AN INTEGRABLE MODEL FOR UNDULAR BORES ON SHALLOW WATER

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Abstract On the basis of the integrable Kaup-Boussinesq version of the shallow water equations, an analytical theory of undular bores is constructed. The problem of the decay of an initial discontinuity is considered.

1. Introduction

It is well-known that the standard procedure for the derivation of system describing one-dimensional nonlinear shallow water waves from the basic equations allows great flexibility, due to the availability of two small parameters (nonlinearity and dispersion) (see Whitham 1974). As a result one has a number of systems, which are asymptotically equivalent but have quite different mathematical structure. In this paper we consider the Kaup – Boussinesq (KB) version of the shallow water equations (Kaup 1976):

$$h_t + (hu)_x + \frac{1}{4}u_{xxx} = 0\,, \qquad u_t + uu_x + h_x = 0. \tag{1}$$

This system, in comparison to other versions of the nonlinear shallow water equations, has an advantage of complete integrability. Our aim is to utilise the integrable structure of the KB system for the exact analytic description of undular bores.

In contrast to another integrable system, the Korteweg – de Vries equation, which is often used as a satisfactory model for the shallow water undular bores, the KB system has a natural two-wave structure, which enables one to capture the effects of interaction of undular bores and/or rarefaction waves arising in the decay of an initial jump discontinuity in the surface elevation and/or horizontal velocity fields, and also to determine their mutual disposition.

2. Lax pair and conservation laws

The Lax pair for the system (1) was found by Kaup (1976) and can be represented in the form (see El, Grimshaw & Pavlov 2000),

$$\varphi_{xxx} = 4[(\lambda - u/2)^2 - h]\varphi_x - 2[(\lambda - u/2)u_x + h_x]\varphi, \tag{2}$$

$$\varphi_t = (\lambda - u/2)\varphi_x + \frac{1}{2}u_x\varphi, \tag{3}$$

where λ is a complex spectral parameter (the KB system (1) represents the compatibility condition $(\varphi_{xxx})_t = (\varphi_t)_{xxx}$) provided λ is constant).

Combining (2) and (3) with the original system (1) we get the generation equation for the infinite series of KB conservation laws

$$\partial_t[2(\lambda - u/2)\varphi] = \partial_x[\frac{1}{2}\varphi_{xx} - 2(2\lambda^2 - \lambda u - h)\varphi]. \tag{4}$$

One can observe that equation (2) can be integrated in x once with the aid of the integrating factor φ to give

$$\varphi\varphi_{xx} - \frac{1}{2}\varphi_x^2 = 2(\lambda^2 - \lambda u - h + u^2/4)\varphi^2 - \nu(\lambda, t), \tag{5}$$

where $\nu(\lambda, t)$ is a 'constant' of integration.

3. One-phase travelling wave solutions

We obtain the one-phase travelling wave solution to the KB system in the 'right' parametrization. 'Right' here means that the parameters (integrals of motion) appearing in the travelling wave solution would be exactly Riemann invariants of the Whitham equations, which will be derived in the next section. Some of these solutions were constructed by Smirnov (1986) using general methods of finite-gap integration. Here we propose a simple straightforward method of obtaining the one-phase real-valued solutions, along with the Whitham equations in Riemann invariants. Our method is based on the simple substitution

$$u = U - \mu(\theta)\,, \qquad h = h(\theta), \tag{6}$$

where $\theta = kx - \omega t$ is the phase, and $U = \omega/k$ is the phase velocity.

Substituting (6) into the original KB system (1) we get, after some integration, the ordinary differential equation,

$$\frac{1}{4}k^2(\mu')^2 = \prod_{j=1}^{4}(\mu - r_j) \equiv R_4(\mu)\,, \tag{7}$$

where we choose the roots of the polynomial $R_4(\mu)$ $r_1 > r_2 > r_3 > r_4$ as the constants of integration. The depth $h(\theta)$ is expressed in terms of $\mu(\theta)$:

$$h = -2\mu^2 + 2\mu U + C\,,$$

where

$$U = \frac{1}{2}\sum_{j=1}^{4} r_j, \quad U^2 - C = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4,$$

Generically, r_j's can be either real or complex. Here, however, we restrict ourselves to the case of real branch points, which corresponds to the shallow water physics. Then, for μ to be real it has to oscillate between the roots r_3 and r_2.

Equation (7) can be integrated in terms of Jacobi elliptic functions

$$\mu(\theta) = r_4 + \frac{\Delta_{24}\Delta_{34}}{\Delta_{24} - \Delta_{23}\mathrm{sn}^2(\sqrt{\Delta_{13}\Delta_{24}}(x - Ut); m)}\,, \tag{8}$$

where the modulus m and the amplitude a are

$$m = \frac{\Delta_{23}\Delta_{14}}{\Delta_{13}\Delta_{24}}\,, \qquad a = \Delta_{23}\,, \qquad \Delta_{ij} = r_i - r_j. \tag{9}$$

The wavelength is given by the periodicity condition $\oint d\theta = 2\pi$, which, together with (7), yields

$$L = 2\pi k^{-1} = \int_{r_3}^{r_2} \frac{d\mu}{\sqrt{R_4(\mu)}} = \frac{2K(m)}{\sqrt{\Delta_{13}\Delta_{24}}}, \tag{10}$$

where $K(m)$ is the complete elliptic integral of the first kind. As usual, the cnoidal wave takes the sinusoidal form as $m \to 0$ ($r_2 \to r_3$) and converts into a soliton as $m \to 1$. In the soliton limit the result depends on the way in which $m \to 1$, as it can appear by two ways; $r_1 \to r_2$ or $r_3 \to r_4$ (see (9)). This corresponds to solitons moving in both directions (see El, Grimshaw & Pavlov 2000 for details)

4. Conservation laws on the one-phase travelling solution

Our aim now is to present the generating equation for the conservation laws (4) in a form suitable for averaging over the period of the travelling wave solution (8). We put $\varphi = \varphi(\theta)$ and substitute the solution (6) of the KB system into the stationary equation (2). After some simple but rather lengthy manipulations, we get

$$\varphi(\mu, \lambda) = \nu(\lambda)\frac{\lambda - \mu}{\sqrt{R_4(\lambda)}}. \tag{11}$$

Now, the conservation equation (4) takes the form for this chosen family of solutions

$$\partial_t \frac{P(\lambda, \mu)}{\sqrt{R_4(\lambda)}} = \partial_x \frac{Q(\lambda, \mu)}{\sqrt{R_4(\lambda)}}, \tag{12}$$

where

$$P(\lambda, \mu) = 2\lambda^2 - U\lambda - \mu^2,$$

$$Q(\lambda, \mu) = -\mu'' - 4\lambda^3 - 2U\lambda^2 + 2\lambda[4\mu(\mu - U) - C] + 2[\mu^2(\mu - U) - C\mu].$$

We emphasize that the normalizing factor $\nu(\lambda)$ cancels in (12) while $\sqrt{R_4(\lambda)}$ cannot be cancelled as we are going to investigate the slow dependence of the integrals of motion r_j on x and t.

5. Whitham-KB equations in Riemann invariants

Now we obtain the modulation equations for the parameters r_j considered as slowly varying functions of x and t. The Whitham prescription for obtaining the modulation equations (Whitham 1974) is that one should average the needed number (four in our case) of conservation laws over the period of the travelling wave solution. The generating equation (12) provides us with the infinite series of the conservation laws. We introduce the averaging procedure by the formula

$$\bar{f}(r_1, r_2, r_3, r_4) = \frac{2}{L}\int_{r_3}^{r_2} \frac{f(\mu, r_1, r_2, r_3, r_4)}{\sqrt{R_4(\mu)}} d\mu. \tag{13}$$

Applying this to (12) we get the generating modulation equation

$$\partial_t \frac{\bar{P}(\lambda, r_1, r_2, r_3, r_4)}{\sqrt{R_4(\lambda)}} = \partial_x \frac{\bar{Q}(\lambda, r_1, r_2, r_3, r_4)}{\sqrt{R_4(\lambda)}}. \tag{14}$$

Multiplying (14) by $(\lambda - r_j)^{3/2}$ and passing to the limit as $\lambda \to r_j$ we obtain the Whitham equations in Riemann form

$$\partial_t r_j + V_j(r_1, r_2, r_3, r_4)\partial_x r_j = 0, \quad j = 1, 2, 3, 4, \tag{15}$$

where the characteristic speeds can be represented in a compact universal form (El, Grimshaw & Pavlov (2000))

$$V_j = \frac{1}{2}\{\sum r_j - L(\partial L/\partial r_j)^{-1}\}\,, \tag{16}$$

where $L(\mathbf{r})$ is the wavelength given by (10).

6. Self-similar solutions to the Whitham-KB system

Let us consider the initial data for the KB system in the form of a step discontinuity: at $t = 0:\quad u = u^-\,, h = h^-$ for $x > 0$; and $u = u^+\,, h = h^+$ for $x < 0$, which implies four free parameters . This type of initial data is known to lead, in the systems of this type, to the onset of a rapidly oscillating nonlinear wave. This wave is the undular bore and it was first analytically described by Gurevich and Pitaevsky (GP) (1974) on the basis of the KdV equation. The GP description implies that the solution in the bore region has the form of a one-phase modulated travelling wave. At one edge of the undular bore the oscillations have a form of solitons $(m = 1)$ and at the opposite edge they degenerate into small-amplitude sinusoidal waves $(m = 0)$. For the problem of decay of an initial discontinuity the desired solution of the Whitham system must be self-similar, i.e. $r_j = r_j(x/t)$. Then, the KB-Whitham system (15) transforms into the system,

$$(V_j - \tau)\frac{dr_j}{d\tau} = 0\,, \qquad \tau = x/t, \qquad j = 1, 2, 3, 4\,, \tag{17}$$

which implies that three of the invariants r_j are constants and for the remaining one (r_k) we have the algebraic equation $V_k = \tau$.

For example, for the initial discontinuity with $h^- = 1\,, u^- = 0\,, h^+ = (c+1)^2/4\,, u^+ = c - 1\,, -1 < c < 1$ the solution is:

$$r_1 = 1,\ r_3 = c,\ r_4 = -1\,,$$

$$\frac{r_2 + c}{2} + \frac{r_2 - c}{1 - \dfrac{1-c}{1-r_2}\dfrac{E(m)}{K(m)}} = \frac{x}{t}\,, \qquad m = \frac{2(r_2 - c)}{(1-c)(r_2+1)}\,. \tag{18}$$

The solution (18) describes the slow modulation for the so-called simple undular bore moving to the right (see Figure 1a). The oscillatory structure of the bore is given by the travelling solution (8). An analogous solution can be constructed for the simple undular bore moving to the left (Figure 1b).

The Whitham-KB system has another family of self-similar solutions coinciding with the standard rarefaction waves in the dispersionless shallow water theory. For example the above initial jump with $c > 1$ is resolved by the rarefaction wave moving to the right (Figure 1c):

$$r_4 \equiv r_- = -1\,, \quad r_3 = r_2 = r_1 \equiv r_+ = 2x/3t + 1/3.$$

Analogously, for the resolution of the appropriate initial data, the rarefaction wave moving to the left can be constructed (Figure 1d).

One can see, that the number of free parameters both in the simple undular bore and in the refraction wave is equal to three. Therefore to resolve an arbitrary, four-parametric, initial jump one needs to combine two different waves which are provided by the two-wave nature of the KB system. These waves may be undular bores as well as rarefaction waves in various combinations. In the next section we consider the most important cases .

7. Decay of an arbitrary initial discontinuity

Without loss of generality we consider the initial jump in the form:

$$t = 0: \quad h = 1, \quad u = 0, \text{ for } x < 0, \text{ and } h = h_0, \quad u = u_0, \text{ for } x > 0,$$

where h_0 and u_0 are constants, $h_0 > 0$. It is convenient to introduce two new constants $c_\pm$, which have the meaning of the Riemann invariants for the dispersionless shallow water equations, instead of h_0, u_0.

$$c_\pm = \frac{u_0}{2} \pm \sqrt{h_0} \tag{19}$$

Then we can illustrate some important cases with the aid of the diagrams shown in Figure 2.

a) $c_+ > 1$, $1 > c_- > -1$ (Figure 2a)

Two rarefaction waves separated by a plateau are produced as a result of the decay. The self-similar coordinates of the weak discontinuites in the solution of the Whitham system (edges of the waves) are

$$\tau_4 = -1, \ \tau_3 = (1 + 3c_-)/2, \ \tau_2 = (3 + c_-)/2, \ \tau_1 = (3c_+ + c_-)/2,$$

The value of h and u at the plateau are: $u_p = 1 + c_-$, $h_p = (1 - c_-)^2/4$.

b) $c_+ > 1$, $-5/3 < c_- < -1$ (Figure 2b)

A leading rarefaction wave and a trailing undular bore, separated by a plateau, are produced . The coordinates of the edges are:

$$\tau_1 = (3c_+ + c_-)/2, \ \tau_2 = (3 + c_-)/2\,, \ \tau_3 = (c_- - 9)/2 + 8/(3 + c_-), \ \tau_4 = c_-.$$

The values of h and u at the plateau are: $u_p = 1 + c_-$, $h_p = (1 - c_-)^2/4$

c) $-1 < c_- < c_+$, $(3 + c_-)/4 < c_+ < 1$ (Figure 2c)

The leading undular bore and the trailing rarefaction wave separated by a plateau are produced . The coordinates of the edges are:

$$\tau_1 = 1+(c_+ + c_-)/2,\ \tau_2 = (2c_+ + 1 + c_-)/2 - 2(1-c_+)(c_+ - c_-)/(2c_+ - 1 - c_-),$$

$$\tau_3 = (1 + 3c_-)/2,\ \tau_4 = -1.$$

The values of h and u at the plateau are $u_p = 1 + c_-$, $h_p = (1 - c_-)^2/4$.

d) $(3c_-^2 + 6c_- + 7)/(4(3 + c_-) < c_+ < 1$, $c_- < -1$ (Figure 2d).

Two undular bores separated by a plateau are produced. The edges are:

$$\tau_1 = 1+(c_+ + c_-)/2,\ \tau_2 = c_+ + (1+c_-)/2 - 2(1-c_+)(c_+ - c_-)/(2c_+ - 1 - c_-),$$

$$\tau_3 = (c_- - 9)/2 + 8/(3 + c_-),\ \tau_4 = -c_-.$$

At the plateau we have $u_p = 1 + c_-$, $h_p = (1 - c_-)^2/4$.

The complete classification for the problem of the decay of an initial discontinuity for the KB equation can be found in (El, Grimshaw & Pavlov 2000).

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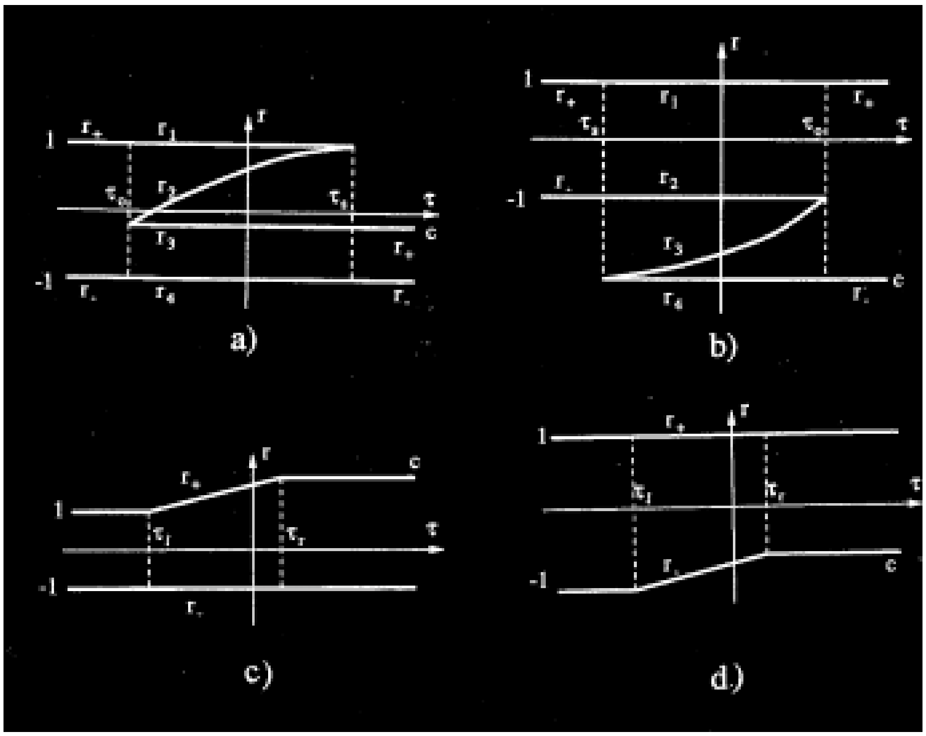

Figure 1. Behaviour of the Riemann invariants in undulars bores moving a) to the right, b) to the left; in rarefaction waves moving: c) to the right, d) to the left

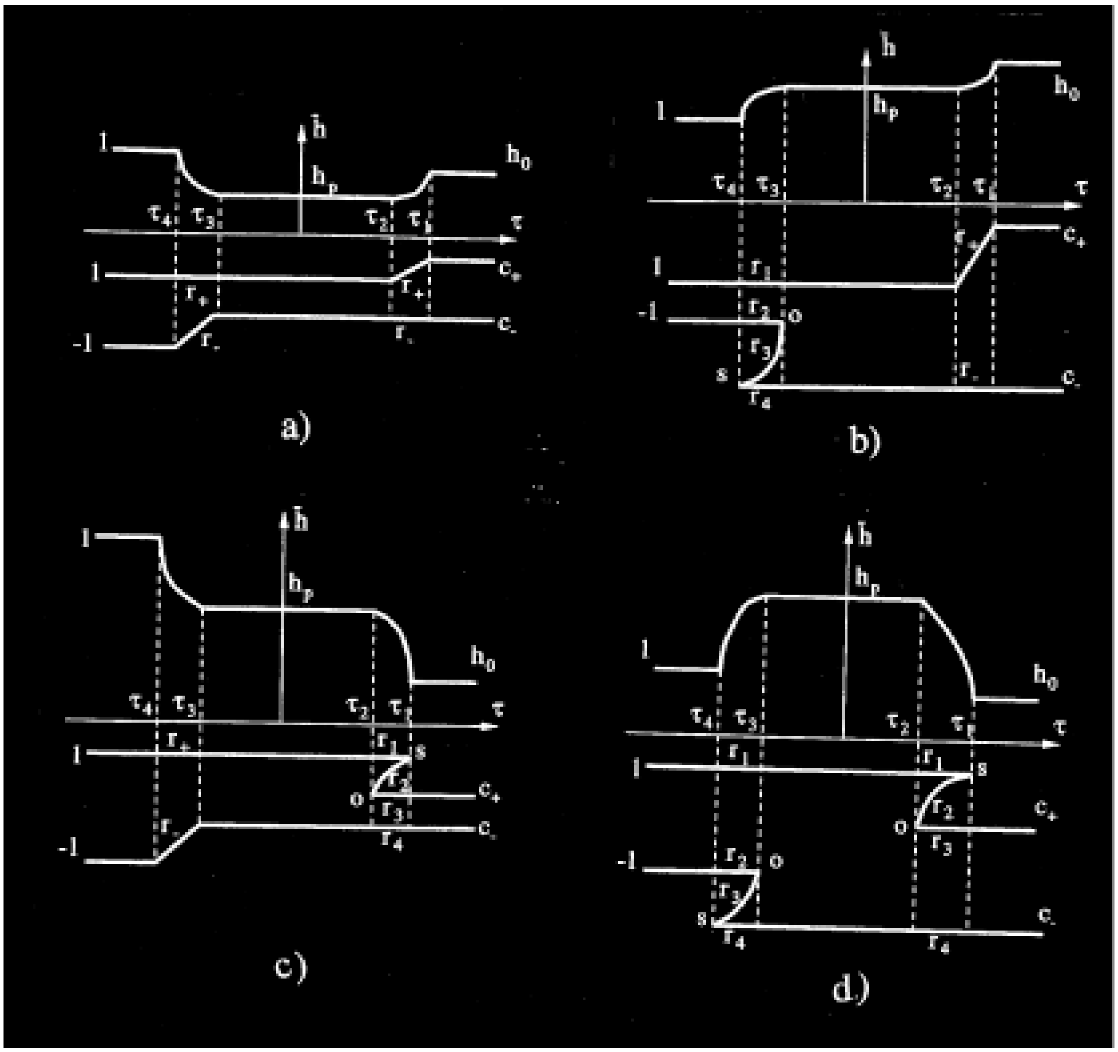

Figure 2. Behaviour of the Riemann invariants and the averaged depth in the decay of an initial discountinuity problem for different values of the initial step