

Article

# Strong Equality of Perfect Roman and Weak Roman Domination in Trees

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**Abstract:** Let  $G = (V, E)$  be a graph and  $f : V \rightarrow \{0, 1, 2\}$  be a function. Given a vertex  $u$  with  $f(u) = 0$ , if all neighbors of  $u$  have zero weights, then  $u$  is called undefended with respect to  $f$ . Furthermore, if every vertex  $u$  with  $f(u) = 0$  has a neighbor  $v$  with  $f(v) > 0$  and the function  $f' : V \rightarrow \{0, 1, 2\}$  with  $f'(u) = 1$ ,  $f'(v) = f(v) - 1$ ,  $f'(w) = f(w)$  if  $w \in V \setminus \{u, v\}$  has no undefended vertex, then  $f$  is called a weak Roman dominating function. Also, the function  $f$  is a perfect Roman dominating function if every vertex  $u$  with  $f(u) = 0$  is adjacent to exactly one vertex  $v$  for which  $f(v) = 2$ . Let the weight of  $f$  be  $w(f) = \sum_{v \in V} f(v)$ . The weak (resp., perfect) Roman domination number, denoted by  $\gamma_r(G)$  (resp.,  $\gamma_r^p(G)$ ), is the minimum weight of the weak (resp., perfect) Roman dominating function in  $G$ . In this paper, we characterize those trees where the perfect Roman domination number strongly equals the weak Roman domination number, in the sense that each weak Roman dominating function of minimum weight is, at the same time, perfect Roman dominating.

**Keywords:** Perfect Roman dominating function; Roman dominating number; weak Roman dominating function

**MSC:** 05C69

## 1. Introduction

In this paper, we study finite, undirected, simple graphs. Let  $G$  be a graph characterized by the vertex set  $V = V(G)$  and the edge set  $E = E(G)$ . The *order* of the graph is the number of vertices in the graph and here it is  $n = n(G)$ . Denote by  $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$  the *open neighborhood* of  $v \in V$ . The cardinality of the open neighborhood of a vertex is referred to as the *degree* of that vertex. For a given tree, vertices with degree one are called *leaves* and those adjacent to leaves are called *support vertices*. If a support vertex is adjacent to only one leaf, then it is called a *weak support vertex*. Otherwise, it becomes a *strong support vertex*. Here, the set of all support vertices of a given tree  $T$  is denoted by  $S(T)$ . All leaves of  $T$  is denoted by  $L(T)$ . Set  $\ell(T) = |L(T)|$  and  $s(T) = |S(T)|$ . Denote by  $L(x)$  the set of all leaves that are neighbors of the support vertex  $x$ . Set  $\ell_x = |L(x)|$ . When a tree  $T$  has a root, let  $T_v$  be the sub-rooted tree rooted at  $v$  for any vertex  $v$ . More definitions and notations can be found in e.g., [1].

Inspired by the work entitled “Defend the Roman Empire!” by Ian Stewart [2], recently Cockayne et al. [3] introduced the Roman dominating function in graphs. For a graph function  $f : V(G) \rightarrow \{0, 1, 2\}$ , the vertex set can be partitioned as  $(V_0, V_1, V_2)$ , where  $V_i = \{v \in V(G) : f(v) = i\}$  ( $i = 0, 1, 2$ ). The functions  $f : V(G) \rightarrow \{0, 1, 2\}$  and the ordered partitions  $(V_0, V_1, V_2)$  form a

one-to-one correspondence. For this reason, it is handy to set  $f = (V_0, V_1, V_2)$ . If every vertex  $u$  with  $f(u) = 0$  has at least one neighbor  $v$  with  $f(v) = 2$ , then the function  $f : V(G) \rightarrow \{0, 1, 2\}$  is referred to as a *Roman dominating function* (RDF). The weight of an RDF  $f$  is  $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$ . We denote by  $\gamma_R(G)$  the minimum weight of an RDF of  $G$ . It is called the *Roman domination number* of  $G$ . For some further results on Roman dominating function in graphs, we recommend the reader to consult the papers [4–7]. RDFs are useful in the study of generalized  $k$ -core percolation [8], where  $V_2 = \alpha$ -removable,  $V_0 = \beta$ -removable, and  $V_1 =$  non-removable vertices.

Henning, Klostermeyer and MacGillivray [9] introduced the concept of perfect Roman domination in graphs. An RDF  $f = (V_0, V_1, V_2)$  is referred to as a *perfect Roman dominating function* (PRDF) when each vertex  $u$  with  $f(u) = 0$  has only one neighbor  $v$  such that  $f(v) = 2$  [9]. The minimum weight of an RDF is represented by  $\gamma_R^P(G)$  and is called the *perfect Roman domination number*. In other words,

$$\gamma_R^P(G) = \min\{w(f) \mid f \text{ is a PRDF in } G\}.$$

Note that for any graph  $G$ ,  $\gamma_R^P(G)$  is valid. This is because  $(\emptyset, V(G), \emptyset)$  is a PRDF for  $G$ . Note also that every graph  $G$  of order  $n$  satisfies  $\gamma_R^P(G) \leq n$ , as one can define a perfect Roman dominating function  $f$  by letting  $f(u) = 1$  for any vertex  $u$  of  $G$ . A PRDF of  $G$  with minimum weight is also called a  $\gamma_R^P(G)$ -function (see Figure 1). For some recent results on perfect Roman domination number of graphs we refer the readers to the papers [10,11].

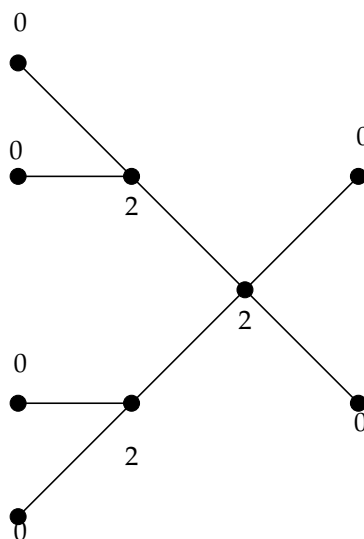


Figure 1.  $\gamma_R^P(T)$ -function of a tree  $T$ .

For graph  $G = (V, E)$  and function  $f : V \rightarrow \{0, 1, 2\}$ , a vertex  $u$  with  $f(u) = 0$  is undefended with respect to  $f$  if all its neighbors have only zero weight. Henning and Hedetinemi [12] defined the weak Roman dominating function as follows. The function  $f$  is a *weak Roman dominating function* (WRDF) if each vertex  $u$  with  $f(u) = 0$  is adjacent to a vertex  $v$  with  $f(v) > 0$  such that the function  $f' : V \rightarrow \{0, 1, 2\}$  defined by  $f'(u) = 1$ ,  $f'(v) = f(v) - 1$  and  $f'(w) = f(w)$  if  $w \in V \setminus \{u, v\}$ , has no undefended vertex. Let the weight of  $f$  be  $w(f) = \sum_{v \in V} f(v)$ . The *weak Roman domination number*, denoted by  $\gamma_r(G)$ , is the minimum weight of a WRDF in  $G$ , that is:

$$\gamma_r(G) = \min\{w(f) \mid f \text{ is a WRDF in } G\}.$$

It was shown by Henning and Hedetinemi [12] that the problem of computing  $\gamma_r(G)$  is NP-hard. This is true even when the graph in consideration is bipartite or chordal. Hence, finding the weak Roman domination number for some special classes of graphs as well as finding some good bounds for

this invariant is of great importance. For some recent results on the weak Roman domination number of graphs we refer the interested reader to the papers [4,13].

Note that  $\gamma_r(G) \leq \gamma_R^P(G)$  for every graph  $G$ . Clearly, if  $G$  is a graph with  $\gamma_r(G) = \gamma_R^P(G)$ , then every  $\gamma_R^P(G)$ -function is a  $\gamma_r(G)$ -function. However, not every  $\gamma_r(G)$ -function is an  $\gamma_R^P(G)$ -function (see Figure 2), even when  $\gamma_r(G) = \gamma_R^P(G)$ . For example, consider the path  $P_3$ . We say that  $\gamma_r(G)$  and  $\gamma_R^P(G)$  are strongly equal, denoted by  $\gamma_r(G) \equiv \gamma_R^P(G)$ , if every  $\gamma_r(G)$ -function is an  $\gamma_R^P(G)$ -function. In this paper, we provide a constructive characterization of trees  $T$  such that  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Other interesting examples of characterizations for strong equalities have been discussed in [4,6,7,14,15].

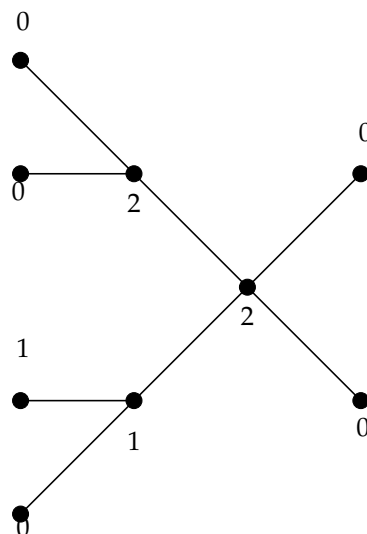


Figure 2.  $\gamma_r(T)$ -function over a tree  $T$ , which is not a perfect Roman dominating function (PRDF).

## 2. Constructive Characterization of Strong Equality

We start with the following essential lemmas.

**Lemma 1.** Let  $T$  be a tree with  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Then  $T$  has no weak support vertex with degree two.

**Proof.** Let  $T$  be a tree with  $\gamma_r(T) \equiv \gamma_R^P(T)$  and  $u$  be a weak support vertex of it with leaf neighbor  $v$ . Let  $f$  be a  $\gamma_r(T)$ -function and  $N(u) - \{v\} = \{w\}$ . If  $f(w) = 2$ , then, clearly  $f(u) + f(v) = 1$  and so the function  $g$  with  $g(u) = 1, g(v) = 0$  and for  $w \in V \setminus \{u, v\}, g(w) = f(w)$  is a  $\gamma_r(T)$ -function, which is not a PRDF. This leads to a contradiction. Thus, we may assume that  $f(w) \neq 2$ . We first assume that  $f(w) = 1$ . Then, it is true that  $f(u) = 1, f(v) = 0$  or  $f(v) = 1, f(u) = 0$ . Hence,  $f$  is not a PRDF, which leads to a contradiction. Now, assume that  $f(w) = 0$ . Then,  $1 \leq f(u) + f(v) \leq 2$ . If  $f(u) + f(v) = 1$ , then  $f(u) = 1, f(v) = 0$  or  $f(v) = 1, f(u) = 0$ . Accordingly,  $f$  is not a PRDF, which is a contradiction. Thus,  $f(u) + f(v) = 2$ . Therefore, function  $f'$  by  $f'(w) = f'(v) = 1$  and  $f'(u) = 0$  and for  $x \notin \{w, v, u\}, f'(x) = f(x)$  is a  $\gamma_r(T)$ -function, which is not a PRDF. This leads to a contradiction. Hence,  $T$  has no weak support vertex with degree two.  $\square$

**Lemma 2.** Fix a tree  $T$ . Assume that  $\gamma_r(T) \equiv \gamma_R^P(T)$  and  $u$  is a strong support vertex with  $\ell_u = 2$ . If  $f$  is a  $\gamma_r(T)$ -function, then  $f(u) = 2$ . Also, if  $\deg(u) = 3$  then any non-leaf neighbor of it has weight 0.

**Proof.** Let  $L(u) = \{x, y\}$ . We first assume that  $f(u) = 1$ . Since  $f$  is a PRDF, we deduce that  $f(x) = f(y) = 1$ . Then by re-assigning to the vertex  $u$  the value 2 and to the vertices  $x, y$  the value 0, we get a new WRDF with weight less than  $\gamma_r(T)$ , which is a contradiction. Next, assume that  $f(u) = 0$ , then  $f(x) = f(y) = 1$ . If for any vertex  $v \in N(u)$  we have  $f(v) \neq 0$ , then by re-assigning to the vertex  $u$  the value 1 and to the vertex  $x$  the value 0, we get a new  $\gamma_r(T)$ -function that is not a PRDF. This is a contradiction. Hence, we can assume that  $w \in N(u)$  and  $f(w) = 0$ . Since  $f$  is a  $\gamma_R^p(T)$ -function, there exists a vertex  $z \neq u$  such that  $z \in N(w)$  and  $f(z) = 2$ . Thus, by re-assigning to the vertex  $u$  the value 2 and to the vertices  $x, y$  the value 0, we obtain a new  $\gamma_r(T)$ -function that is not a PRDF. This is a contradiction. Hence  $f(u) = 2$ . Assume that  $\deg(u) = 3$  and  $N(u) = \{x, y, w\}$ . If  $f(w) \neq 0$ , then by re-assigning to the vertices  $u$  and  $x$  the value 1, the  $\gamma_r(T)$ -function is not a PRDF. This is a contradiction. Therefore,  $f(w) = 0$ .  $\square$

In the case of  $\gamma_R^p(T) \equiv \gamma_r(T)$ , a family of trees can be defined as follows:  
 Let  $F$  be the tree depicted in Figure 3 and  $\mathcal{T}$  be the collection of trees  $T$  that can be obtained from a sequence  $T_1, T_2, \dots, T_k = T (k \geq 1)$  of trees, where  $T_1$  is a star  $K_{1,t}$  with  $t \geq 3$ , and  $T_{i+1}$  can be obtained recursively from  $T' = T_i$  by one of the following six operations  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$  defined below and illustrated in Figure 4.

- **Operation  $\mathcal{O}_1$**  : Add a star  $K_{1,t}$ , ( $t \geq 3$ ) and link its central vertex  $v$  to a vertex  $u$  of  $T'$  such that  $\gamma_r(T' - u) > \gamma_r(T')$ .
- **Operation  $\mathcal{O}_2$**  : Add a new vertex  $v$  to  $T'$  and link it to a strong support vertex  $u$  of  $T'$  with at least three leaf neighbors.
- **Operation  $\mathcal{O}_3$**  : Add a new vertex  $v$  to  $T'$  and link it to a strong support vertex  $u$  of  $T'$  with  $\deg(u) = 3$ .
- **Operation  $\mathcal{O}_4$**  : Add a star  $K_{1,3}$ , and link a leaf  $v$  to a leaf  $u$  of  $T'$  that is adjacent to a strong support vertex of  $T'$ .
- **Operation  $\mathcal{O}_5$**  : Add a star  $K_{1,3}$  and join a leaf to vertex  $u$  of  $T'$  such that every  $\gamma_r(T')$ -function of  $T'$  assigns to the vertex  $u$  the value 0.
- **Operation  $\mathcal{O}_6$**  : Add a tree  $F$  (as depicted in Figure 3) and join its support vertex  $v$  with degree two to a vertex  $u$  of  $T'$  such that  $\deg(u) \geq 2$  and every  $\gamma_r(T')$ -function of  $T'$  assigns to the vertex  $u$  the value 2.

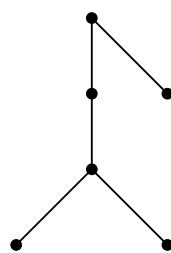


Figure 3. The tree  $F$ .

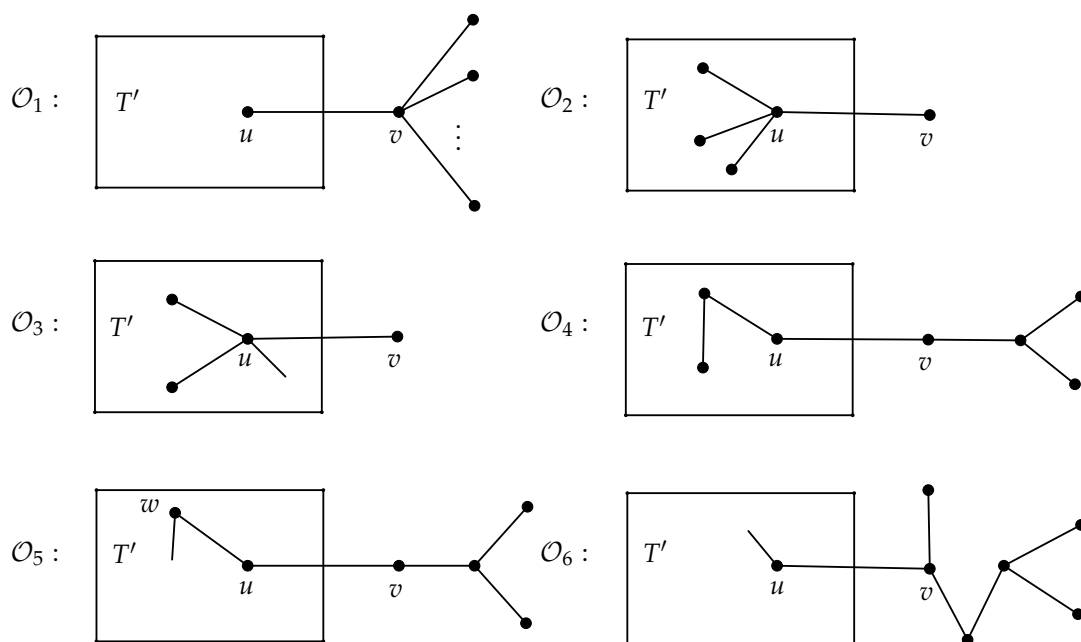


Figure 4. The operations  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$ .

Next, we will show that for every tree in the family  $\mathcal{T}$  (see Figure 5 for an example of a tree in this family), we have  $\gamma_R^p(T) \equiv \gamma_r(T)$ .

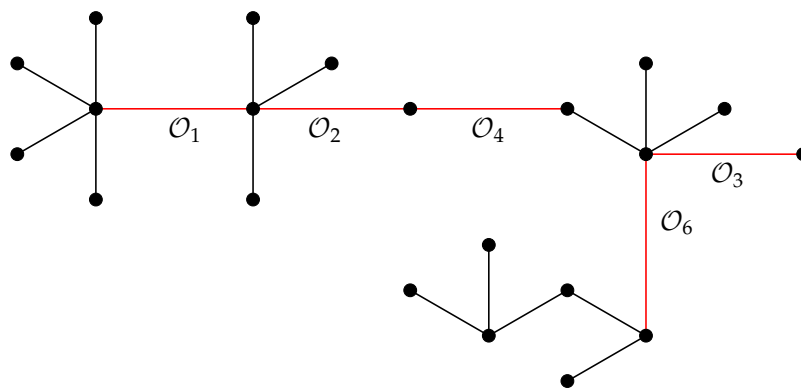


Figure 5. An example of a tree  $T$  in the family  $\mathcal{T}$ , obtained from tree  $K_{1,4}$  by Operations  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_3$  and  $\mathcal{O}_6$ .

**Lemma 3.** Suppose that  $T$  is a tree in the family  $\mathcal{T}$ , then  $\gamma_R^p(T) \equiv \gamma_r(T)$ .

**Proof.** We proceed by induction on the order  $n \geq 4$  of a tree  $T \in \mathcal{T}$ . If  $n = 4$ , then  $T = K_{1,3}$  and clearly  $\gamma_R^p(T) \equiv \gamma_r(T)$ . Suppose that  $n \geq 5$  and that for every tree in  $\mathcal{T}$  of order  $n'$ , where  $5 \leq n' \leq n$ , we have  $\gamma_R^p(T) \equiv \gamma_r(T)$ . Assume that  $T \in \mathcal{T}$  has order  $n$ . Thus,  $T$  can be obtained from a sequence of trees  $T_1, \dots, T_k$ , where  $k \geq 1$  and where  $T_1 = K_{1,3}$  and  $T = T_k$ . For each  $i \leq k - 1$ , the tree  $T_i$  can be constructed by  $T_{i-1}$  via one of the operations  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$ .

Set  $T' = T_{k-1}$ . We know that  $T' \in \mathcal{T}$  has less than  $n$  vertices. Applying the inductive hypothesis to  $T'$ , we know that  $\gamma_R^p(T') \equiv \gamma_r(T')$ . Suppose that  $f$  is a  $\gamma_r$ -function of  $T$ , where the sum of the values assigned to all leaves under  $f$  is the minimum. We restrict the function  $f$  to the tree  $T'$  and denote it by

$f'$ . For any vertex  $w \in V(T')$ , the equality  $f'(w) = f(w)$  holds. Six different situations are considered in the sequel.

**Case 1.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_1$ .

Assume  $T$  is obtained from  $T'$  by adding a star  $K_{1,3}$ , with central vertex  $v$  and leaves  $x, y, z$ , and adding the edge  $uv$  such that  $\gamma_r(T' - u) > \gamma_r(T')$ . Then  $u$  can not get the value 0 under any  $\gamma_r(T')$ -function. Hence every PRDF on  $T'$  can be extended to a PRDF on  $T$  by assigning the weight 2 to  $v$  and the weight 0 to the three neighbors of  $v$ . Hence, by the statement above, and inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{1}$$

Conversely, the vertex  $v$  is a strong support vertex of  $T$  and so we can see that  $f(v) = 2$  and  $f(x) = f(y) = f(z) = 0$ . If  $f(u) = 0$ , then  $f|_{T'-u}$  is a WRDF on  $T' - u$  and so

$$\gamma_r(T') < \gamma_r(T' - u) \leq w(f|_{T'-u}) = w(f) - 2 = \gamma_r(T) - 2. \tag{2}$$

Now assume that  $f(u) \neq 0$ , then  $f|_{T'}$  is a WRDF on tree  $T'$  and so

$$\gamma_r(T') \leq w(f|_{T'}) = w(f) - 2 = \gamma_r(T) - 2. \tag{3}$$

Therefore, we must have equalities throughout the inequality chain (1). In particular,  $\gamma_r(T) = \gamma_R^P(T)$ .

Now, assume that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Then there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $v$  is a strong support vertex, hence  $g(v) = 2$ . If  $g(u) \neq 0$ , then the function  $g|_{T'}$  is a  $\gamma_r(T')$ -function that is not a PRDF. This is a contradiction. Hence,  $g(u) = 0$ . Thus, the function  $g|_{T'-u}$  is a WRDF on tree  $T' - u$  and so  $\gamma_r(T') < \gamma_r(T' - u) = w(g|_{T'-u}) = w(g) - 2 = \gamma_r(T) - 2 = \gamma_r(T')$ . This is a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ .

**Case 2.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_2$ .

Assume that  $f'$  is a  $\gamma_R^P(T')$ -function. We have  $f'(u) = 2$  and therefore  $f'$  is extended to a PRDF on  $T$  by giving the weight 0 to  $v$ . Hence, by inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') = \gamma_r(T'). \tag{4}$$

Conversely, the vertex  $u$  is a strong support vertex of  $T$  and so we can see that  $f(u) = 2$  and  $f(v) = 0$ . Hence,  $f'$  is a WRDF on  $T'$  and so  $\gamma_r(T') \leq w(f') = w(f) = \gamma_r(T)$ . Therefore, we must have equalities throughout the inequality chain (4). In particular,  $\gamma_r(T) = \gamma_R^P(T)$ . Assume that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Thus, there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $u$  is a strong support vertex, then  $g(u) = 2$  and  $g(v) = 0$ . Hence, the function  $g|_{T'}$  is a  $\gamma_r(T')$ -function that is not a PRDF. This is a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ .

**Case 3.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_3$ .

Assume that  $f'$  is a  $\gamma_R^P(T')$ -function and  $L(u) = \{x, y\}$ . Hence, we assume that  $f'(u) = 2$  and  $f(x) = f(y) = 0$ . Then  $f'$  can be extended to a PRDF on  $T$  by assigning the weight 0 to  $v$ . Hence, by the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') = \gamma_r(T'). \tag{5}$$

Conversely, the vertex  $u$  is a strong support vertex of  $T$  and so we obtain that  $f(u) = 2$  and  $f(v) = 0$ . Hence,  $f'$  is a WRDF on  $T'$  and so  $\gamma_r(T') \leq w(f') = w(f) = \gamma_r(T)$ . Therefore, we must have equalities throughout the inequality chain (5). In particular,  $\gamma_r(T) = \gamma_R^P(T)$ . Assume that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Thus, there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $u$  is a strong support vertex, then  $g(u) = 2$  and  $g(v) = 0$ . Hence, the function  $g|_{T'}$  is a  $\gamma_r(T')$ -function that is not a PRDF. This is a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ .

**Case 4.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_4$ .

Assume  $T$  is obtained from  $T'$  by adding a star  $K_{1,3}$ , with central vertex  $x$  and leaves  $v, y, z$ , and adding the edge  $uv$  where  $u$  is a leaf that is adjacent to a strong support vertex  $w$  of  $T'$ . Assume that  $f'$  is a  $\gamma_r(T')$ -function. Note that  $w$  is a strong support vertex, we assume that  $f'(w) = 2$  and  $f'(u) = 0$ . By the inductive hypothesis,  $f'$  is a  $\gamma_R^P(T')$ -function. Then, we extend  $f'$  to a PRDF on  $T$  by giving the weight 2 to  $x$  and the weight 0 to the vertices  $v, y$  and  $z$ . Hence, by the statement above, and the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{6}$$

Conversely, the vertex  $x$  is a strong support vertex of  $T$ . Also,  $f(x) = 2$  and  $f(v) = f(y) = f(z) = 0$  hold. Hence,  $f'$  is a WRDF on  $T'$  and therefore

$$\gamma_r(T') \leq w(f') = w(f) - 2 = \gamma_r(T) - 2. \tag{7}$$

Therefore, we must have equalities throughout the inequality chain (6). In particular,  $\gamma_r(T) = \gamma_R^P(T)$ .

Assume that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Then there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $x$  is a strong support vertex, we may assume that  $g(x) = 2$  and  $g(v) = 0$ . Then the function  $g|_{T'}$  is a  $\gamma_r(T')$ -function on tree  $T'$  that is not a PRDF. This is a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ .

**Case 5.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_5$ .

If  $T$  is constructed from  $T'$  via adding a star  $K_{1,3}$  with central vertex  $x$  and leaves  $v, y, z$ , and joining leaf  $v$  of the star to vertex  $u$  of  $T'$  such that every  $\gamma_r(T')$ -function of  $T'$  gives to the vertex  $u$  the value 0. Assume that  $g'$  is a  $\gamma_r(T')$ -function. By the hypothesis,  $g'(u) = 0$ . On the other hand,  $g'$  is a  $\gamma_R^P(T')$ -function since  $\gamma_r(T') \equiv \gamma_R^P(T')$ . Thus,  $g'$  can be extended to a PRDF on  $T$  by assigning the weight 2 to  $x$  and the weight 0 to  $v, y$  and  $z$ . Hence, by the statement above, and inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{8}$$

Conversely, the vertex  $x$  is a strong support vertex of  $T$  and so we can see that  $f(x) = 2$  and  $f(v) = f(y) = f(z) = 0$ . Hence,  $f'$  is a WRDF on  $T'$  and therefore

$$\gamma_r(T') \leq w(f') = w(f) - 2 = \gamma_r(T) - 2. \tag{9}$$

Accordingly, we know these equalities across the inequality chain (8). Accordingly,  $\gamma_r(T) = \gamma_R^P(T)$ .

Assume that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Then there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $x$  is a strong support vertex, we may assume that  $g(x) = 2$  and  $g(v) = 0$ . Then, the function  $g|_{T'}$  is a  $\gamma_r(T')$ -function on tree  $T'$  and so  $g(u) = 0$ . Since  $g$  is not a PRDF, we deduce that  $g|_{T'}$  is not a PRDF, which leads to a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ .

**Case 6.**  $T$  is derived by  $T'$  through Operation  $\mathcal{O}_6$ .

Suppose that  $T$  is derived from  $T'$  via adding a tree  $F$  and joining its support vertex  $v$  with degree two to a vertex  $u$  of  $T'$  such that  $\deg(u) \geq 2$  and every  $\gamma_r(T')$ -function of  $T'$  assigns to the vertex  $u$  the value 2. Let  $N(v) = \{u, x, w\}$ ,  $L(v) = \{w\}$ ,  $N(x) = \{v, r\}$  and  $L(r) = \{y, z\}$ .

Suppose that  $g'$  is a  $\gamma_R^P(T')$ -function. Then  $g'(u) = 2$ . Hence, we extend  $g'$  to a PRDF on  $T$  by assigning the weight 0 to  $v, x, y, z$ , the weight 1 to  $w$  and the weight 2 to  $r$ . Hence, by the statement above, and the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 3 = \gamma_r(T') + 3. \tag{10}$$

Conversely, the vertex  $r$  is a strong support vertex of  $T$  and so we can assume that  $f(r) = 2$  and  $f(x) = f(y) = f(z) = 0$ . Without loss of generality, we may assume that  $f(v) = 0$  or  $f(v) = 2$ . If  $f(v) = 0$ , then  $f(w) \geq 1$  and also  $f'$  is a WRDF on  $T'$  and therefore

$$\gamma_r(T') \leq w(f') \leq w(f) - 3 = \gamma_r(T) - 3.$$

Next, suppose that  $f(v) = 2$ , then  $f(w) = 0$  and we may assume that  $f(u) \neq 2$ . Then the function  $g'$  such that  $g'(u) = \max\{f(u), 1\}$  and for  $w \neq u$ , we have  $g'(w) = f(w)$ , is a WRDF on  $T'$  and so

$$\gamma_r(T') \leq w(g') \leq w(f) - 3 = \gamma_r(T) - 3.$$

Accordingly, we must have equalities throughout the inequality chain (10). In particular,  $\gamma_r(T) = \gamma_R^P(T)$ .

Now, suppose that  $\gamma_r(T) \neq \gamma_R^P(T)$ . Then there is a  $\gamma_r(T)$ -function  $g$  such that  $g$  is not a PRDF. Since  $r$  is a strong support vertex, we may assume that  $g(r) = 2$  and  $g(x) = g(y) = g(z) = 0$ . If  $g(v) = 0$ , then we can assume that  $g(w) = 1$ . Then  $g|_{T'}$  is  $\gamma_r(T')$ -function that is not a PRDF, which leads to a contradiction. Hence, we assume that  $g(v) \neq 0$ . We may assume that  $g(v) = 2$  and  $g(w) = 0$ . If  $g(u) \neq 0$ , then  $g|_{T'}$  is  $\gamma_r(T')$ -function with weight  $\gamma_r(T) - 4 = \gamma_r(T') - 1$ . This is a contradiction. Hence, we assume that  $g(u) = 0$ . Thus, function  $g'$  by  $g'(u) = 1$  and  $g'(a) = g(a)$  for  $a \neq u$  is a  $\gamma_r(T')$ -function such that  $g'(u) \neq 2$ , which is a contradiction. Therefore,  $\gamma_r(T) \equiv \gamma_R^P(T)$ . This completes the proof.  $\square$

Next we have our main results as follows.

**Theorem 1.** *Let  $T$  be a tree. We have  $\gamma_r(T) \equiv \gamma_R^P(T)$  if and only if  $T$  is  $K_1$  or  $T \in \mathcal{T}$ .*

**Proof.** Suppose  $T$  is a tree of order  $n$ . Moreover, we have  $\gamma_r(T) \equiv \gamma_R^P(T)$ . When  $n = 1$ ,  $T = K_1$  holds. Without loss of generality, we suppose  $T$  is a tree of order  $n \geq 2$ . The induction method will be used in terms of  $n$  in the following. We can safely assume that  $n \geq 3$  because a path  $P_2$  has a  $\gamma_r(P_2)$ -function which is not a PRDF. In the case of  $n = 3$ ,  $T = P_3$ . Furthermore,  $\gamma_r(T) \neq \gamma_R^P(T)$ . What remains to be shown is the case of  $n \geq 4$ . Assume that every tree  $T'$  of order  $2 \leq n' < n$  having  $\gamma_r(T) \equiv \gamma_R^P(T)$  lies in  $\mathcal{F}$ . Suppose that  $T$  has  $n$  vertices and  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Let  $f$  be a  $\gamma_r(T)$ -function. It is also safe to view  $T$  as a tree with a diameter of at least three because any star having at least four vertices is a member of  $\mathcal{F}$ . In the case of  $diam(T) = 3$ ,  $T$  becomes a double star  $S_{a,b}$ . We can safely suppose that  $a \leq b$ . If  $b = 2$ , then  $T = P_4$  and clearly  $\gamma_r(T) \neq \gamma_R^P(T)$ , which is a contradiction. Hence, we assume that  $b \geq 3$ . If  $a = 2$ , then  $T$  has a  $\gamma_r(P_2)$ -function that is not a PRDF, which is a contradiction. Hence, we assume that  $a \geq 3$ . Then  $T \in \mathcal{F}$  because it is obtained from a star  $K_{1,b}$  by using Operation  $\mathcal{O}_1$ . Therefore, we assume that  $diam(T) \geq 4$  and  $T$  has a strong support vertex  $u$  with  $\ell_u \geq 4$ .

Let  $v \in L(u)$ ,  $T' = T - v$  and  $f$  be a  $\gamma_r^P(T)$ -function. Then  $f(u) = 2$  and  $f|_{T'}$  is a PRDF for tree  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) = \gamma_r(T)$ . Now, assume that  $g'$  is a  $\gamma_r(T')$ -function, then  $g'$  can be extended to a WRDF on  $T$  by assigning the weight 0 to the vertex  $v$ , and then  $\gamma_r(T) \leq \gamma_r(T')$ . Accordingly,  $\gamma_r(T) = \gamma_r(T')$ . On the other hand,  $\gamma_R^P(T) = \gamma_r(T) = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_R^P(T) = \gamma_R^P(T')$ . Therefore,  $\gamma_r(T') = \gamma_R^P(T')$ . If  $\gamma_r(T')$  and  $\gamma_R^P(T')$  are not strongly equal, every  $\gamma_r(T')$ -function that is not a PRDF can be extended to a  $\gamma_r(T)$ -function by giving 0 to  $v$ , that is not a PRDF. This is a contradiction with  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$  and by induction on  $T'$ , we have  $T' \in \mathcal{F}$ . We conclude that  $T \in \mathcal{F}$  as it is obtained from  $T'$  by using Operation  $\mathcal{O}_2$ . Hence, we assume that the following fact holds.

**Fact 1.** *If  $u$  is a strong support vertex of a tree  $T$ , then  $\ell_u \leq 3$ .*

We root  $T$  at a leaf  $x_0$  of a diametral path  $x_0x_1 \dots x_d$  from  $x_0$  to a leaf  $x_d$  farthest from  $x_0$ . Then Lemma 1 and Fact 1, implies that  $3 \leq \deg(x_{d-1}) \leq 4$ . Among all  $\gamma_r(T)$ -functions, let  $f$  be chosen so that the weight assigned to  $x_{d-1}$  is as large as possible. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , it then follows that  $f$  is a  $\gamma_R^P(T)$ -function. We consider the following two cases:

**Case 1.**  $\deg(x_{d-1}) = 4$ .

Let  $T' = T - x_d$ . Clearly  $f(x_{d-1}) = 2$  and  $f|_{T'}$  is a PRDF on  $T'$ . Hence,  $\gamma_r(T') \leq \gamma_R^P(T') \leq w(f|_{T'}) = \gamma_R^P(T) = \gamma_r(T)$ . Assume that  $g$  is a  $\gamma_r(T')$ -function. Then we can extend  $g$  to a WRDF for  $T$  by assigning 0 to  $x_d$ , and so  $\gamma_r(T) \leq \gamma_r(T')$ . Consequently,  $\gamma_r(T) = \gamma_r(T')$ . Hence  $\gamma_R^P(T) = \gamma_r(T) = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_R^P(T) = \gamma_R^P(T')$ . Therefore,  $\gamma_r(T') = \gamma_R^P(T')$ . If  $\gamma_r(T')$  and  $\gamma_R^P(T')$  are not



strongly equal, then every  $\gamma_r(T')$ -function which is not a PRDF can be extended to a  $\gamma_r(T)$ -function, by assigning 0 to  $x_d$ , that is not a PRDF. This leads to a contradiction with  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Therefore  $\gamma_r(T') \equiv \gamma_R^P(T')$  and by induction on  $T', T' \in \mathcal{F}$  must be true. Therefore,  $T \in \mathcal{F}$  must hold as it is derived from  $T'$  by applying Operation  $\mathcal{O}_3$ .

**Case 2.**  $\deg(x_{d-1}) = 3$ .

Then Lemma 2 implies that  $f(x_{d-1}) = 2$ . If  $f(x_{d-2}) \neq 0$ , then re-assigning to the vertices  $x_d$  and  $x_{d-1}$  the weight 1 produces a new  $\gamma_r(T)$ -function which is not a PRDF. This is a contradiction. Hence,  $f(x_{d-2}) = 0$ . If  $u \neq x_{d-1}$  is a child strong support vertex of  $x_{d-2}$ , then we can assume that  $f(u) = 2$  and so  $f$  is not a PRDF, which is a contradiction. On the other hand, Lemma 1 implies that  $x_{d-2}$  has not a weak support vertex as child. Thus every child  $u \neq x_{d-1}$  of  $x_d$  is a leaf. Since  $f(x_{d-2}) = 0$  holds,  $x_{d-2}$  is not a strong support vertex. Assume that  $x_{d-2}$  is a weak support vertex. Let  $L(x_{d-2}) = \{x\}$  and  $L(x_{d-1}) = \{x_d, z\}$ . Without loss of generality, we may assume that  $f(x) = 1$ . Then re-assigning to the vertices  $x_d$  and  $z$  the weight 1 and to the vertex  $x_{d-1}$  the weight 0, we obtain a new  $\gamma_r(T)$ -function which is not a PRDF. This is a contradiction. Hence, we assume that  $\deg(x_{d-2}) = 2$ . Thus, Lemma 2 implies that  $f(x_{d-1}) = 2$  and  $f(x_d) = f(z) = f(x_{d-2}) = 0$ . Since  $f$  is also a  $\gamma_R^P(T)$ -function, then  $f(x_{d-3}) \neq 2$ . Assume that  $f(x_{d-3}) = 1$ . Since  $f$  is a  $\gamma_R^P(T)$ -function, every vertex with weight 0 is adjacent to a vertex with weight 2, and then by giving to the vertex  $x_{d-3}$  the weight 0 and to the vertex  $x_{d-2}$  the weight 1, we obtain a new  $\gamma_r(T)$ -function, contradicting Lemma 2. Hence,  $f(x_{d-3}) = 0$  and so  $f(x_{d-4}) = 2$ . Two subcases follow as follows:

**Subcase 2.1**  $\deg(x_{d-3}) \geq 3$ .

First, we assume that  $x_{d-3}$  is a support vertex. Since  $f(x_{d-3}) = 0$ , it is not a strong support vertex so  $x_{d-3}$  is a weak support vertex. Let  $L(x_{d-3}) = x'$ . Without loss of generality, we can assume that  $f(x') = 1$  since  $f(x_{d-3}) = 0$ . If  $x_{d-3}$  has a child  $u$  that is a strong support vertex, then we can assume that  $f(u) = 2$ , contradicting that  $f$  is a  $\gamma_R^P(T)$ -function of  $T$ . Thus,  $x_{d-3}$  does not have a strong support vertex as a child. On the other hand, Lemma 1 implies that  $x_{d-3}$  does not have a weak support vertex as a child. Hence, every child of  $x_{d-3}$ , except  $x'$ , plays a role similar as  $x_{d-2}$ . Assume that  $\deg(x_{d-3}) \geq 4$  and  $u$  is a child of  $x_{d-3}$  other than  $x'$  and  $x_{d-2}$ . Let  $T' = T - T_{x_{d-2}}$ . Since  $f$  is a  $\gamma_R^P(T)$ -function and  $f(x_{d-2}) = 0$ , hence  $f|_{T'}$  is a PRDF on  $T'$ , and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Assume that  $f'$  is a  $\gamma_r(T')$ -function. Let  $N(u) = \{x_{d-3}, x\}$  and  $N(x) = \{u, y, z\}$ . Then, we can assume that  $f'(x) = 2$  and  $f'(y) = f'(z) = f'(u) = 0$ . If  $f'(x_{d-3}) \neq 1$ , then we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Next, assume that  $f'(x_{d-3}) = 1$ . Since  $f'(u) = 0$ , for every  $w \in N(x_{d-3}) \setminus \{u\}$  such that  $f'(w) = 0$ , there exists at least one vertex  $w' \in N(w) \setminus \{x_{d-3}\}$  such that  $f'(w') \neq 0$ . Hence, we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in both cases,  $\gamma_r(T') = \gamma_r(T) - 2$ . Therefore,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . If  $\gamma_r(T')$  and  $\gamma_R^P(T')$  are not strongly equal, then every  $\gamma_r(T')$ -function which is not a PRDF is extended to a  $\gamma_r(T)$ -function by giving the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it, that is not a PRDF. This leads to a contradiction with  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$  and by induction on  $T'$ , we have  $T' \in \mathcal{F}$ . We conclude that  $T \in \mathcal{F}$ , as it is obtained from  $T'$  by using Operation  $\mathcal{O}_5$ .

Thus, we can assume that  $\deg(x_{d-3}) = 3$ . If  $\deg(x_{d-4}) = 2$ , then clearly  $f(x_{d-5}) = 0$  and so by re-assigning to the vertices  $x_{d-3}$  and  $x_{d-5}$  the value 1 and to the vertex  $x_{d-4}$  the value 0 we get a new  $\gamma_r(T)$ -function which is not a PRDF. This is a contradiction. Hence, we can assume that  $\deg(x_{d-4}) \geq 3$ . Let  $T' = T - T_{x_{d-3}}$ . Since  $f$  is a  $\gamma_R^P(T)$ -function and  $f(x_{d-3}) = 0$ , then  $f|_{T'}$  is a PRDF on  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 3 = \gamma_r(T) - 3$ . Assume that  $f'$  is a  $\gamma_r(T')$ -function. If  $f'(x_{d-4}) \neq 1$ , then we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$ , the weight 1 to  $x'$  and the weight 0 to the remaining vertices of  $T$ , which is a WRDF of  $T$  of weight  $\gamma_r(T') + 3$ , and so  $\gamma_r(T) \leq \gamma_r(T') + 3$ . Next, assume that  $f'(x_{d-4}) = 1$ . If for every vertex  $u \in N(x_{d-4})$  with  $f'(u) = 0$ , there exists at least one vertex  $w \in N(u)$  with  $f'(w) > 0$ , then we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$ , the weight 1 to  $x'$  and the weight 0 to the remaining vertices of  $T$ ,

which is a WRDF of  $T$  of weight  $\gamma_r(T') + 3$ , and so  $\gamma_r(T) \leq \gamma_r(T') + 3$ . We assume that there exists a vertex  $u \in N(x_{d-4})$  with  $f'(u) = 0$  such that for every  $w \in N(u)$ , we have  $f'(w) = 0$ . Then for every vertex  $y \in N(x_{d-4}) - \{u\}$ , we get  $f'(y) \neq 0$ , since  $f'$  is a  $\gamma_r(T')$ -function. Hence, there exists a vertex  $y \in N(x_{d-4})$  such that  $f'(y) \neq 0$ , since  $\deg(x_{d-4}) \geq 3$ . Then by re-assigning to the vertices  $x_{d-4}$  and  $y$  the value 1, we get a new  $\gamma_r(T)$ -function  $g'$ . Consequently,  $\gamma_r(T') = \gamma_r(T) - 3$ . Hence,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 3 = \gamma_r(T) - 3 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ .

Now assume that  $g'$  is a  $\gamma_r(T')$ -function. We first assume that  $g'(x_{d-4}) = 0$ . Then we can extend  $g'$  to a  $\gamma_r(T)$ -function by assigning the weight 2 to  $x_{d-1}$ , the weight 1 to  $x'$  and the weight 0 to the remaining vertices of  $T$ . This is a contradiction since every  $\gamma_r(T)$ -function assigns to the vertex  $x_{d-4}$  the value 2. Next, assume that  $g'(x_{d-4}) = 1$ . If for every vertex  $u \in N(x_{d-4})$  with  $g'(u) = 0$ , there exists at least one vertex  $w \in N(u)$  with  $g'(w) > 0$ , then we can extend  $g'$  to a  $\gamma_r(T)$ -function by assigning the weight 2 to  $x_{d-1}$ , the weight 1 to  $x'$  and the weight 0 to the remaining vertices of  $T$ . This is a contradiction since every  $\gamma_r(T)$ -function assigns to the vertex  $x_{d-4}$  the value 2. Hence, we can assume that every  $\gamma_r(T')$ -function assigns to the vertex  $x_{d-4}$  the value 2.

Also, if  $\gamma_r(T')$  and  $\gamma_R^P(T')$  are not strongly equal, then every  $\gamma_r(T')$ -function which is not a PRDF, can be extended to a  $\gamma_r(T)$ -function by assigning the weight 2 to  $x_{d-1}$ , the weight 1 to  $x'$  and the weight 0 to the remaining vertices of  $T$ , that is not a PRDF on tree  $T$ , which leads to a contradiction with  $\gamma_r(T) \equiv \gamma_R^P(T)$ . Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$  and so by induction on  $T'$ , we have  $T' \in \mathcal{F}$ . We conclude that  $T \in \mathcal{F}$  since it is obtained from  $T'$  by using Operation  $\mathcal{O}_6$ .

Now, assume that  $x_{d-3}$  is not a strong support vertex. Then, we observe that every child of  $x_{d-3}$  plays a similar role as  $x_{d-2}$ . Let  $T' = T - T_{x_{d-2}}$ . Since  $f$  is a  $\gamma_R^P(T)$ -function and  $f(x_{d-2}) = 0$ , then  $f|_{T'}$  is a PRDF on  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Assume that  $f'$  is a  $\gamma_r(T')$ -function. Let  $N(u) = \{x_{d-3}, x\}$  and  $N(x) = \{u, y, z\}$ . Then, clearly we can assume that  $f'(x) = 2$  and  $f'(y) = f'(z) = f'(u) = 0$ . If  $f'(x_{d-3}) \neq 1$ , then we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Hence  $\gamma_r(T) \equiv \gamma_r(T') + 2$ . Next, assume that  $f'(x_{d-3}) = 1$ . Since  $f'(u) = 0$ , for every  $w \in N(x_{d-3}) - \{u\}$  such that  $f'(w) = 0$ , there exists at least one vertex  $w' \in N(w) - \{x_{d-3}\}$  such that  $f'(w') \neq 0$ . Hence, we can extend  $f'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in both cases,  $\gamma_r(T') = \gamma_r(T) - 2$ . Therefore,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . Assume that  $g'$  is an arbitrary  $\gamma_r(T')$ -function. Then we can assume that  $g'(u) = 0$ . Hence, as before, it can be extended to a  $\gamma_r(T)$ -function  $g$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , we deduce that  $g$  is a  $\gamma_R^P(T)$ -function and so  $g'$  is  $\gamma_R^P(T')$ -function. Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$  and via induction on  $T'$ ,  $T' \in \mathcal{F}$  holds. Therefore,  $T \in \mathcal{F}$  as it is derived via  $T'$  by applying Operation  $\mathcal{O}_5$ .

**Subcase 2.2**  $\deg(x_{d-3}) = 2$ .

If  $\deg(x_{d-4}) = 2$ , then by re-assigning to the vertex  $x_{d-4}$  the value 0, to the vertex  $x_{d-3}$  the value 1 and to the vertex  $x_{d-5}$  the value  $\max\{f(x_{d-5}), 1\}$ , we arrive at a new WRDF  $g$  with weight at most  $\gamma_r(T)$  such that  $g(x_{d-3}) \neq 0$ . This is a contradiction. Thus, we may assume that  $g(x_{d-4}) \geq 3$ .

First, we assume that  $x_{d-4}$  is a support vertex. Let  $T' = T - T_{x_{d-2}}$ . Since  $f(x_{d-2}) = 0$ , hence  $f|_{T'}$  is a PRDF for  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Now, assume that  $g'$  be a  $\gamma_r(T')$ -function. Since  $x_{d-4}$  in tree  $T'$  is a strong support vertex, we can assume that  $g'(x_{d-4}) = 2$ . Then, we can extend  $g'$  to a WRDF on  $T$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in both cases,  $\gamma_r(T') = \gamma_r(T) - 2$ . Therefore,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . Assume that  $g'$  is an arbitrary  $\gamma_r(T')$ -function. Then, we can assume that  $g'(x_{d-4}) = 2$  and  $g'(x_{d-3}) = 0$ . Hence, as before, it can be extended to a  $\gamma_r(T)$ -function  $g$  by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to the three neighbors of it. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , we deduce that  $g$  is a  $\gamma_R^P(T)$ -function and so  $g'$  is  $\gamma_R^P(T')$ -function. Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$  and via induction on  $T'$ ,  $T' \in \mathcal{F}$  holds true. Similarly, it is easy to see that  $T \in \mathcal{F}$  holds. This can be seen as it is derived by  $T'$  applying Operation  $\mathcal{O}_4$ .

Next, suppose  $x_{d-4}$  is not a support vertex. By Lemma 1, the vertex  $x_{d-2}$  does not have a weak support vertex as a child. Also, by Lemma 2, the vertex  $x_{d-2}$  does not have a strong support vertex as a child with degree three. We assume that  $x_{d-4}$  has as a child, a strong support vertex  $v$  with  $\ell_v = \deg(v) - 1 \geq 3$ . Let  $T' = T - T_v$ . Then, clearly  $f(v) = 2$  and  $f|_{T'}$  is a PRDF for  $T'$ . So  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Assume that  $g'$  is a  $\gamma_r(T')$ -function.  $g'$  can be extended to a WRDF on  $T$  by giving the weight 2 to  $v$  and the weight 0 to the all leaves neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in both cases,  $\gamma_r(T') = \gamma_r(T) - 2$ . Hence,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . Now, assume that  $g'$  is an arbitrary  $\gamma_r(T')$ -function. It can then be extended to a  $\gamma_r(T)$ -function  $g$  by assigning the weight 2 to  $v$  and the weight 0 to the all leaves neighbors of it. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , we deduce that  $g$  is a  $\gamma_R^P(T)$ -function and so  $g'$  is a  $\gamma_R^P(T')$ -function. Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$ . Using induction on  $T', T' \in \mathcal{F}$  holds true. Similarly,  $T \in \mathcal{F}$ . This is because  $T'$  and the Operation  $\mathcal{O}_1$ .

Next, we suppose that for any child support vertex  $v$  of  $x_{d-4}$ , we have  $\ell_v < \deg(v) - 1$ . Let  $N(x_{d-1}) = \{x_d, y, x_{d-2}\}$ . Assume that  $y_0$  is a leaf vertex of a tree  $T_{x_{d-4}}$ , other than  $x_d$  and  $y$ , and at maximum distance from  $x_{d-4}$ . Let  $y_0y_1 \dots y_t x_{d-4}$  be the shortest path from  $y_0$  to  $x_{d-4}$ . Clearly,  $t \leq 3$ . As noted earlier,  $t \notin \{0, 1\}$ . We proceed depending on  $t$ .

We first suppose that  $t = 2$ . Then Lemma 1 implies that  $\deg(y_1) \geq 3$  and so  $y_1$  is a strong support vertex. Without loss of generality, we may assume that  $f(y_1) = 2$  and so  $f(y_2) \neq 0$ , since  $f$  is a  $\gamma_R^P(T)$ -function and  $f(x_{d-4}) = 2$ . Hence,  $\deg(y_2) \geq 3$ . If  $y_2$  is not a support vertex, then by Lemma 1 any child  $u$  of  $y_2$  is a strong support vertex and so we can assume that  $f(u) = 2$ . Then by re-assigning to the vertex  $y_2$  the value 0, we get a new WRDF with weight less than  $\gamma_r(T)$ . This is a contradiction. Assume that  $y_2$  is a weak support vertex. Let  $L(y_2) = \{y'_2\}$ . Without loss of generality, we may assume that  $f(y_2) = 1$  and  $f(y'_2) = 0$ . Then by re-assigning to the vertex  $y_2$  the value 0 and to the vertex  $y'_2$  the value 1, we produce a new  $\gamma_r(T)$ -function which is not a PRDF. This is a contradiction. Thus, we may assume that  $y_2$  is a strong support vertex. If  $\ell_{y_2} = 2$  and  $L(y_2) = \{y'_2, y''_2\}$ , then we can assume that  $f(y_2) = 2$  and  $f(y'_2) = f(y''_2) = 0$ . Then by re-assigning to the vertex  $y_2$  the value 0 and to the vertices  $y'_2$  and  $y''_2$  the value 1, we get a new  $\gamma_r(T)$ -function which is not a PRDF. This is a contradiction. Hence, we may assume that  $\ell_{y_2} \geq 3$  and so  $f(y_2) = 2$ . Then, Lemma 2 implies that  $\deg(y_1) \geq 4$ . Let  $T' = T - T_{y_1}$ . Then, clearly  $f(y_1) = 2$  and  $f|_{T'}$  is a PRDF for  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Assume that  $g'$  is a  $\gamma_r(T')$ -function. Then, we can extend  $g'$  to a WRDF on  $T$  by assigning the weight 2 to  $y_1$  and the weight 0 to the all leaves neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in both cases,  $\gamma_r(T') = \gamma_r(T) - 2$ . Therefore,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . Now, assume that  $g'$  is an arbitrary  $\gamma_r(T')$ -function. Then it can be extended to a  $\gamma_r(T)$ -function  $g$  by assigning the weight 2 to  $y_1$  and the weight 0 to all leaves neighbors of it. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , we deduce that  $g$  is a  $\gamma_R^P(T)$ -function and so  $g'$  is a  $\gamma_R^P(T')$ -function. Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$ . Using induction on  $T', T' \in \mathcal{F}$  must be true. Therefore, we have  $T \in \mathcal{F}$ . This is due to  $T'$  and the Operation  $\mathcal{O}_1$ .

Suppose that  $t = 3$ . Since  $y_0$  plays the same role as  $x_d$ , we may assume that  $\deg(y_2) = \deg(y_3) = 2$  and  $y_1$  is a strong support vertex of degree three. Let  $T' = T - T_{x_{d-2}}$  and  $L(y_1) = \{y_0, y'_0\}$ . Then  $f|_{T'}$  is a PRDF for  $T'$  and so  $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$ . Now, assume that  $g'$  is a  $\gamma_r(T')$ -function. If  $g'(x_{d-3}) \neq 1$ ,  $g'$  can be generalized to a WRDF on  $T$  by giving the weight 2 to  $x_d$  and the weight 0 to all neighbors of it. Hence,  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Assume that  $g'(x_{d-3}) = 1$ . Then, we can assume that  $g(y_1) = 2, g'(y_0) = g'(y'_0) = g'(y_2) = 0$  and  $g'(x_{d-4}) \neq 2$ . If  $g'(x_{d-4}) = 1$ , then we can extend  $g'$  to a  $\gamma_r(T)$ -function  $g$  by re-assigning to the vertex  $x_{d-4}$  the value 2 and to  $x_{d-3}$  the value 0 and by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to all neighbors of it, and so  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Assume that  $g'(x_{d-4}) = 0$ . Then we can assume that  $g'(y_3) = 1$ . We can extend  $g'$  to a  $\gamma_r(T)$ -function  $g$  by re-assigning to the vertex  $x_{d-4}$  the value 2 and to the vertices  $x_{d-3}$  and  $y_3$  the value 0 and by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to all neighbors of it, and so  $\gamma_r(T) \leq \gamma_r(T') + 2$ . Consequently, in all cases,  $\gamma_r(T') \leq \gamma_r(T) - 2$ . Hence,  $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$  and so  $\gamma_r(T') = \gamma_R^P(T')$ . Now, assume that  $g'$  be

an arbitrary  $\gamma_r(T')$ -function. Then, as before, it can be extended to a  $\gamma_r(T)$ -function  $g$  by re-assigning to the vertex  $x_{d-4}$  the value 2 and to the vertices  $x_{d-3}$  and  $y_3$  the value 0 and by assigning the weight 2 to  $x_{d-1}$  and the weight 0 to all neighbors of it. Since  $\gamma_r(T) \equiv \gamma_R^P(T)$ , we deduce that  $g$  is a  $\gamma_R^P(T)$ -function and so  $g'$  is a  $\gamma_R^P(T')$ -function. Therefore,  $\gamma_r(T') \equiv \gamma_R^P(T')$ . Via induction on  $T'$ ,  $T' \in \mathcal{F}$  holds true. Consequently, it can be seen that  $T \in \mathcal{F}$ . This is again due to  $T'$  as well as the Operation  $\mathcal{O}_5$ . The proof is then completed as desired.  $\square$

### 3. Conclusions

Ian Stewart's work "Defend the Roman Empire!" [2] initially introduced the notion of Roman dominating functions. In recent years, many variations of Roman domination have been proposed due to different conditions on Roman domination, e.g., weak Roman domination [12] and perfect Roman domination [9]. Among the problems of Roman domination, the strong equality results have attracted great attention. In this paper, we studied the class of trees in which the perfect Roman domination number is strongly equal to the weak Roman domination number. This strongly equality means that each weak Roman dominating function of minimum weight is equivalent to a function that is deemed as perfect Roman dominating. We formulated some interesting conditions which we hope can be useful in stimulating relevant research.

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