$-th benzenoid graph from the so-called coronene/circumcoronene series, the polyacenes, honeycomb and square lattices $P_n \times P_n$, the wheel graph $W_n$ and the graph $W_n - e$, where $e$ is the edge not incident on the dominant vertex in $W_n$, is one.
(6). The number of positive $D$ eigenvalues of $W_n - E_k$, where $E_k, 1 \leq k \leq n-1$ is any subset of $E(W_n - w_0)$, $w_0$ is the dominant vertex in $W_n$, is one. The number of positive $D$ eigenvalues of the Cartesian product of $W_n - E_{k_1}$ and $W_t - E_{k_2}$, where $1 \leq k_1 \leq n-1$ and $1 \leq k_2 \leq t-1$, is one.
(7). The number of positive $D$ eigenvalues of the Cartesian product of trees and the Cartesian product of a tree with a unicyclic graph is one.
(8). The number of positive $D$ eigenvalues of the clique trees is one.
(9). The number of positive $D$ eigenvalues of the generalized barbell graphs is one.
(10). A distance regular graph $G$ has one positive $D$ eigenvalue iff $G$ is isomorphic to the following graphs: the Gosset graph, a cocktail party graph, the Schläfli graph, a Johnson graph, a halved cube, one of the three Chang graphs, a Doob graph, a Hamming graph, the icosahedron, a double odd graph, a polygon, the dodecahedron and the Petersen graph.

Let $\Gamma_n$ be the family of connected graphs of order $n$ which consists of all the graphs belonging to (1)-(10), then we have the following result from Theorem 2.8.

**Corollary 2.9.** *For a graph $G \in \Gamma_n$, we always have $DSLE(G) \geq DE(G)$.*

Using the fact that $\upsilon_1^Q = \frac{4W(G)}{n}$, iff $G$ is a transmission regular graph together with Lemma 2.7, the following observation is also immediate from the proof of Theorem 2.8.

**Corollary 2.10.** *For a non-transmission regular graphs with one positive distance eigenvalue, we always have $DSLE(G) \neq DE(G)$.*

The next result gives the relation between $D^L$ energy and the $D$ energy for graphs with 1 positive $D$ eigenvalue.

**Theorem 2.11.** *Let G be a graph of order $n \geq 3$ with distance spectral radius $\upsilon_1^D$ and $D^L$ energy $DLE(G)$. Then the following holds.*

(1) *If G has 1 positive D eigenvalue, then $DLE(G) \geq DE(G)$, provided that $\upsilon_1^D(G) \leq \frac{4W(G)}{n} - \frac{n}{n-1}Tr_{\min}$.*
(2) *If G has 1 positive D eigenvalue, then $DLE(G) \geq DE(G)$, provided that $\upsilon_1^D(G) \leq \frac{4W(G)}{n} - \min_{w_iw_j \in E(G)}\{Tr_i + Tr_j + 2\}$.*

**Proof.** Since for a graph with 1 positive $D$ eigenvalue, the distance energy is simply twice the $D$ spectral radius, the first part now follows from the inequality (2.6). Similarly, the second part follows from the inequality (2.3). □

For the path graph $P_n$, it is well known that $W(P_n)=\binom{n+1}{3}=\frac{n(n^2-1)}{6}$, the minimum transmission is $Tr_{\min}=\frac{n^2-1}{4}$, if $n$ is odd and $Tr_{\min}=\frac{n(n-2)}{4}+1$, if $n$ is even. Let us suppose that $n$ is odd, then $\frac{4W(P_n)}{n}-\frac{n}{n-1}Tr_{\min}=\frac{4n(n^2-1)}{6n}-\frac{n(n^2-1)}{4(n-1)}=\frac{5n^2}{12}-\frac{n}{4}-\frac{2}{3}\approx 0.4166n^2-0.25n-0.67$. Note the fact that $\upsilon_1^D(P_n)=\frac{n^2}{2a^2}-\frac{(2+a^2)}{6a^2}+0\left(\frac{1}{n^2}\right)$, where $a=1.199679$, see [24]. Using the value of $a$ together with the fact $0\left(\frac{1}{n^2}\right)<1$, it follows that $\upsilon_1^D(P_n)\leq 0.34862n^2+0.6$. Now, $\upsilon_1^D(P_n)\leq\frac{4W(P_n)}{n}-\frac{n}{n-1}Tr_{\min}$ gives that $0.06804n^2-0.25n-1.2666\geq 0$. It is easy to verify that this last inequality holds for all $n\geq 7$. This shows that for $n\geq 7$ with $n$ is odd, the inequality $\upsilon_1^D(P_n)\leq\frac{4W(P_n)}{n}-\frac{n}{n-1}Tr_{\min}$ always holds. Similarly, we can show that for $n$ is even the inequality $\upsilon_1^D(P_n)\leq\frac{4W(P_n)}{n}-\frac{n}{n-1}Tr_{\min}$ holds. Thus, it follows that for $n\geq 7$, the inequality $\upsilon_1^D(G)\leq\frac{4W(G)}{n}-\frac{n}{n-1}Tr_{\min}$ holds for path graph. Since path graph $P_n$ has the maximal $D$ spectral radius among all connected graphs of order $n$, it seems that the inequality $\upsilon_1^D(G)\leq\frac{4W(G)}{n}-\frac{n}{n-1}Tr_{\min}$ always holds.

Let $S(G)$ denote the sum of the squares of the distances between all unordered pairs of vertices in the graph $G$, that is, $S(G)=\sum_{1\leq i<j\leq n} d_{ij}^2$. The following upper bound for the $D$ spectral radius $\upsilon_1^D$, in terms of order $n$ and the parameter $S(G)$ was obtained in [28].

$$\upsilon_1^D(G)\leq\sqrt{\frac{2(n-1)S(G)}{n}},\tag{2.15}$$

with equality iff $G\cong K_n$.

The following theorem gives another necessary condition for the inequality $DLE(G)\geq DE(G)$, when $G$ is a graph with 1 positive $D$ eigenvalue.

**Theorem 2.12.** *Let $G$ be a graph of order $n\geq 3$ having Wiener index $W(G)$ and $S(G)$, the sum of the squares of the distances between all unordered pairs of vertices. Then $DLE(G)\geq DE(G)$, provided that $2(n-1)S(G)\leq n\left(\frac{4W(G)}{n}-\frac{n}{n-1}Tr_{\min}\right)^2$ or $2(n-1)S(G)\leq n\left(\frac{4W(G)}{n}-\min_{w_iw_j\in E(G)}\{Tr_i+Tr_j+2\}\right)^2$, holds.*

**Proof.** The proof follows directly by Theorem 2.11 and Inequality (2.15). □

Consider the Wheel graph $W_n$ with the vertex $w_1$ adjacent to all other vertices $w_2,w_3,\dots,w_n$ on $C_{n-1}$. For this graph we have $Tr_{\min}=Tr_1=n-1, Tr_i=2n-5$, for $i\geq 2$, $2W(W_n)=2(n-1)(n-2)$ and $2S(W_n)=4(n-1)(n-3)$. With this information it is now easy to verify that the inequality $2(n-1)S(W_n)\leq n\left(\frac{4W(W_n)}{n}-\frac{n}{n-1}Tr_{\min}\right)^2$ holds for all $n\geq 6$. Since the Wheel graph $W_n$ has just one positive eigenvalue, it follows from Theorem 2.12 that $DLE(W_n)\geq DE(W_n)$.

We note that for graphs with 1 positive $D$ eigenvalue, unlike distance signless Laplacian energy which is always greater than or equal to corresponding distance energy, the distance Laplacian energy can be less than the corresponding distance energy. This fact is clear from the graphs $G_{10},G_{14},G_{21},G_{67},G_{77},G_{89},G_{91},G_{102}$ and $G_{109}$. For these graphs it is shown in the Table 1 that $DLE(G)<DE(G)$.

The following Perron-Frobenius theorem can be found in [15].

**Lemma 2.13.** *Let $B=(s_{ij})$ be a complex matrix, and $X$ be an irreducible non-negative matrix with same order as $B$. Let $|B|$ denote the matrix whose $(i,j)$-entry is $|s_{ij}|$. If $|B|\leq X$ and $B$ has $t$ as an eigenvalue, then $|t|\leq\curlywedge_1(X)$, where $\curlywedge_1(X)$ is the largest eigenvalue of $X$. If the equality occurs, then $|B|=X$, and there is a diagonal matrix $E$ with diagonal entries of absolute value 1 and a constant $c$ of absolute value 1, such that $B=cEXE^{-1}$.*

The following result gives some necessary conditions for the inequalities $DSLE(G)>DLE(G)$ and $DLE(G)\geq DSLE(G)$ to hold.

**Theorem 2.14.** *Let $G$ be a graph with $n\geq 3$ vertices. Let $\zeta$ and $\zeta^+$ be respectively the number of distance Laplacian eigenvalues and the number of $D^Q$ eigenvalues of $G$ which are greater than or equal to $\frac{2W(G)}{n}$.*

(i) *If $\zeta=1$, then $DSLE(G)>DLE(G)$.*
(ii) *If $\zeta^+=n-1$, then $DLE(G)>DSLE(G)$.*
(iii) *If $\zeta^+=n-2$ and $\upsilon_{n-1}^Q+\upsilon_n^Q\geq\upsilon_{n-1}^L$, then $DLE(G)\geq DSLE(G)$.*

**Proof.** Since $|D^L(G)|=D^Q(G)$ and the matrix $D^Q(G)$ is irreducible, it follows from Lemma 2.13 that $\upsilon_1^L(G)=|\upsilon_1^L(G)|\leq\upsilon_1^Q(G)$, as $D^L(G)$ is a positive semi-definite matrix. Since 0 is the smallest eigenvalue of the matrix $D^L(G)$ and the smallest eigenvalue of the matrix $D^Q(G)$ is positive, it follows that it is not possible to find a diagonal matrix $E$ with diagonal entries of unit modulus and a constant $c$ with $|c|=1$, such that $D^L(G)=cED^Q(G)E^{-1}$. This gives that equality can not occur in $\upsilon_1^L(G)\leq\upsilon_1^Q(G)$. Therefore, we have $\upsilon_1^L(G)<\upsilon_1^Q(G)$. If $\zeta=1$, then by the definitions of distance Laplacian energy and the distance signless Laplacian energy, we have

**Table 1**
$DE(G)$, $DLE(G)$, $DSLE(G)$, $\zeta$, $n_+$, and $\zeta^+$ of graphs up to order 6.

| Name | $DE(G)$ | $DLE(G)$ | $DSLE(G)$ | $\zeta$ | $n_+$ | $\zeta^+$ | Name | $DE(G)$ | $DLE(G)$ | $DSLE(G)$ | $\zeta$ | $n_+$ | $\zeta^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Graphs of order 2 | | | | | | | | | | | | | |
| $G_1$ | 2 | 2 | 2 | 1 | 1 | 1 | | | | | | | |
| Graphs of order 3 | | | | | | | | | | | | | |
| $G_1$ | 5.4641 | 5.3333 | 5.7897 | 2 | 1 | 1 | $G_2$ | 4 | 4 | 4 | 2 | 1 | 1 |
| Graphs of order 4 | | | | | | | | | | | | | |
| $G_1$ | 9.2915 | 10 | 9.9282 | 2 | 1 | 1 | $G_2$ | 7.1231 | 7 | 7.4721 | 3 | 1 | 1 |
| $G_3$ | 10.3246 | 10.4721 | 11.2111 | 2 | 1 | 1 | $G_4$ | 8.1993 | 8 | 8.7445 | 3 | 1 | 1 |
| $G_5$ | 8 | 8 | 8 | 3 | 1 | 2 | $G_6$ | 6 | 6 | 6 | 3 | 1 | 1 |
| Graphs of order 5 | | | | | | | | | | | | | |
| $G_1$ | 10.6789 | 10.8 | 10.9765 | 3 | 2 | 2 | $G_2$ | 13.0762 | 14.1848 | 13.8883 | 3 | 1 | 1 |
| $G_3$ | 13.275 | 14.0212 | 14.6053 | 2 | 1 | 1 | $G_4$ | 11.4031 | 12.4 | 11.8623 | 3 | 1 | 1 |
| $G_5$ | 12 | 12.4 | 11.6 | 3 | 2 | 2 | $G_6$ | 10.7446 | 11.2 | 11.6 | 2 | 1 | 1 |
| $G_7$ | 12.3529 | 14 | 13.1652 | 3 | 1 | 1 | $G_8$ | 12 | 12 | 12 | 4 | 1 | 1 |
| $G_9$ | 12.4322 | 12.5848 | 13.7393 | 3 | 1 | 1 | $G_{10}$ | 8.8989 | 8.8 | 9.2 | 4 | 1 | 1 |
| $G_{11}$ | 14.9186 | 16.8 | 16.2239 | 3 | 1 | 1 | $G_{12}$ | 10.5852 | 10.8 | 11.0833 | 3 | 1 | 1 |
| $G_{13}$ | 11.2703 | 11.6361 | 11.5054 | 3 | 1 | 2 | $G_{14}$ | 9.8036 | 9.6 | 10.4 | 4 | 1 | 1 |
| $G_{15}$ | 12.4322 | 12.5848 | 13.7393 | 3 | 1 | 1 | $G_{16}$ | 10.6882 | 10.8 | 11.6 | 3 | 1 | 1 |
| $G_{17}$ | 16.5764 | 18 | 18.2304 | 3 | 1 | 1 | $G_{18}$ | 8 | 8 | 8 | 4 | 1 | 1 |
| $G_{19}$ | 13.2111 | 15.6 | 14.0489 | 3 | 1 | 1 | $G_{20}$ | 14.0172 | 15.2 | 15.2404 | 3 | 1 | 1 |
| $G_{21}$ | 9.6568 | 9.6 | 10.2031 | 4 | 1 | 2 | | | | | | | |
| Graphs of order 6 | | | | | | | | | | | | | |
| $G_1$ | 16.4594 | 20 | 17.4295 | 4 | 1 | 1 | $G_2$ | 17.9818 | 21.4931 | 19.6589 | 3 | 1 | 1 |
| $G_3$ | 19.3137 | 21.3333 | 21.3333 | 4 | 1 | 1 | $G_4$ | 16.5764 | 19.7887 | 18.1269 | 3 | 1 | 1 |
| $G_5$ | 17.1051 | 18.6418 | 17.8815 | 4 | 1 | 1 | $G_6$ | 15.1346 | 18 | 16.2528 | 3 | 1 | 1 |
| $G_7$ | 20.1096 | 23.4214 | 22.8 | 3 | 1 | 2 | $G_8$ | 15.7053 | 18.6667 | 16.5191 | 4 | 1 | 1 |
| $G_9$ | 19.2177 | 22.5498 | 20.5791 | 3 | 1 | 1 | $G_{10}$ | 20.1096 | 23.4214 | 22.8 | 3 | 1 | 2 |
| $G_{11}$ | 17.1872 | 19.4931 | 18.6683 | 3 | 1 | 1 | $G_{12}$ | 17.9389 | 20.5778 | 19.7589 | 4 | 1 | 1 |
| $G_{13}$ | 14.9727 | 17.3333 | 14.9727 | 4 | 1 | 1 | $G_{14}$ | 17.2755 | 20.1284 | 18.8713 | 3 | 1 | 1 |
| $G_{15}$ | 15.1091 | 17.3333 | 16.2528 | 4 | 1 | 1 | $G_{16}$ | 17.3265 | 20.3871 | 19.156 | 3 | 1 | 1 |
| $G_{17}$ | 16.3643 | 18.3179 | 17.3908 | 4 | 1 | 2 | $G_{18}$ | 15.0444 | 17.3333 | 15.9748 | 4 | 1 | 1 |
| $G_{19}$ | 15.9553 | 17.7887 | 17.8022 | 3 | 1 | 1 | $G_{20}$ | 16.0156 | 17.3084 | 17.1282 | 4 | 2 | 2 |
| $G_{21}$ | 14.4113 | 16 | 15.5241 | 4 | 1 | 1 | $G_{22}$ | 15.5768 | 16.1308 | 15.5501 | 4 | 2 | 1 |
| $G_{23}$ | 14.8835 | 16.2925 | 15.6061 | 3 | 1 | 2 | $G_{24}$ | 15.8302 | 18.1097 | 17.8878 | 3 | 1 | 1 |
| $G_{25}$ | 15.9352 | 18 | 17.2584 | 3 | 1 | 1 | $G_{26}$ | 15.5399 | 16.4564 | 15.4029 | 4 | 2 | 2 |
| $G_{27}$ | 14.3484 | 16.3402 | 15.2097 | 3 | 1 | 1 | $G_{28}$ | 15.0796 | 16 | 14.7278 | 4 | 2 | 1 |
| $G_{29}$ | 13.7717 | 16 | 14.8623 | 3 | 1 | 1 | $G_{30}$ | 17.0475 | 18.619 | 18.2843 | 4 | 1 | 1 |
| $G_{31}$ | 16.1481 | 16.8284 | 16.8814 | 4 | 2 | 1 | $G_{32}$ | 18.8489 | 22 | 21.8885 | 3 | 1 | 2 |
| $G_{33}$ | 16.5445 | 18.5941 | 18.2412 | 3 | 1 | 1 | $G_{34}$ | 15.6087 | 16.9848 | 17.0607 | 4 | 1 | 2 |
| $G_{35}$ | 14.4949 | 15.1738 | 14.9039 | 4 | 2 | 2 | $G_{36}$ | 13.6577 | 14.6667 | 14.5443 | 4 | 1 | 1 |
| $G_{37}$ | 17.8539 | 19.3333 | 18.5803 | 4 | 2 | 1 | $G_{38}$ | 16.0741 | 17.5274 | 17.1195 | 3 | 2 | 2 |
| $G_{39}$ | 14.3295 | 16 | 15.247 | 4 | 1 | 2 | $G_{40}$ | 15.2469 | 16.8484 | 17.0222 | 3 | 1 | 1 |
| $G_{41}$ | 15.0909 | 16.6418 | 16.4739 | 3 | 1 | 1 | $G_{42}$ | 13.7201 | 14.6667 | 14.9011 | 4 | 1 | 1 |
| $G_{43}$ | 16.7288 | 18 | 17.8362 | 4 | 2 | 1 | $G_{44}$ | 15.4416 | 16.7446 | 16.8749 | 3 | 2 | 2 |
| $G_{45}$ | 15.0338 | 15.3503 | 15.1997 | 4 | 2 | 2 | $G_{46}$ | 13.7813 | 14.9624 | 14.5525 | 3 | 2 | 2 |
| $G_{47}$ | 13.0218 | 14 | 13.9959 | 3 | 1 | 1 | $G_{48}$ | 14.2142 | 14.6667 | 14.1268 | 4 | 2 | 2 |
| $G_{49}$ | 14.7698 | 14.6667 | 15.0865 | 4 | 2 | 2 | $G_{50}$ | 19.3928 | 22.0565 | 22.3741 | 3 | 1 | 2 |
| $G_{51}$ | 16.434 | 18.1284 | 17.6459 | 4 | 1 | 2 | $G_{52}$ | 14.2492 | 16 | 14.8489 | 4 | 1 | 2 |
| $G_{53}$ | 14.1493 | 14.8284 | 15.4049 | 4 | 1 | 2 | $G_{54}$ | 15.544 | 16.6667 | 16.2107 | 4 | 1 | 1 |
| $G_{55}$ | 15.9352 | 18 | 17.8878 | 3 | 1 | 1 | $G_{56}$ | 14.4078 | 15.1231 | 14.3295 | 4 | 2 | 2 |
| $G_{57}$ | 13.566 | 14.6667 | 14.2993 | 4 | 1 | 2 | $G_{58}$ | 14 | 14.6667 | 13.5949 | 4 | 2 | 1 |
| $G_{59}$ | 14.535 | 14.6418 | 16.3326 | 4 | 1 | 1 | $G_{60}$ | 14.4853 | 14.7446 | 16 | 4 | 1 | 2 |
| $G_{61}$ | 13.8383 | 14.0169 | 14.7912 | 4 | 2 | 2 | $G_{62}$ | 12.9047 | 13.3333 | 13.7722 | 4 | 1 | 2 |
| $G_{63}$ | 12.2849 | 12 | 13.2111 | 5 | 1 | 1 | $G_{64}$ | 14.7107 | 15.7974 | 15.0092 | 4 | 1 | 2 |
| $G_{65}$ | 15.1027 | 15.3333 | 15.0805 | 4 | 2 | 2 | $G_{66}$ | 13.4727 | 13.9027 | 13.9099 | 4 | 1 | 1 |
| $G_{67}$ | 11.5132 | 11.3333 | 12.0698 | 5 | 1 | 1 | $G_{68}$ | 13.5462 | 13.3333 | 14.2107 | 4 | 2 | 2 |
| $G_{69}$ | 13.2771 | 13.3333 | 14.0036 | 4 | 2 | 2 | $G_{70}$ | 18.5212 | 20.9282 | 19.9485 | 4 | 1 | 2 |
| $G_{71}$ | 17.1086 | 19.153 | 18.6485 | 3 | 1 | 2 | $G_{72}$ | 18.3923 | 20.5498 | 19.3693 | 4 | 1 | 3 |
| $G_{73}$ | 12.4311 | 12 | 13.1877 | 5 | 2 | 2 | $G_{74}$ | 14.8004 | 15.9398 | 15.1664 | 4 | 1 | 1 |
| $G_{75}$ | 10 | 10 | 10 | 5 | 1 | 1 | $G_{76}$ | 16 | 17.3333 | 15.5795 | 4 | 2 | 1 |
| $G_{77}$ | 12.3264 | 12 | 13.2111 | 5 | 1 | 1 | $G_{78}$ | 14.5498 | 18 | 15.798 | 3 | 1 | 1 |
| $G_{79}$ | 13.6112 | 14 | 14.0036 | 3 | 2 | 2 | $G_{80}$ | 19.3404 | 23.2445 | 20.9983 | 4 | 1 | 1 |
| $G_{81}$ | 18 | 18 | 18 | 5 | 1 | 3 | $G_{82}$ | 16.8489 | 18.4564 | 18.6667 | 4 | 1 | 2 |

**Table 1** (*continued*)

| Name | $DE(G)$ | $DLE(G)$ | $DSLE(G)$ | $\zeta$ | $n_+$ | $\zeta^+$ | Name | $DE(G)$ | $DLE(G)$ | $DSLE(G)$ | $\zeta$ | $n_+$ | $\zeta^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G_{83}$ | 21.4848 | 25.1224 | 23.6662 | 4 | 1 | 1 | $G_{84}$ | 14.2574 | 16 | 14.8847 | 4 | 1 | 1 |
| $G_{85}$ | 13.1308 | 13.3333 | 13.1916 | 4 | 2 | 2 | $G_{86}$ | 16.9939 | 19.4617 | 18.4252 | 4 | 1 | 2 |
| $G_{87}$ | 16.6671 | 17.4112 | 16.8773 | 4 | 2 | 2 | $G_{88}$ | 12.8228 | 13.3333 | 13.8079 | 4 | 1 | 2 |
| $G_{89}$ | 12.2023 | 12 | 12.7755 | 5 | 1 | 2 | $G_{90}$ | 20 | 24.5498 | 21.5591 | 3 | 1 | 1 |
| $G_{91}$ | 12.2023 | 12 | 12.7755 | 5 | 1 | 2 | $G_{92}$ | 17.2558 | 19.2445 | 19.3880 | 4 | 1 | 2 |
| $G_{93}$ | 13.0484 | 13.3333 | 14.3956 | 4 | 1 | 1 | $G_{94}$ | 15.7776 | 18.6667 | 16.7984 | 4 | 1 | 1 |
| $G_{95}$ | 18.7703 | 22.6491 | 20.8064 | 3 | 1 | 1 | $G_{96}$ | 12 | 12 | 12 | 5 | 1 | 3 |
| $G_{97}$ | 17.6437 | 18.9912 | 18.6161 | 4 | 1 | 1 | $G_{98}$ | 24.2186 | 28.4922 | 27.8521 | 3 | 1 | 2 |
| $G_{99}$ | 13.8845 | 14.1308 | 14.2173 | 4 | 2 | 2 | $G_{100}$ | 15.0331 | 16.4204 | 16.1337 | 3 | 1 | 1 |
| $G_{101}$ | 14 | 14 | 14 | 5 | 1 | 3 | $G_{102}$ | 11.4031 | 11.3333 | 12.2615 | 5 | 1 | 2 |
| $G_{103}$ | 17.7980 | 20.7950 | 18.9572 | 4 | 1 | 1 | $G_{104}$ | 17.1652 | 21.3333 | 18.1396 | 4 | 1 | 1 |
| $G_{105}$ | 21.3359 | 24.5223 | 23.8378 | 3 | 1 | 2 | $G_{106}$ | 19.9425 | 22.7232 | 22.0700 | 4 | 1 | 2 |
| $G_{107}$ | 22.1176 | 26.5223 | 24.3810 | 3 | 1 | 2 | $G_{108}$ | 14.2462 | 15.2111 | 14.9443 | 3 | 1 | 1 |
| $G_{109}$ | 10.7446 | 10.6667 | 10.9902 | 5 | 1 | 1 | $G_{110}$ | 14.0744 | 14.6056 | 14.7226 | 4 | 1 | 2 |
| $G_{111}$ | 16 | 16 | 16 | 4 | 2 | 2 | $G_{112}$ | 13.4833 | 14.6667 | 13.8381 | 4 | 1 | 1 |

$$\begin{aligned}DLE(G) =& 2\sum_{i=1}^{\zeta}\Big(\upsilon_i^L-\frac{2W(G)}{n}\Big)=2\Big(\upsilon_1^L-\frac{2W(G)}{n}\Big)\\ &<2\Big(\upsilon_1^Q-\frac{2W(G)}{n}\Big)\le 2\max_{1\le k\le n}\Big\{\sum_{i=1}^{k}\upsilon_i^Q-\frac{2kW(G)}{n}\Big\}=DSLE(G).\end{aligned}$$

Since $\upsilon_1^L+\upsilon_2^L+\cdots+\upsilon_{n-1}^L=\upsilon_1^Q+\upsilon_2^Q+\cdots+\upsilon_{n-1}^Q+\upsilon_n^Q=2W(G)$, it follows that if $\zeta^+=n-1$, then again by the definitions of distance Laplacian energy and the distance signless Laplacian energy, we have

$$\begin{aligned}DSLE(G) =& 2\sum_{i=1}^{\zeta^+}\Big(\upsilon_i^Q-\frac{2W(G)}{n}\Big)=2\sum_{i=1}^{n-1}\Big(\upsilon_i^Q-\frac{2W(G)}{n}\Big)\\ &<2\Big(2W(G)-\frac{2(n-1)W(G)}{n}\Big)=2\sum_{i=1}^{n-1}\Big(\upsilon_i^L-\frac{2W(G)}{n}\Big)\\ &\le 2\max_{1\le k\le n}\Big\{\sum_{i=1}^{k}\upsilon_i^L-\frac{2kW(G)}{n}\Big\}=DLE(G).\end{aligned}$$

Again since $\upsilon_1^L+\upsilon_2^L+\cdots+\upsilon_{n-2}^L=2W(G)-\upsilon_{n-1}^L$, $\upsilon_1^Q+\upsilon_2^Q+\cdots+\upsilon_{n-2}^Q=2W(G)-\upsilon_{n-1}^Q-\upsilon_n^Q$ and $\upsilon_{n-1}^Q+\upsilon_n^Q\ge\upsilon_{n-1}^L$, it follows that $2W(G)-\upsilon_{n-1}^Q-\upsilon_n^Q\le 2W(G)-\upsilon_{n-1}^L$. Therefore, if $\zeta^+=n-2$, then by the definitions of $D^L$ energy and the $D^Q$ energy, we have

$$\begin{aligned}DSLE(G) =& 2\sum_{i=1}^{\zeta^+}\Big(\upsilon_i^Q-\frac{2W(G)}{n}\Big)=2\sum_{i=1}^{n-2}\Big(\upsilon_i^Q-\frac{2W(G)}{n}\Big)\\ =& 2\Big(2W(G)-\upsilon_{n-1}^Q-\upsilon_n^Q-\frac{2(n-2)W(G)}{n}\Big)\\ \le& 2\Big(2W(G)-\upsilon_{n-1}^L-\frac{2(n-2)W(G)}{n}\Big)=2\sum_{i=1}^{n-2}\Big(\upsilon_i^L-\frac{2W(G)}{n}\Big)\\ \le& 2\max_{1\le k\le n}\Big\{\sum_{i=1}^{k}\upsilon_i^L-\frac{2kW(G)}{n}\Big\}=DLE(G).\quad\square\end{aligned}$$

## 3. Some computational results

The below table calculates the values of $DE(G)$, $DLE(G)$, $DSLE(G)$, $n_+$, $\zeta$, and $\zeta^+$ of connected graphs of order up to 6. Connected undirected graphs up to order 6 can be generated by `Wolfram Mathematica` [18] using the following command:

```
ConnectedUG[n_] := Module[graphs, graphs=GraphData /@GraphData[n];
Cases[graphs,x_/; Length@WeaklyConnectedComponents[x]==1]];
```

Next, enter the following command

```
Ci=GraphData["Connected",i];
Row[Column[GraphData[#,"StandardName","Image"]]&/@Ci]
```

For, $i=2,3,4,5,6$, Mathematica displays generating connected graphs of orders $2,3,4,5$ and 6, respectively.

We list these graphs as Mathematica orders them, see also [5]. The spectral parameters $DE(G)$, $DLE(G)$, $DSLE(G)$, $n_+$, $\zeta$, and $\zeta^+$ have been calculated using the AutoGraphiX III [4] (available at https://www.gerad.ca/Gilles.Caporossi/agx/AGX/AutoGraphiX.html).

## CRediT authorship contribution statement

**Hilal A. Ganie:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Bilal Ahmad Rather:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Yilun Shang:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: The authors declare that Y. Shang is a Section editor for Heliyon. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

## References


[1] A. Alhevaz, M. Baghipur, K.C. Das, Y. Shang, Sharp bounds on (generalized) distance energy of graphs, Mathematics 8 (3) (2020) 426.
[2] A. Alhevaz, M. Baghipur, E. Hashemi, Y. Shang, Some upper and lower bounds for $D_\alpha$-energy of graphs, J. Algebra Combin. Discrete Struct. Appl. 10 (2) (2023) 73–86.
[3] M. Aouchiche, P. Hansen, Distance spectra of graphs: a survey, Linear Algebra Appl. 458 (2014) 301–386.
[4] G. Caporossi, Variable neighborhood search for extremal vertices: the AutoGraphiX-III system, Comput. Oper. Res. 78 (2017) 431–438.
[5] D. Cvetković, M. Petrić, A table of connected graphs on six vertices, Discrete Math. 50 (1984) 37–49.
[6] K.C. Das, M. Aouchiche, P. Hansen, On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs, Discrete Appl. Math. 243 (2018) 172–185.
[7] R.C. Diaz, O. Rojo, Sharp upper bounds on the distance energies of a graph, Linear Algebra Appl. 545 (2018) 55–75.
[8] K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, Proc. Natl. Acad. Sci. USA 37 (1951) 760–766.
[9] H.A. Ganie, On the distance Laplacian energy ordering of trees, Appl. Math. Comput. 394 (2021), https://doi.org/10.1016/j.amc.2020.125762.
[10] H.A. Ganie, On the distance Laplacian spectrum (energy) of graphs, Discrete Math. Algorithms Appl. (2020), https://doi.org/10.1142/S1793830920500615.
[11] H.A. Ganie, B.A. Chat, S. Pirzada, On the signless Laplacian energy of a graph and energy of line graph, Linear Algebra Appl. 544 (2018) 306–324.
[12] H.A. Ganie, R.U. Shaban, Bilal A. Rather, S. Pirzada, On distance Laplacian energy, vertex connectivity and independence number of graphs, Czechoslov. Math. J. 73 (2) (2023) 335–353.
[13] H.A. Ganie, Y. Shang, On the spectral radius and energy of signless Laplacian matrix of digraphs, Heliyon 8 (2022) e09186.
[14] I. Gutman, The energy of a graph, Ber. Math.-Stat. Sekt. Forsch-Zent. Graz 103 (1978) 1–22.
[15] R. Horn, C. Johnson, Matrix Analysis, second edition, Cambridge University Press, New York, 2013.
[16] G. Indulal, I. Gutman, A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 461–472.
[17] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.
[18] Wolfram Research, Inc., System Modeler, Version 13.0.0, Champaign, Illinois, https://www.wolfram.com/mathematica, 2021.
[19] V. Nikiforov, The energy of graphs and matrices, J. Math. Appl. 326 (2007) 1472–1475.
[20] H. Lin, R. Liu, X. Lu, The inertia and energy of the distance matrix of a connected graph, Linear Algebra Appl. 467 (2015) 29–39.
[21] V. Nikiforov, Beyond graph energy: norms of graphs and matrices, Linear Algebra Appl. 506 (2016) 82–138.
[22] S. Pirzada, Bilal A. Rather, R. Ul Shaban, Merajuddin, On graphs with minimal distance signless Laplacian energy, Acta Univ. Sapientiae Math. 13 (2) (2021) 450–467.
[23] Bilal A. Rather, Hilal A. Ganie, Y. Shang, Distance Laplacian eigenvalues of sun graphs, Appl. Math. Comput. 445 (2023) 127847, https://doi.org/10.1016/j.amc.2023.127847.
[24] S.N. Ruzieh, D.L. Powers, The distance spectrum of the path $P_n$ and the first distance eigenvector of connected graphs, Linear Multilinear Algebra 28 (1990) 75–81.
[25] W. So, Commutativity and spectra of Hermitian matrices, Linear Algebra Appl. 212 (213) (1994) 121–129.
[26] S. Sun, K.C. Das, Y. Shang, On maximal distance energy, Mathematics 9 (4) (2021) 360.
[27] J. Yang, L. You, I. Gutman, Bounds on the distance Laplacian energy of graphs, Kragujev. J. Math. 37 (2013) 245–255.
[28] B. Zhou, N. Trinajstić, On the largest eigenvalue of the distance matrix of a connected graph, Chem. Phys. Lett. 447 (2007) 384–387.