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The Generalized Distance Spectrum of the Join of Graphs

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Abstract: Let G be a simple connected graph. In this paper, we study the spectral properties of the generalized distance matrix of graphs, the convex combination of the symmetric distance matrix $D(G)$ and diagonal matrix of the vertex transmissions $Tr(G)$. We determine the spectrum of the join of two graphs and of the join of a regular graph with another graph, which is the union of two different regular graphs. Moreover, thanks to the symmetry of the matrices involved, we study the generalized distance spectrum of the graphs obtained by generalization of the join graph operation through their eigenvalues of adjacency matrices and some auxiliary matrices.

Keywords: generalized distance matrix (spectrum); distance signless Laplacian matrix; joined union; lexicographic product; complete split graph; graph operation

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1. Introduction

Complicated graph structures can often be built from relatively simple graphs via graph-theoretic binary operations such as products. Graph spectrum provides a unique way of characterizing graph structures, sometimes even identifying the entire graph classes. Moreover, using simple graph operations, the spectra of complicated graphs may be constructed from those of small and simple graphs. The interplay between graph spectra (including adjacency, Laplacian, etc.) and various binary graph operations such as corona, edge corona, and disjoint union has been extensively studied in the literature; see e.g., [1–6].

In this paper, we consider simple connected graphs [7]. A graph G is represented by $G = (V(G), E(G))$, in which the set $V(G) = \{v_1, v_2, \ldots, v_n\}$ represents its vertex set and $E(G)$ is the edge set connecting pairs of distinct vertices. The number $n = |V(G)|$ is referred to as the *order* of G and $|E(G)|$ is the *size* of it. A vertix adjacent to a vertex $v \in V(G)$ is called the *neighborhood* of v and is presented by $N(v)$. The *degree* of a vertex v is the cardinality of its neighborhood and denoted by $d_G(v)$ or simply d_v. A *regular* graph has the same degree for all vertices. The *distance* d_{uv} is the length of a shortest path between two vertices u and v. The maximum distance between two vertices is called the *diameter* of a graph. The matrix $D(G) = (d_{uv})_{u,v \in V(G)}$ is called the *distance matrix* of G. As usual, $\overline{G}$ is the *complement* of the graph G. Moreover, the complete graph K_n, the complete bipartite graph $K_{s,t}$, the path P_n, the cycle C_n, and the wheel graph W_n are defined in the conventional way. The sum of the distances from a vertex v to all other vertices, $Tr_G(v) = \sum_{u \in V(G)} d_{uv}$, is called the *transmission degree* of v. A *k-transmission regular* graph admits $Tr_G(v) = k$ for any vertex v. Let $Tr_i = Tr_G(v_i)$. Then the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is said to be the *transmission degree sequence*. The quantity $T_i := \sum_{j=1}^{n} d_{ij} Tr_j$ is referred to as the *second transmission degree* of v_i.

The diagonal matrix $Tr(G) := diag(Tr_1, Tr_2, \ldots, Tr_n)$ characterizes the vertex transmissions of G. For a connected graph, M. Aouchiche and P. Hansen [8,9] studied the Laplacian and the signless Laplacian for its distance matrix. The *distance Laplacian matrix* $D^L(G) = Tr(G) - D(G)$ and the *distance signless Laplacian matrix* $D^Q(G) = Tr(G) + D(G)$ have attracted great recent research attention due to their usefulness in spectrum theory. Recently, Cui et al. [10] investigated a convex combination of $Tr(G)$ and $D(G)$ in the form of $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, $0 \leq \alpha \leq 1$, which is called the *generalized distance matrix*. Through the study of generalized distance matrix, not only new results can be derived but existing results can be looked into in a new unified point of view.

Let I be the identity matrix of order n. The characteristic polynomial of $D_\alpha(G)$ can be written as $\psi(G : \partial) = \det(\partial I - D_\alpha(G))$. The *generalized distance eigenvalues* of G are the zeros of $\psi(G : \partial)$. Noting that $D_\alpha(G)$ is real and symmetric, we arrange the eigenvalues as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. We call ∂_1 the *generalized distance spectral radius* of G. The generalized distance spectrum and energy have been recently scoped in [11,12].

The rest of the paper is organized as follows. In Section 2, we study the generalized distance spectrum of join of regular graphs. We will show that the generalized distance spectrum of join of two regular graphs can be obtained from their adjacency spectrum. Again using adjacency eigenvalues, we determine the generalized distance spectrum of join of a regular graph with the union of two different regular graphs. In Section 3, we use the adjacency matrix eigenvalues and auxiliary matrices to characterize the generalized distance spectrum of the joined union of regular graphs.

2. On the Generalized Distance Spectrum of Join of Graphs

In this section, we study the generalized distance spectrum of join of regular graphs. We will establish new relationship between generalized distance spectrum and adjacency spectrum. As applications, we obtain the generalized distance spectrum of some special graph classes including complete bipartite graph, complete split graph, wheel graph and some derived graphs from a complete graph.

Consider two disjoint vertex sets V_1 and V_2 with $|V_1| = n_1$ and $|V_2| = n_2$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union* is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of them is denoted by $G_1 \nabla G_2$ consisting of $G_1 \cup G_2$ and all edges joining each vertex in V_1 and each vertex in V_2. In other words, the join of them can be obtained by connecting each vertex of G_1 to all vertices of G_2.

The following gives the generalized distance spectrum of join of two regular graphs in terms of their eigenvalues of adjacency matrices.

Theorem 1. *Let G_i be an r_i-regular graph of order n_i, for $i = 1, 2$. Let $r_1 = \lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ and $r_2 = \mu_1, \mu_2, \ldots, \mu_{n_2}$ are the adjacency eigenvalues of G_1 and G_2, respectively. The characteristic polynomial of the generalized distance matrix of $G_1 \nabla G_2$ is given by*

$$\begin{aligned} \psi(G_1 \nabla G_2 : x) = \quad & [x^2 - (\gamma_1 + \gamma_2 - (1-\alpha)(n_1+n_2))x + \gamma_1\gamma_2 - \gamma_1 n_1(1-\alpha) - \gamma_2 n_2(1-\alpha)] \\ & \prod_{i=2}^{n_1} \Big(x - \alpha\gamma_1 + (1-\alpha)(\lambda_i + 2)\Big) \prod_{j=2}^{n_2} \Big(x - \alpha\gamma_2 + (1-\alpha)(\mu_j + 2)\Big), \end{aligned}$$

where $\gamma_1 = 2n_1 + n_2 - r_1 - 2$ and $\gamma_2 = 2n_2 + n_1 - r_2 - 2$.

Proof. For $i = 1, 2$, let G_i be an r_i-regular graph of order n_i. Let $G = G_1 \nabla G_2$ be the join of the graphs G_1 and G_2. It is clear that G is graph of diameter 2. Let $V(G_i) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ be the vertex set of the graph G_i, then the vertex set of G is $V(G) = V(G_1) \cup V(G_2)$. For all $v \in V(G_1)$, we have $Tr(v) = 2n_1 + n_2 - r_1 - 2$ and for all $u \in V(G_2)$, we have $Tr(u) = 2n_2 + n_1 - r_2 - 2$. Let us label the vertices of G, so that the first n_1 vertices are from G_1. Under this labelling, it can be seen that the generalized distance matrix of G can be written as

$$D_\alpha(G) = \begin{pmatrix} \alpha\gamma_1 I_{n_1} + (1-\alpha)(A_1 + 2\overline{A}_1) & (1-\alpha)J_{n_1\times n_2} \\ (1-\alpha)J_{n_2\times n_1} & \alpha\gamma_2 I_{n_2} + (1-\alpha)(A_2 + 2\overline{A}_2) \end{pmatrix},$$

where $\gamma_1 = 2n_1 + n_2 - r_1 - 2$, $\gamma_2 = 2n_2 + n_1 - r_2 - 2$, $J_{n_1\times n_2}$ is an all one matrix, I_{n_i} is the identity matrix of order n_i, A_i is the adjacency matrix of G_i and $\overline{A}_i$ is the adjacency matrix of the complement $\overline{G}_i$, for $i = 1, 2$.

Since G_i is an r_i-regular graph, it follows that $e_{n_i} = (1, 1, \ldots, 1)^T$, the all ones vector of order n_i, is an eigenvector corresponding to the eigenvalue r_i of A_i and corresponding to the eigenvalue $n_i - 1 - r_i$ of $\overline{A}_i$. Let x be a vector orthogonal to e_{n_1}, satisfying $A_1 x = \lambda x$, then $\overline{A}_1 x = (-\lambda - 1)x$. Taking $X = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and using $J_{n_1\times n_2} x = 0$, we have $D_\alpha(G)X = [\alpha\gamma_1 - (1-\alpha)(\lambda+2)]X$. This shows that $\alpha\gamma_1 - (1-\alpha)(\lambda+2)$ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvalue λ of A_1. Let y be a vector orthogonal to e_{n_2}, satisfying $A_2 y = \mu y$, then $\overline{A}_2 y = (-\mu - 1)y$. Taking $Y = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and using $J_{n_2\times n_1} y = 0$, we have $D_\alpha(G)Y = [\alpha\gamma_2 - (1-\alpha)(\mu+2)]Y$. This shows that $\alpha\gamma_2 - (1-\alpha)(\mu+2)$ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvalue μ of A_2. The equitable quotient matrix of $D_\alpha(G)$ is

$$M = \begin{pmatrix} \alpha n_2 + 2n_1 - r_1 - 2 & (1-\alpha)n_2 \\ (1-\alpha)n_1 & \alpha n_1 + 2n_2 - r_2 - 2 \end{pmatrix}.$$

Since the characteristic polynomial of M is $x^2 - (\gamma_1 + \gamma_2 - (1-\alpha)(n_1+n_2))x + \gamma_1\gamma_2 - \gamma_1 n_1(1-\alpha) - \gamma_2 n_2(1-\alpha)$ and any eigenvalue of M is an eigenvalue of $D_\alpha(G)$ [13], the result follows. □

Let $K_{r,s}$ be the *complete bipartite graph*. It is well-known that $K_{r,s} = \overline{K}_r \nabla \overline{K}_s$. We have the following observation from Theorem 1, which gives the generalized distance spectrum of $K_{r,s}$.

Corollary 1. *The generalized distance eigenvalues of $K_{r,s}$ consists of the eigenvalue $\alpha(2r+s) - 2$ with multiplicity $r-1$, the eigenvalue $\alpha(2s+r) - 2$ with multiplicity $s-1$ and the eigenvalues $x_1, x_2 = \frac{\alpha(s+r)+2(s+r)-4\pm\sqrt{(r^2+s^2)(\alpha-2)^2+2rs(\alpha^2-2)}}{2}$.*

Proof. Similarly as in Theorem 1, this can be proved by taking $n_1 = r, n_2 = s, r_1 = r_2 = 0$ and $\lambda_i = \mu_j = 0$, for all i, j. □

Let W_{n+1} be the *wheel graph* of order $n+1$. It is well known that $W_{n+1} = C_n \nabla K_1$. Using the fact that the adjacency spectrum of C_n is $\{2\cos(\frac{2\pi(j-1)}{n}) : j = 1, 2, \ldots, n\}$, we have the following observation from Theorem 1, which gives the generalized distance spectrum of W_{n+1}.

Corollary 2. *The generalized distance eigenvalues of the wheel graph W_{n+1} consists of the eigenvalues $\alpha(2n-3) - (1-\alpha)(2 + 2\cos(\frac{2\pi(i-1)}{2}))$, $i = 2, 3, \ldots, n$ and also the eigenvalues $x_1, x_2 = \frac{(\alpha+2)n+\alpha-4\pm\sqrt{[(\alpha+2)n+\alpha-4]^2-8\alpha n(n-1)+4n}}{2}$.*

Proof. Proof follows from Theorem 1, by taking $n_1 = n, n_2 = 1, r_1 = 2, r_2 = 0$ and $\lambda_i = 2\cos(\frac{2\pi(i-1)}{n})$, for $i = 2, 3, \ldots, n$. □

The graph $CS_{t,n-t}$ of order n is called *complete split graph*. It is constructed by linking each vertex of a clique of t vertices to each vertex of an independent set of $n-t$ vertices. It is clear that

$CS_{t,n-t} = K_t \nabla \overline{K}_{n-t}$. Using the fact that the adjacency spectrum of K_t is $\{t-1, -1^{[t-1]}\}$, we have the following observation from Theorem 1, which gives the generalized distance spectrum of $CS_{t,n-t}$.

Corollary 3. *The generalized distance eigenvalues of $CS_{t,n-t}$ consists of the eigenvalues $\alpha n - 1$ with multiplicity $t-1$, the eigenvalue $\alpha(2n-t)-2$ with multiplicity $n-t-1$ and the eigenvalues* $x_1, x_2 = \frac{2n-t+\alpha n-3\pm\sqrt{\theta}}{2}$, $\theta = (5-4\alpha)t^2 + (6\alpha n - 8n - 4\alpha + 6)t + n^2(\alpha-2)^2 + 2n\alpha - 4n + 1.$

Proof. Similarly as in Theorem 1, this can be shown by taking $n_1 = t, n_2 = n-t, r_1 = t-1, r_2 = 0$, $\lambda_i = -1$, for $i = 2, 3, \ldots, t$ and $\mu_j = 0$, for $j = 2, 3, \ldots, n-t$. □

In the next result, we work out the relationship between the generalized distance spectrum of the join of regular graphs and their adjacency spectra.

Theorem 2. *For $i = 0, 1, 2$, let G_i be r_i-regular with order n_i. Let $A(G_i)$ be their adjacency matrices and the adjacency eigenvalues are $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,n_i}$. We have that the generalized distance spectrum of $G_0\nabla(G_1 \cup G_2)$ is eigenvalues $\alpha(m + n_0 + \lambda_{0,j} - r_0) - \lambda_{0,j} - 2$ for $j = 2, \ldots, n_0$, and $\alpha(2m - n_0 + \lambda_{i,j} - r_i) - \lambda_{i,j} - 2$, for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$, where $m = \sum_{i=0}^{2} n_i$, and three extra eigenvalues defined by the eigenvalues of the following matrix*

$$\begin{pmatrix} \Theta_0 & (1-\alpha)n_1 & (1-\alpha)n_2 \\ (1-\alpha)n_0 & \Theta_1 & 2(1-\alpha)n_2 \\ (1-\alpha)n_0 & 2(1-\alpha)n_1 & \Theta_2 \end{pmatrix}, \tag{1}$$

where $\Theta_0 = \alpha(m - n_0) + 2n_0 - r_0 - 2$, and $\Theta_i = \alpha(2m - n_0 - 2n_i) + 2n_i - r_i - 2$, $i = 1, 2$.

Proof. Given $i = 0, 1, 2$. Assume G_i is r_i-regular and has n_i vertices. Let $G = G_0\nabla(G_1 \cup G_2)$ be the join of the graphs G_0 and $G_1 \cup G_2$. Obviously, G has diameter 2. Let $V(G_i) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ be the vertex set of the graph G_i, then the vertex set of G is $V(G) = V(G_0) \cup V(G_1) \cup V(G_2)$. For all $v \in V(G_0)$, we have $Tr(v) = m + n_0 - r_0 - 2$, for all $u \in V(G_1)$, we have $Tr(v) = 2m - n_0 - r_1 - 2$ and for all $w \in V(G_2)$, we have $Tr(w) = 2m - n_0 - r_2 - 2$. Let us label the vertices of G, so that the first n_0 vertices are from G_0, the next n_1 vertices are from G_1 and the next n_2 vertices are from G_2. Under this labelling, the generalized distance matrix of G has the form

$$D_\alpha(G) = \begin{pmatrix} S_0 & (1-\alpha)J_{n_0\times n_1} & (1-\alpha)J_{n_0\times n_2} \\ (1-\alpha)J_{n_1\times n_0} & S_1 & 2(1-\alpha)J_{n_1\times n_2} \\ (1-\alpha)J_{n_2\times n_0} & 2(1-\alpha)J_{n_2\times n_1} & S_2 \end{pmatrix},$$

where $S_0 = \alpha\Big((m + n_0 - r_0)I_{n_0} + A(G_0) - 2J_{n_0}\Big) + 2(J_{n_0} - I_{n_0}) - A(G_0)$, and $S_i = \alpha\Big((2m - n_0 - r_i)I_{n_i} + A(G_i) - 2J_{n_i}\Big) + 2(J_{n_i} - I_{n_i}) - A(G_i)$, for $i = 1, 2$.

For a regular graph G_i, the all ones vector $e_{n_i} = (1, 1, \ldots, 1)^T$ of order n_i is an eigenvector corresponding to the eigenvalue r_i. Other eigenvectors are orthogonal to e_{n_i}. Therefore, the all ones vector $e_{n_0} = (1, 1, \ldots, 1)^T$ of order n_0 is an eigenvector corresponding to the eigenvalue r_0. Other eigenvectors are orthogonal to e_{n_0}. Suppose that λ be an eigenvalue of adjacency matrix of G_0 and its eigenvector is x satisfying $e_{n_0}^T x = 0$, then $(x^T \quad 0_{1\times n_1} \quad 0_{1\times n_2})^T$ is an eigenvector of $D_\alpha(G)$ with the eigenvalue $\alpha(m + n_0 + \lambda - r_0) - \lambda - 2$. Let μ, ξ be any eigenvalues of the adjacency matrix of G_1 and G_2 with associated eigenvector y and z satisfying $e_{n_1}^T y = 0, e_{n_2}^T z = 0$, respectively. In a similar way, it can be seen that the vectors $(0_{1\times n_0} \quad y^T \quad 0_{1\times n_2})^T$ and $(0_{1\times n_0} \quad 0_{1\times n_1} \quad z^T)^T$ are eigenvectors of $D_\alpha(G)$ with corresponding eigenvalues $\alpha(2m - n_0 + \mu - r_1) - \mu - 2$ and $\alpha(2m - n_0 + \xi - r_2) - \xi - 2$, respectively.

Hence, we obtained eigenvectors $(x^T \quad 0_{1\times n_1} \quad 0_{1\times n_2})^T$, $(0_{1\times n_0} \quad y^T \quad 0_{1\times n_2})^T$ and $(0_{1\times n_0} \quad 0_{1\times n_1} \quad z^T)^T$. They are $m-3$ eigenvectors. It is easy to see that they are orthogonal to $(e_{n_0}^T \quad 0_{1\times n_1} \quad 0_{1\times n_2})^T$,

$(0_{1\times n_0} \quad e_{n_1}^T \quad 0_{1\times n_2})^T$ and $(0_{1\times n_0} \quad 0_{1\times n_1} \quad e_{n_2}^T)^T$. All other three eigenvectors of $D_\alpha(G)$ can be represented by $(\beta e_{n_0}^T \quad \gamma e_{n_1}^T \quad \theta e_{n_2}^T)^T$ for some $(\beta,\gamma,\theta) \neq (0,0,0)$.

Suppose that ν is an eigenvalue of the matrix $D_\alpha(G)$ with associated eigenvector $X = (\beta e_{n_0}^T, \gamma e_{n_1}^T, \theta e_{n_2}^T)^T$. Recall that $D_\alpha(G)X = \nu X$, and $A(G_i)e_{n_i} = r_i e_{n_i}$ $(i = 0,1,2)$. We obtain:

$$\begin{aligned}
&(\alpha(m-n_0)+2n_0-r_0-2)\beta+(1-\alpha)n_1\gamma+(1-\alpha)n_2\theta=\nu\beta,\\
&(1-\alpha)n_0\beta+(\alpha(2m-n_0-2n_1)+2n_1-r_1-2)\gamma+2(1-\alpha)n_2\theta=\nu\gamma,\\
&(1-\alpha)n_0\beta+2(1-\alpha)n_1\gamma+(\alpha(2m-n_0-2n_2)+2n_2-r_2-2)\theta=\nu\theta.
\end{aligned}$$

These equations admit a nontrivial solution only if (1) has an eigenvalue ν. Moreover, any nontrivial solution of the equations is an eigenvector of $D_\alpha(G)$ associated to ν. As the remaining three eigenvectors of $D_\alpha(G)$ are formed like this, it is obvious that any eigenvalue of (1) is also an eigenvalue of $D_\alpha(G)$. □

Consider the graph $G(n_0,n_1,n_2) = K_{n_0}\nabla(K_{n_1}\cup K_{n_2})$. We have the following observation from Theorem 2, which gives the generalized distance spectrum of $G(n_0,n_1,n_2)$.

Corollary 4. *The generalized distance eigenvalues of $G(n_0,n_1,n_2)$ consists of eigenvalue $\alpha m-1$, with multiplicity n_0-1, the eigenvalue $\alpha(2m-n_0-n_1)-1$, with multiplicity n_1-1, the eigenvalue $\alpha(2m-n_0-n_2)-1$, with multiplicity n_2-1 and three more eigenvalues which are the eigenvalues of the matrix*

$$\begin{pmatrix}
\alpha(m-n_0)+n_0-1 & (1-\alpha)n_1 & (1-\alpha)n_2\\
(1-\alpha)n_0 & \alpha(2m-n_0-2n_1)+n_1-1 & 2(1-\alpha)n_2\\
(1-\alpha)n_0 & 2(1-\alpha)n_1 & \alpha(2m-n_0-2n_2)+n_2-1
\end{pmatrix},$$

where $m=\sum_{i=0}^{2} n_i$.

Proof. Proof follows from Theorem 2, by taking $r_0=n_0-1, r_1=n_1-1, r_2=n_2-1, \lambda_{i,j}=-1$, for all $i=0,1,2$ and $j=2,3,\ldots,n_i$. □

Suppose we have a complete graph K_n of order n. The graph K_n-e is obtained by removing an edge e from K_n. Taking $n_0=n-2, n_1=n_2=1$ and $m=n$, in Corollary 4, we obtain the generalized distance spectrum of the graph K_n-e given by $\{\alpha n-1^{[n-3]}, x_1, x_2, x_3\}$, where x_1, x_2 and x_3 are the roots of the equation $f(x)=x^3-[2\alpha(n+1)+n-3]x^2+[(n^2+2n)\alpha^2+2n(n-1)\alpha-2n]x-[(n^3+n^2+4)\alpha^2-2\alpha(n^2+4)+4]=0$.

3. The Generalized Distance Spectrum of the Joined Union

In this section, we describe the relationship between generalized distance spectrum and the adjacency spectrum of the joined union of regular graphs.

The spectrum of a graph may determine the class of graphs that share the same properties. There have been some different names for the binary graph operation to be introduced below. We will call it joined union following [4,6]. This operation is also called generalized composition [14] or H-join [3]. Let $G=(V,E)$ have order n and $G_i=(V_i,E_i)$ have order m_i, for $i=1,\ldots,n$. The *joined union* $G[G_1,\ldots,G_n]$ is the graph $H=(W,F)$ satisfying:

$$W = \bigcup_{i=1}^{n} V_i$$

and

$$F = \bigcup_{i=1}^{n} E_i \cup \bigcup_{\{v_i,v_j\}\in E} V_i \times V_j.$$

Clearly, the joined union graph can be constructed by taking the union of $G_1, \ldots, G_n$ and linking any pair of vertices between G_i and G_j if v_i and v_j are neighbors in G. By this definition, the usual join of G_1 and G_2 can be viewed as $K_2[G_1, G_2]$, which is a special joined union graph.

Theorem 3. *Suppose G is a graph with diameter at most 2 over $V(G) = \{v_1, \ldots, v_n\}$. Denote by G_i an r_i-regular graph of order m_i and adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{im_i}$, where $i = 1, 2, \ldots, n$. The generalized distance spectrum of the joined union $G[G_1, \ldots, G_n]$ consists of the eigenvalues $\alpha(2m + \lambda_{ik} - m'_i - r_i) - \lambda_{ik} - 2$ for $i = 1, \ldots, n$ and $k = 2, 3, \ldots, m_i$, where $m = \sum_{i=1}^{n} m_i$ and $m'_i = \sum_{v_i v_j \in E(G)} m_j$. The remaining n eigenvalues are given by the matrix*

$$\begin{pmatrix} M_{1,1} & (1-\alpha)m_2 d_G(v_1, v_2) & \ldots & (1-\alpha)m_n d_G(v_1, v_n) \\ (1-\alpha)m_1 d_G(v_2, v_1) & M_{2,2} & \ldots & (1-\alpha)m_n d_G(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)m_1 d_G(v_n, v_1) & (1-\alpha)m_2 d_G(v_n, v_2) & \ldots & M_{n,n} \end{pmatrix}, \tag{2}$$

where $M_{i,i} = \alpha(2m - 2m_i - m'_i) + 2m_i - r_i - 2$.

Proof. Let G be a graph over $V(G) = \{v_1, \ldots, v_n\}$ and let $V(G_i) = \{v_{i1}, \ldots, v_{in_i}\}$ be the vertex set of graph G_i, for $i = 1, 2, \ldots, n_i$. Suppose that $H = G[G_1, \ldots, G_n]$ is the joined union of the graphs $G_1, G_2, \ldots, G_n$. By appropriately labelling the vertices of the graph H, we see that the generalized distance matrix $D_\alpha(H)$ of the graph H can be put into the form

$$D_\alpha(H) = \begin{pmatrix} S_1 & (1-\alpha)d_G(v_1, v_2)J_{n_1 \times n_2} & \ldots & (1-\alpha)d_G(v_1, v_n)J_{n_1 \times n_n} \\ (1-\alpha)d_G(v_2, v_1)J_{n_2 \times n_1} & S_2 & \ldots & (1-\alpha)d_G(v_2, v_n)J_{n_2 \times n_n} \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)d_G(v_n, v_1)J_{n_n \times n_1} & (1-\alpha)d_G(v_n, v_2)J_{n_n \times n_{n-1}} & \ldots & S_n \end{pmatrix},$$

where for $i = 1, 2, \ldots, n$,

$$\begin{aligned} S_i &= (1-\alpha)(2(J_{n_i} - I_{n_i}) - A(G_i)) + \alpha(2m - 2 - r_i - m'_i)I_{n_i} \\ &= \alpha\left((2m - r_i - m'_i)I_{n_i} - 2J_{n_i} + A(G_i)\right) + 2J_{n_i} - 2I_{n_i} - A(G_i), \end{aligned}$$

J_{n_i} is the all-one matrix, $A(G_i)$ is the adjacency matrix, and I_{n_i} is the identity matrix of order n_i.

Since G_i is r_i-regular, the all-one vector e_{m_i} is an eigenvector of $A(G_i)$ associated to eigenvalue r_i. The rest of the eigenvectors turn out to be orthogonal to e_{m_i}. We do not require connectivity of G_i and likewise we do not require r_i to be a simple eigenvalue. Suppose that λ is an eigenvalue of $A(G_i)$ associated with the eigenvector $X = (x_{i1}, x_{i2}, \ldots, x_{in_i})^T$ satisfying $e_{m_i}^T X = 0$. Note that X is essentially defined over $V(G_i)$ and allows a correspondence from v_{ij} to x_{ij}. Namely, $X(v_{ij}) = x_{ij}$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n_i$). Given the vector $Y = (y_1, y_2, \ldots, y_m)^T$, where

$$y_j = \begin{cases} x_{ij} & \text{if } v_{ij} \in V(G_i) \\ 0 & \text{otherwise.} \end{cases}$$

It can seen that the vector Y is an eigenvector of $D_\alpha(H)$ corresponding to the eigenvalue $\alpha(2m + \lambda - m'_i - r_i) - \lambda - 2$. There exists a total of $m - n$ mutually orthogonal eigenvectors of $D_\alpha(H)$ in this manner. They turn out to be orthogonal to the vectors $\mathbf{1}^i = (z_1^i, z_2^i, \ldots, z_m^i)^T$, where $i = 1, \ldots, n$, and

$$z_j^i = \begin{cases} 1 & \text{if } v_{ij} \in V(G_i) \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the rest n eigenvectors of $D_\alpha(H)$ are spanned by the vectors $\mathbf{1}^1, \mathbf{1}^2, \ldots, \mathbf{1}^n$, which due to the fact that $\mathbf{1}^1, \mathbf{1}^2, \ldots, \mathbf{1}^n$ appear to be linearly independent, suggests that the rest eigenvectors of $D_\alpha(H)$ are $\sum_{i=1}^{n} \beta_i \mathbf{1}^i$ for some coefficients $\beta_1, \ldots, \beta_n$.

Assume that μ is an eigenvalue of $D_\alpha(H)$ associated to an eigenvector $\sum_{i=1}^{n} \beta_i \mathbf{1}^i$. As $A(G_i)e_{m_i} = r_i e_{m_i}$, $(i = 1, \ldots, n,)$

$$\begin{aligned} D_\alpha(H) \sum_{i=1}^{n} \beta_i \mathbf{1}^i &= \sum_{i=1}^{n} \beta_i D_\alpha(H) \mathbf{1}^i \\ &= \sum_{i=1}^{n} \beta_i \Big(\alpha(2m - 2m_i - m_i') + 2m_i - r_i - 2\Big) \mathbf{1}^i + \sum_{k \neq i} d_G(v_k, v_i) m_i \mathbf{1}^k \\ &= \sum_{i=1}^{n} \Big((\alpha(2m - 2m_i - m_i') + 2m_i - r_i - 2)\beta_i + \sum_{k \neq i} d_G(v_k, v_i) m_k \beta_k \Big) \mathbf{1}^i \\ &= \mu \sum_{i=1}^{n} \beta_i \mathbf{1}^i. \end{aligned}$$

We derive the following equations involving $\beta_1, \ldots, \beta_n$:

$$\left(\alpha(2m - 2m_i - m_i') + 2m_i - r_i - 2 - \mu\right) \beta_i + \sum_{k \neq i} d_G(v_k, v_i) m_k \beta_k = 0, \quad i = 1, \ldots, n. \tag{3}$$

This set of equations admits a nontrivial solution only if μ becomes an eigenvalue of (2). Moreover, any nontrivial solution of (3) appears to be an eigenvector of $D_\alpha(H)$ associated to the eigenvalue μ. We see that each eigenvalue of (2) must also be an eigenvalue of $D_\alpha(H)$ since the rest n eigenvectors of $D_\alpha(H)$ are represented in this manner. □

The *lexicographic product* $G[H]$ of two graphs G and H can be constructed in the following way. The vertex set of $G[H]$ is equivalent to the product set $V(G) \times V(H)$. If $ab \in E(G)$, or $a = b$ and $xy \in E(H)$, then (a, x) and (b, y) are connected, namely, they form an edge in $E(G[H])$. We know that $G[H]$ is a special case of joined union $G[G_1, G_2, \ldots, G_n]$ with $G_i = H$ $(1 \leq i \leq n)$. When $G_i = K_1$, it can be seen that $G[K_1, K_1, \ldots, K_1] = G$. In view of Theorem 3, the generalized distance spectrum of the joined union $G[G_1, G_2, \ldots, G_n]$ can be written using eigenvalues of $A(G_i)$'s as well as those of (2). The relationship between the eigenvalues of $A(G)$ and the generalized distance spectrum of the joined union $G[G_1, G_2, \ldots, G_n]$ is not explicit though. The following example should shed a light on this relationship. When both G and H are regular graphs and G is a graph of diameter less than or equal to 2, the general distance spectrum of $G[H]$ can be calculated via Theorem 3.

Corollary 5. *Suppose that G is s-regular over n vertices with adjacency eigenvalues $\mu_1 = s \geq \mu_2 \geq \ldots \geq \mu_n$ and diameter less than or equal to 2. Assume that H is r-regular over m vertices with adjacency eigenvalues $\lambda_1 = r \geq \lambda_2 \ldots \geq \lambda_m$. Therefore, the generalized distance spectrum of $D_\alpha(G[H])$ contains $\alpha(2nm + \lambda_k - r - sm) - \lambda_k - 2$ for $2 \leq k \leq m$ each (n times) together with the eigenvalues of the matrix $m(1-\alpha)(2J - A(G)) + (\alpha(2nm - sm) - r - 2)I$, which are $2nm - sm - r - 2$ and $\alpha(2nm + m\mu_j - sm) - m\mu_j - r - 2$ for $2 \leq j \leq n$.*

It is clear that the *complete t-partite graph* $K_{m_1, m_2, \ldots, m_t}$ is a joined union of the graphs $G_i = \overline{K}_{m_i}$, when the parent graph is $G = K_t$. That is, $K_{m_1, m_2, \ldots, m_t} = K_t[\overline{K}_{m_1}, \overline{K}_{m_2}, \ldots, \overline{K}_{m_t}]$. The following observation is a result of Theorem 3 and gives the generalized distance spectrum of, $K_{m_1, m_2, \ldots, m_t}$, the complete t-partite graph.

Corollary 6. *The generalized distance spectrum of* $K_{m_1,m_2,\dots,m_t}$ *with* $m = \sum_{i=1}^{t} m_i$ *consists of the eigenvalue* $\alpha(m+m_i)-2$, *for* $i=1,2,\dots,t$ *each* m_i*(times) and the k eigenvalues of the matrix*

$$\begin{pmatrix} M_{1,1} & (1-\alpha)m_2 & \dots & (1-\alpha)m_n \\ (1-\alpha)m_1 & M_{2,2} & \dots & (1-\alpha)m_n \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)m_1 & (1-\alpha)m_2 & \dots & M_{n,n} \end{pmatrix},$$

where $M_{i,i} = \alpha(m-m_i)+2m_i-2$.

Proof. Proof follows from Theorem 3 by using $r_i = 0, m_i' = m - m_i$ and the fact that the eigenvalues of $\overline{K}_{m_i}$ are 0 with multiplicity m_i ($i = 1,2,\dots,t$). □

Example 1. *Considering the family of graphs* $F = \{G_1, G_2, G_3\}$ *as depicted in Figure 1 and the graph* $G = P_3$, *the path of order 3, the generalized distance matrix* $D_\alpha(H)$ *of the joined union* $H = P_3[G_1, G_2, G_3]$ *is a block matrix of the form*

$$\begin{pmatrix} S_1 & J(1-\alpha) & 2J(1-\alpha) \\ J(1-\alpha) & S_2 & J(1-\alpha) \\ 2J(1-\alpha) & J(1-\alpha) & S_3 \end{pmatrix},$$

where $S_i = \alpha(14I - 2J + A(G_i)) + 2J - 2I - A(G_i), i = 1,3$ *and* $S_2 = \alpha(10I - 2J + A(G_2)) + 2J - 2I + A(G_2)$.

Since the adjacency spectrums of G_1, G_2, G_3 *are* $spec_A(G_1) = \{(-1)^{[2]}, 2\}, spec_A(G_2) = \{-1, 1\}$ *and* $spec_A(G_3) = \{-2, 0^{[2]}, 2\}$, *respectively, then from Theorem 3, the generalized distance spectrum of H, consists of the eigenvalues* $\left\{13\alpha - 1^{[2]}, 14\alpha - 2^{[2]}, 9\alpha - 1, 12\alpha\right\}$, *also with the eigenvalues of the matrix*

$$\begin{pmatrix} 10\alpha+2 & 2(1-\alpha) & 8(1-\alpha) \\ 3(1-\alpha) & 7\alpha+1 & 4(1-\alpha) \\ 6(1-\alpha) & 2(1-\alpha) & 8\alpha+4 \end{pmatrix}.$$

Therefore, $spec_{D_\alpha}(H) = \left\{13\alpha - 1^{[2]}, 14\alpha - 2^{[2]}, 9\alpha - 1, 12\alpha, 16\alpha - 4, \frac{9\alpha+11\pm\sqrt{81\alpha^2-202\alpha+137}}{2}\right\}$.
Note that, as $D_0(H) = D(H)$, *then the distance spectrum of H is*

$$spec_D(H) = \left\{-1^{[2]}, -2^{[2]}, -1, 0, -4, \frac{11\pm\sqrt{137}}{2}\right\}.$$

Also, as $D_{\frac{1}{2}}(H) = \frac{1}{2}D^Q(H)$, *then the distance signless Laplacian spectrum of H is*

$$spec_{D^Q}(H) = \left\{8^{[2]}, 10^{[2]}, 11^{[2]}, 7, 12, 23\right\}.$$

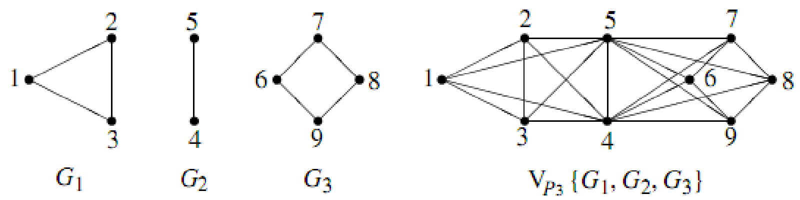

Figure 1. The joined union $H = P_3[G_1, G_2, G_3]$.

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