

Spectral theory of soliton and breather gases for the focusing nonlinear Schrödinger equationGennady El^{1,*} and Alexander Tovbis²¹*Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne NE1 8ST, United Kingdom*²*Department of Mathematics, University of Central Florida, Orlando, Florida 32816, USA*

(Received 16 November 2019; accepted 3 April 2020; published 8 May 2020)

Solitons and breathers are localized solutions of integrable systems that can be viewed as “particles” of complex statistical objects called soliton and breather gases. In view of the growing evidence of their ubiquity in fluids and nonlinear optical media, these “integrable” gases present a fundamental interest for nonlinear physics. We develop an analytical theory of breather and soliton gases by considering a special, thermodynamic-type limit of the wave-number–frequency relations for multiphase (finite-gap) solutions of the focusing nonlinear Schrödinger equation. This limit is defined by the locus and the critical scaling of the band spectrum of the associated Zakharov-Shabat operator, and it yields the nonlinear dispersion relations for a spatially homogeneous breather or soliton gas, depending on the presence or absence of the “background” Stokes mode. The key quantity of interest is the density of states defining, in principle, all spectral and statistical properties of a soliton (breather) gas. The balance of terms in the nonlinear dispersion relations determines the nature of the gas: from an ideal gas of well separated, noninteracting breathers (solitons) to a special limiting state, which we term a breather (soliton) condensate, and whose properties are entirely determined by the pairwise interactions between breathers (solitons). For a nonhomogeneous breather gas, we derive a full set of kinetic equations describing the slow evolution of the density of states and of its carrier wave counterpart. The kinetic equation for soliton gas is recovered by collapsing the Stokes spectral band. A number of concrete examples of breather and soliton gases are considered, demonstrating the efficacy of the developed general theory with broad implications for nonlinear optics, superfluids, and oceanography. In particular, our work provides the theoretical underpinning for the recently observed remarkable connection of the soliton gas dynamics with the long-term evolution of spontaneous modulational instability.

DOI: [10.1103/PhysRevE.101.052207](https://doi.org/10.1103/PhysRevE.101.052207)**I. INTRODUCTION**

There is rapidly growing interest in the subject of random solutions to integrable nonlinear dispersive equations prompted by Zakharov’s paper “Turbulence in integrable systems” [1]. The unlikely marriage of integrability and randomness within the framework of “integrable turbulence” is motivated by the complexity of many nonlinear wave phenomena in physical systems that can be successfully modeled by integrable partial differential equations. Despite the integrability of the mathematical model, physically reasonable results for such systems can often be obtained only in statistical terms (such as a probability density function, a power spectrum, a correlation function, etc.). This is particularly true for modulationally unstable media, where small random perturbations, inevitably present in any physical system, rapidly grow, leading to disintegration of a constant or slowly varying background and the establishment of a turbulent nonlinear wave field exhibiting spontaneous emergence of localized coherent structures such as solitons and breathers [2]. Applications of integrable turbulence range from oceanography to nonlinear fiber optics and Bose-Einstein condensates. Indeed, recent observations in ocean waves [3,4] and laboratory experiments

in optical media [5–7] and classical fluids [8,9] provide growing evidence of the ubiquity and pervasiveness of integrable turbulence in physical systems. Due to the complexity of turbulent nonlinear wave fields, the majority of the existing studies of integrable turbulence involve extensive numerical simulations, while an analytical development, vital for the understanding of this fundamental physical phenomenon, is rather limited.

Our paper develops an analytical theory of an important class of integrable turbulence called soliton gas, and its natural yet nontrivial generalization, which we term breather gas, in the framework of the one-dimensional focusing nonlinear Schrödinger (fNLS) equation, a canonical model for the description of the envelope dynamics of weakly nonlinear quasimonochromatic waves propagating in dispersive, modulationally unstable media when dissipative processes are negligible.

The notion of soliton gas—an infinite statistical ensemble of interacting solitons—was first introduced by Zakharov [10], who derived a kinetic equation for a “rarefied” gas of KdV solitons by considering the modification of the soliton velocity due to the position shifts in its pairwise collisions with other solitons in the gas. The generalization of Zakharov’s kinetic equation to the case of the KdV soliton gas of finite density was obtained by one of the authors in [11] by considering the infinite-phase, thermodynamic-type limit of the Whitham modulation equations. In [11] the soliton

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distribution function was identified with the *density of states*, the fundamental quantity in the spectral theory of random potentials [12,13]. The finite-gap theory derivation in [11] served as a motivation for a more intuitive, physical construction of the kinetic equation for a dense soliton gas of the fNLS equation in [14]. Very recently, the kinetic equation having the same structure as the kinetic equation for soliton gas was derived in the framework of the “generalized hydrodynamics” for quantum many-body integrable systems [15–17]. These theoretical studies, along with already mentioned observations in a variety of physical media, strongly indicate that soliton gases represent a fundamental object of nonlinear physics, providing a number of intriguing, novel connections between soliton theory, dispersive hydrodynamics [18], and statistical mechanics. In particular, the dynamics of a soliton gas in the fNLS equation recently attracted considerable attention in relation with the description of the nonlinear stage of spontaneous modulational instability [2,19] and the rogue wave formation [20].

Solitons represent spatially localized, decaying at infinity solutions of the fNLS equation. The presence of a nonzero background gives rise to rich families of space-time localized fNLS solutions called breathers. Thus the fNLS soliton gas dynamics in the presence of a nonzero background can be viewed as breather gas. If the background of a breather is a plane wave, the corresponding fNLS solution in a general case is the so-called Tajiri-Watanabe breather [21], with the “standard” Akhmediev, Kuznetsov-Ma, and Peregrine breathers being its particular cases. The background of a breather can be given by one of the nonlinear multiphase fNLS solutions, also known as finite-gap potentials [22,23] (see [24] for a description of rogue waves within finite-gap potentials, and [25,26] for the explicit constructions of fNLS breather solutions on periodic (elliptic) and two-phase backgrounds, respectively).

Our paper is concerned with the analytical description of soliton and breather gases using the tools of nonlinear spectral theory, also known as finite-gap theory, which represents an extension of the celebrated inverse scattering transform (IST) method [27] to problems with periodic and quasiperiodic boundary conditions [22,28]. While the mathematical development of the paper involves some technical aspects of the finite-gap theory, the basic ideas behind the proposed construction are very general, physically transparent, and fundamental. In fact, we show that the kinetic theory of breather gas can be viewed as a broad generalization of the well-known kinematic wave theory [29] to the case of nonlinear dispersive random waves described by the fNLS equation. Below we outline the organization of the paper and present a high-level description of the main ideas and results. The mathematical underpinnings of more technical aspects of the paper can be found in the Appendix.

First we derive nonlinear dispersion relations for finite-gap potentials of the fNLS equation, which generalize the well-known notion of the dispersion relation $\omega = \omega_0(k)$ for linearized waves. The finite-gap potential $\psi_n(x, t)$ is characterized by n -component wave number \mathbf{k} and frequency ω vectors, where $n \in \mathbb{N}$ is the *genus* of the solution. The genus of the solution is determined by the number of nonlinear wave modes or phases comprising the wave field $\psi(x, t)$ described by the fNLS equation. Within this classification, the

plane wave (condensate) with $|\psi| = 1$ represents a genus 0 solution. The well-known elliptic solutions of the fNLS equation are genus 1 solutions, with the standard (fundamental) solitons being a degenerate case of the genus 1. The standard breathers (Akhmediev, Kuznetsov-Ma, and Peregrine) all represent degenerate genus 2 solutions [23]. In our construction of breather gas, we assume an even genus $n = 2N$; the results for the potentials of an odd genus (yielding soliton gas in the appropriate limit) are obtained by “collapsing” the breather gas background to zero.

Nonlinear dispersion relations for the fNLS n -gap potentials represent a system of linear equations (18) and (19) relating \mathbf{k} and ω with other parameters of the solution, which are most conveniently expressed in terms of the band spectrum $\Sigma_n \in \mathbb{C}$ of the Zakharov-Shabat (ZS) operator associated with the fNLS equation [27]. We can symbolically represent these relations in a parametric form,

$$\omega = \Omega(\Sigma_n), \quad \mathbf{k} = \mathbf{K}(\Sigma_n). \quad (1)$$

The core of the paper is the derivation and analysis of the nonlinear dispersion relations (25) and (26) for a breather or soliton gas, which are obtained from relations (1) by applying a special infinite-genus, thermodynamic-type limit. The crucial role in our analysis is played by (i) a special choice of the wave-number–frequency set in (1), and (ii) the critical, n -dependent scalings of the band-gap distributions in the finite-gap potentials. We distinguish between three such scalings: exponential, superexponential, and subexponential, each corresponding to a distinct type of breather or soliton gas. Each type of scaling implies a specific balance of terms in the nonlinear dispersion relations (1), resulting in certain distinct properties of the corresponding soliton or breather gases.

We show that the superexponential scaling corresponds to an “ideal gas” of noninteracting, isolated quasiparticles (breathers or solitons), whose dynamics is determined by secular (nonintegral) terms in the dispersion relations. In the opposite case of the subexponential spectral scaling, the properties of the gas are entirely determined by the integral, interaction terms, and the individual characterization of quasiparticles is suppressed. We call the corresponding gas a breather (soliton) condensate. We show that a particular case of the condensate representing a critically dense bound-state soliton gas is characterized by a special density-of-states distribution (63), which coincides with the appropriately normalized semiclassical distribution of the discrete spectrum in the ZS scattering problem for a rectangular potential [27,30,31]. We also present a nontrivial example of a non-bound-state soliton condensate characterized by a circular spectral locus in the complex plane, leading to the group velocity of quasiparticles in the condensate being twice the speed of free solitons for the same spectral parameter. Finally, the exponential spectral scaling corresponds to the general case of a soliton or breather gas of finite density, in which the effects related to the individual motion of quasiparticles and their pairwise interactions are in balance.

As a straightforward consequence of the nonlinear dispersion relations of breather or soliton gas, we derive an integral equation (37) relating the velocity $s(\eta)$ of quasiparticles in a breather or soliton gas (we call them the “tracer” breathers or solitons) with the fundamental quantity $u(\eta)$ called the density

of states [12,13], $\eta \in \mathbb{C}$ being the ZS spectral parameter. This integral equation specifying $s = \mathcal{F}[u]$, where \mathcal{F} is a functional, can be viewed as the *equation of state* of a spatially homogeneous (equilibrium) gas. The equations of state for breather and soliton gases have the same structure but are characterized by different forms of both secular (free-particle) and integral (pairwise-interaction) terms. In both cases, the interaction kernel is determined by the position shift in a breather-breather and soliton-soliton pairwise interaction, respectively. We also derive a “satellite” system, Eqs. (88), (30), and (47), describing the spectral distribution of the phase velocity $\tilde{s}(\eta)$ of the carrier wave in a breather or soliton gas with a given density of states $u(\eta)$.

The evolution of slowly modulated n -gap potentials is known to be described by the so-called Whitham modulation equations [29,32] representing a system of quasilinear partial differential equations for weak spatiotemporal deformations of the finite-gap ZS spectrum $\Sigma_n(x, t)$ [33,34]. The modulation system for the fNLS equation necessarily includes the n -component wave conservation law [29,33,35]

$$\mathbf{k}_t = \omega_x, \quad (2)$$

which should be complemented by the nonlinear dispersion relations (1). The application of the thermodynamic limit to the “nonlinear kinematic wave system” (2), (1) results in a kinetic equation for the density of states $u(\eta, x, t)$ in a spatially nonhomogeneous breather or soliton gas. This equation has the form of a transport equation $u_t + (us)_x = 0$, complemented by the x, t -dependent equation of state $s(\eta, x, t) = \mathcal{F}[u(\eta, x, t)]$. Another consequence of the application of the thermodynamic limit to the system (2), (1) is the satellite transport equation $\tilde{u}_t + (\tilde{u}\tilde{s})_x = 0$ for the (continuous interpolation of) carrier wave number $\tilde{u}(\eta, x, t)$ in a breather or soliton gas.

Finally, we derive multicomponent hydrodynamic reductions of the kinetic equation for breather gas and obtain the solution to a “shock tube” problem consisting of three disparate constant states separated by two propagating contact discontinuities satisfying appropriate Rankine-Hugoniot conditions.

In this work, we do not consider particular realizations $\psi_n(x, t)$, $n \gg 1$, of the nonlinear random wave field in a soliton or breather gas, which would depend on a specific choice of the initial phase vector $\theta^{(0)} \in \mathbb{T}^n$ of ψ_n (see Sec. II for details). In the construction of a spatially homogeneous breather or soliton gas, it would be natural to assume that the components of the initial phase vector $\theta^{(0)}$ are independent random variables, with $\theta^{(0)}$ being uniformly distributed on \mathbb{T}^n . In the soliton gas limit $n \rightarrow \infty$, the uniform distribution of phases on \mathbb{T}^n gets replaced by a suitably normalized Poisson distribution on \mathbb{R} ; see [53]. The spectral theory developed in this paper, however, is not based on any assumptions about the phases of finite-gap solutions involved.

Due to the fundamental nature of the fNLS equation as a universal mathematical model describing nonlinear wave processes in a broad range of dispersive media, the theory developed in this paper can find applications in various physical contexts, particularly nonlinear optics and deep water waves, where complex statistical ensembles of breathers or

solitons represent ubiquitous phenomena observed in both experimental and natural conditions.

II. FINITE-GAP SOLUTIONS: BASIC SPECTRAL CONFIGURATION AND NONLINEAR DISPERSION RELATIONS

In this section, we shall describe the basic mathematical objects necessary for the derivation of the nonlinear dispersion relations and the equation of state of the breather gas. We consider the fNLS equation in the form

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \quad (3)$$

where $\psi(x, t)$ is a complex wave field. Various families of exact solutions to the fNLS equation are available due to its integrability via the IST [27]. The key step in the integration of the fNLS equation by the IST is the determination of the spectrum of a linear (Dirac) operator with the potential $\psi(x, t)$ —the ZS scattering problem. The fNLS evolution (3) is characterized by the ZS spectrum, which has a very simple time dependence.

The original IST method enables the construction of fNLS solutions in the class of functions (potentials) decaying sufficiently rapidly at infinity. The long-time asymptotics of such solutions include solitons (discrete ZS spectrum) and linear dispersive waves (continuous spectrum). Various methods including the Darboux transformation and Hirota’s bilinear method enable the construction of exact solutions describing solitons on finite background, also known as breathers (see, e.g., [21,36–39]).

An extension of the IST to a certain class of periodic and quasiperiodic potentials, the so-called finite-gap theory (FGT) [22,28], enables the construction of a broad range of nondecaying NLS solutions, which include solitons and breathers as some particular, limiting cases. The ZS spectrum of finite-gap solutions consists of a finite Schwarz-symmetric collection of curvilinear segments $\gamma_j \subset \mathbb{C}$ called bands. Here Schwarz symmetry means that if $z \in \mathbb{C}$ is a point of the spectrum, then so is the complex conjugate (c.c.) point \bar{z} .

An n -gap solution $\psi = \psi_n(x, t)$ of (3) is defined by a fixed set of $2(n+1)$ end points of spectral bands γ_j , $j = 1, \dots, n+1$, and depends on n real phases $\theta(x, t) = \mathbf{k}x + \omega t + \theta^0$ with the initial phase vector $\theta^0 \in \mathbb{T}^n$, so that $|\psi_n(x, t)| = F_n(\theta(x, t))$, where F_n is a multiphase (quasiperiodic) function in both x and t that can be expressed in terms of the Riemann theta functions [40]. The n -component wave number \mathbf{k} and the frequency ω vectors depend on the endpoints $\{\alpha_j, j = 0, 1, 2, \dots, n\}$ of the spectral bands, which define a hyperelliptic Riemann surface \mathcal{R} of genus n given by

$$R(z) = \prod_{j=0}^n (z - \alpha_j)^{\frac{1}{2}} (z - \bar{\alpha}_j)^{\frac{1}{2}}, \quad \alpha_j = a_j + ib_j, \quad b_j > 0, \quad (4)$$

$z \in \mathbb{C}$ being a complex spectral parameter in the ZS scattering problem; $R(z) \sim z^{2n+1}$ as $z \rightarrow \infty$. The branch cuts of $R(z)$ will be specified below. Finite-gap theory of the focusing NLS equation, originally developed in [40,41], has been realized in [42] as a powerful analytical tool for the understanding

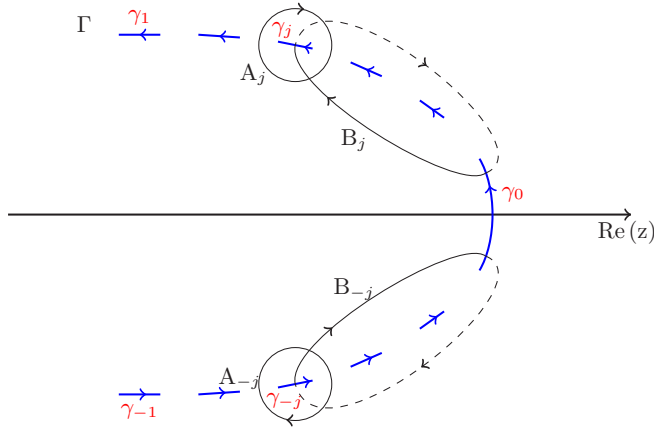


FIG. 1. The spectral bands $\gamma_{\pm j}$ and the basis of cycles $A_{\pm j}, B_{\pm j}$. The 1D curve Γ consists of the bands γ_j and gaps between the bands.

of the fundamental phenomenon of modulational instability. Finite-gap NLS solutions have since been used in a number of physical applications, notably in water waves (see [23] and references therein) and in fiber optics [43,44]. We note here that the finite-gap theory provides a natural framework for the construction of random solutions to the NLS equation by assuming a uniform distribution of the initial phase vector $\theta^0 \in \mathbb{T}^n$ [24,45,46].

We initially assume that all spectral bands lie along a finite, one-dimensional (1D) Schwarz-symmetric curve Γ (see Fig. 1); later this restriction will be removed to allow the bands to be located in a 2D compact subset of \mathbb{C} . The order we use to enumerate the bands along Γ is shown in Fig. 1. The curve Γ does not have to be a connected curve; the exact meaning of this statement will become clear in Sec. III. We also assume that the genus n is even, $n = 2N$, $N \in \mathbb{N}$; the transition to an odd genus case will be described below. We enumerate the branch points according to the following notations for the spectral bands. The spectral band γ_j , $j = 1, \dots, N$, is defined as the segment of Γ in the upper half-plane \mathbb{C}^+ connecting the branch points α_{2j} and α_{2j+1} ; the spectral band γ_{-j} is Schwarz symmetric to γ_j —it connects, naturally, the branch points $\bar{\alpha}_{2j}$ and $\bar{\alpha}_{2j+1}$, which, for convenience, we will denote as α_{-2j} and α_{-2j-1} , respectively. Finally, there is an exceptional band γ_0 crossing the real axis and connecting the branch points α_1 and $\alpha_{-1} = \bar{\alpha}_1$. An odd genus case can then formally be considered by collapsing the exceptional band into a point on the real axis.

Note that generally one can have more than one exceptional band, $\gamma_0^1, \gamma_0^2, \dots, \gamma_0^m$. We remark that, in the literature, the exceptional bands are sometimes associated with the so-called Stokes modes [23] since, considered in isolation, γ_0 represents a “spectral portrait” of the plane-wave solution of the fNLS equation, also called a Stokes wave.

We can now define spectral gaps c_j , $j = 1, \dots, N$ as segments of Γ connecting the end point α_{2j+1} of γ_j with the end point α_{2j+2} of the neighboring band γ_{j+1} (or α_1 of the exceptional band γ_0 if $j = N$), and their c.c. (with $j < 0$).

We define two sets of contours on the Riemann surface \mathcal{R} : the A_j -contours surrounding the bands γ_j clockwise, $|j| =$

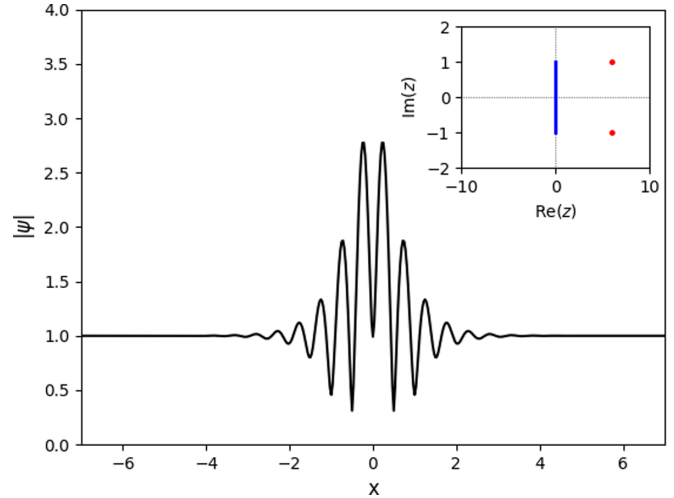


FIG. 2. Tajiri-Watanabe (TW) breather (a soliton on finite background). Shown is $|\psi_{\text{TW}}(x, 0)|$ and the spectral portrait (inset). The spectral parameters are $\alpha_1 = i, \alpha_2 = 6 + 0.996i$.

$0, \dots, N$, and the B_j -contours, $j = 1, \dots, N$, so that B_j starts at the exceptional band γ_0 , goes to the band γ_j on the upper sheet clockwise, and returns to the band γ_0 on the lower sheet of the Riemann surface. For $j < 0$ we define $B_j = \bar{B}_{-j}$ with the same clockwise orientation; see Fig. 1.

We notice that the contours A_j , $|j| = 1, \dots, N$, and $-B_j$, B_{-j} , $j = 1, \dots, N$, form a homology basis of the Riemann surface \mathcal{R} with the branch cuts along γ_j , $|j| = 0, 1, \dots, N$.

Collapsing a single pair of bands $\gamma_{\pm j}$, $j > 0$, into a pair of double points ($\alpha_{\pm(2j+1)} \rightarrow \alpha_{\pm 2j}$) implies the appearance of a soliton on an $(n - 1)$ -gap solution background. Shrinking all bands $\gamma_{\pm j}$, $j = 0, 1, \dots, N$, to points corresponds to the transformation of a $2N$ -gap solution into a $2N$ -soliton solution with b_j corresponding to soliton amplitudes and $-4a_j$ corresponding to their velocities at $t \rightarrow \infty$ [cf. Eq. (4) for the definitions of a_j, b_j] provided that all a_j are distinct. If all a_j are equal (without loss of generality, one can assume that $a_j = 0 \forall j$, i.e., each γ_j is an interval on the imaginary axis), the limiting N -soliton solution is a bound state [27,47], in which case all the solitons do not separate as $t \rightarrow \infty$. We note that in the soliton limit the exceptional band γ_0 collapses to the origin (the collapsed band can be associated with a zero-amplitude soliton) and so it does not contribute to the limiting N -soliton solution. If the exceptional band γ_0 remains finite, then collapsing all the other bands into double points corresponds to an N -breather limit of a $2N$ -gap solution with the finite band γ_0 being “responsible” for the background (indeed, the genus 0 solution with a single band $\gamma_0 = [-iq, iq]$, $q > 0$, in the spectrum is the so-called plane wave, or the condensate, given by $\psi = qe^{i2q^2t}$). The generic “elementary” breather corresponding to a degenerate genus 2 solution is the so-called Tajiri-Watanabe (TW) breather [21] with the typical behavior of the amplitude $|\psi_{\text{TW}}(x, t)|$ shown in Fig. 2. The spectral portrait of the TW breather (shown in the inset of Fig. 2) consists of the vertical band connecting the points $\pm iq$, and two double points: $\lambda = a + ib$ and its c.c. (note that $\lambda = \alpha_2, \bar{\lambda} = \alpha_{-2}$ within our general finite-gap construction). The analytical expression for the TW

breather solution is available elsewhere (see, e.g., [21,48,49]). Here we only present its group (envelope) and phase (carrier) velocities:

$$c_g = -2 \frac{\text{Im}[\lambda R_0(\lambda)]}{\text{Im}R_0(\lambda)} \equiv s_{\text{TW}}(\lambda), \quad c_p = -\frac{2 \text{Re}[\lambda R_0(\lambda)]}{\text{Re}R_0(\lambda)}, \quad (5)$$

where $R_0(\lambda) = \sqrt{\lambda^2 + q^2}$. The “standard” breathers, such as the Akhmediev breather (AB), the Kuznetsov-Ma (KM) breather, and the Peregrine soliton (PS), all of which play a prominent role in rogue-wave theories [50,51], are particular cases of the TW breather with $\gamma_0 = [-iq; iq]$ for some $q > 0$, and the double points $\alpha_{\pm 2} = \pm ip$, $p > 0$, with $p < q$ (AB), $p > q$ (KM), and $p = q$ (PS). The transition from the TW breather solution to the fundamental soliton is achieved by $q \rightarrow 0$. The fNLS fundamental soliton solution is given by [27]

$$\psi_S(x, t) = 2ib \text{sech}[2b(x + 4at - x_0)] e^{-2i[ax + 2(a^2 - b^2)t + i\phi_0]}, \quad (6)$$

where x_0 is the initial position of the soliton, and ϕ_0 is its initial phase. The soliton (envelope) group velocity is $c_g = -4a = -4 \text{Re}\lambda$ and the carrier phase velocity of a moving ($a \neq 0$) soliton is $c_p = (b^2 - a^2)/a = -2 \text{Re}(\lambda^2)/\text{Re}\lambda$, in full agreement with the respective TW breather expressions (5) in the limit $q \rightarrow 0$.

The wave number and frequency vectors, \mathbf{k} and $\boldsymbol{\omega}$, respectively, associated with a given finite-gap solution $\psi_{2N}(x, t)$ are not uniquely defined, as any linear combination of the wave-number (frequency) vector components with integer coefficients is also a wave number (frequency). Here we introduce two special vectors: $\mathbf{k} = (k_1, \dots, k_N, \tilde{k}_1, \dots, \tilde{k}_N)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N, \tilde{\omega}_1, \dots, \tilde{\omega}_N)$, whose components are defined as follows (see Appendix A 2 for details):

$$k_j = - \oint_{A_j} dp, \quad \omega_j = - \oint_{A_j} dq, \quad j = 1, \dots, N, \quad (7)$$

$$\tilde{k}_j = \oint_{B_j} dp, \quad \tilde{\omega}_j = \oint_{B_j} dq, \quad j = 1, \dots, N. \quad (8)$$

The signs of the integrals in (7) and (8) will be opposite if we replace j by $-j$. Here $dp(z)$ and $dq(z)$ are the meromorphic quasimomentum and quasienergy differentials with the only poles at $z = \infty$ on both sheets, and defined by (see, e.g., [33,35])

$$dp = 1 + O(z^{-2}), \quad dq = 4z + O(z^{-2}) \quad (9)$$

near $z = \infty$ on the main sheet, respectively, and the normalization conditions requiring that all the periods of dp, dq are real (real normalized differentials). The wave numbers and frequencies can be symmetrically extended to negative indices by $k_{-j} = k_j, \omega_{-j} = \omega_j, j = 1, \dots, N$, and similar equations for the “tilded” quantities. They also satisfy the corresponding Eqs. (7) and (8), but the signs of the integrals in (7) and (8) will be opposite when we replace j by $-j$.

The proof that k_j, \tilde{k}_j, ω , and $\tilde{\omega}$ defined by (7), (8) are indeed wave numbers and frequencies of the finite-gap fNLS solution associated with the spectral surface \mathcal{R} of (4) can be found in Appendix B. We shall call the special set of wave numbers and frequencies defined by (7), (8) *the fundamental wave-number-*

frequency set. We note that the wave numbers and frequencies defined by (7) and those defined by (8) are of an essentially different nature, which is clarified below.

Let us introduce two new quantities,

$$\eta_j = \frac{1}{2}(\alpha_{2j} + \alpha_{2j+1}), \quad \delta_j = \frac{1}{2}(\alpha_{2j} - \alpha_{2j+1}), \quad (10)$$

where $j = 1, \dots, N$. We shall call the point η_j the center of the j th band γ_j , and $2|\delta_j|$ the j th bandwidth. In the lower half-plane, we denote $\eta_{-j} = \bar{\eta}_j$ and $\delta_{-j} = \bar{\delta}_j$. We also denote the point of intersection of Γ with the real axis as η_0 and the end of the exceptional band α_1 as $\eta_0 + \delta_0$.

It then follows that the wave numbers and frequencies defined by (7) and (8) have drastically different asymptotic properties in the soliton or breather limit, when one of the nonexceptional spectral bands collapses into a double point, $\alpha_{2i+1}, \alpha_{2j} \rightarrow \eta_j$ (see the end of this section for a qualitative explanation):

$$\begin{aligned} \delta_j \rightarrow 0 &\Rightarrow k_j, \omega_j \rightarrow 0, \\ \tilde{k}_j, \tilde{\omega}_j &= O(1), \\ j &= 1, \dots, N. \end{aligned} \quad (11)$$

In particular, for $N = 1$ (genus 2), the limit (11) ($k_1 \rightarrow 0, \omega_1 \rightarrow 0$) with a nonzero band γ_0 (i.e., $\alpha_1 \neq \bar{\alpha}_1$) corresponds to the breather limit of the corresponding two-phase nonlinear wave solution. The remaining wave number and frequency $\tilde{k}_1 = O(1), \tilde{\omega}_1 = O(1)$ correspond to the “carrier” wave of the TW breather (see Fig. 2).

Motivated by these properties for $N = 1$, we shall call the components k_j, ω_j of the wave number and the frequency vectors \mathbf{k} and $\boldsymbol{\omega}$ the *solitonic components*, and we shall call the components $\tilde{k}_j, \tilde{\omega}_j$ the *carrier components*.

It is instructive to characterize the limiting transitions from the TW breather to the AB, KM, and PS in terms of appropriate limits of the fundamental wave-number–frequency set. This will enable us later to identify special cases of breather gas such as, say, PS gas or AB gas. The limiting transitions to the standard breathers are achieved in the following ways (assuming $\text{Im}\delta_0 \neq 0$):

$$\text{TW} \rightarrow \text{AB} : \tilde{\omega}_1 \rightarrow 0, \quad \tilde{k}_1 = O(1), \quad (12)$$

$$\text{TW} \rightarrow \text{KM} : \tilde{\omega}_1 = O(1), \quad \tilde{k}_1 \rightarrow 0, \quad (13)$$

$$\text{TW} \rightarrow \text{PS} : \tilde{\omega}_1 \rightarrow 0, \quad \tilde{k}_1 \rightarrow 0. \quad (14)$$

The key role in our construction of a breather gas is played by the nonlinear dispersion relations for finite-gap fNLS solutions. In the linear wave theory, the dispersion relation connects the frequency of the linearized mode with its wave number. For nonlinear waves, these relations are more complicated, involving other parameters such as mean, amplitude, etc. [29]. In the case of integrable equations, the most natural parametrization occurs in terms of the finite-gap spectrum so that the nonlinear dispersion relations have the form $\mathbf{k} = \mathbf{k}(\boldsymbol{\alpha}), \boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{\alpha})$ [34], where the vector $\boldsymbol{\alpha}$ components are the branch points $\alpha_j, |j| = 1, \dots, 2N + 1$.

One can show that for the wave numbers and frequencies (7), (8) associated with the finite-gap fNLS solution, the nonlinear dispersion relations are given by (see Appendix C

for the proof

$$\begin{aligned} \tilde{k}_j + \sum_{|m|=1}^N k_m \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= -2 \oint_{\hat{\gamma}} \frac{\zeta P_j(\zeta)d\zeta}{R(\zeta)}, \\ \tilde{\omega}_j + \sum_{|m|=1}^N \omega_m \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= -4 \oint_{\hat{\gamma}} \frac{\zeta^2 P_j(\zeta)d\zeta}{R(\zeta)}, \end{aligned} \quad (15)$$

$$|j| = 1, \dots, N,$$

where $\hat{\gamma}$ is a large clockwise oriented contour containing Γ ,

$$P_j(z) = \varkappa_{j,1} z^{2N-1} + \varkappa_{j,2} z^{2M-2} + \dots + \varkappa_{j,2N}, \quad (16)$$

and $\varkappa_{i,j}$ are the coefficients of the normalized holomorphic differentials w_j defined by

$$w_j = [P_j(z)/R(z)]dz, \quad \oint_{A_i} w_j = \delta_{ij}, \quad i, j = \pm 1, \dots, \pm N. \quad (17)$$

Taking real and imaginary parts of (15) and using the residues on the right-hand side, we obtain separate systems for the solitonic components k_m, ω_m (7):

$$\begin{aligned} \sum_{|m|=1}^N k_m \text{Im} \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= 4\pi \text{Re} \varkappa_{j,1}, \\ \sum_{|m|=1}^N \omega_m \text{Im} \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= 8\pi \text{Re} \left(\varkappa_{j,1} \sum_{k=1}^{2N+1} \text{Re} \alpha_k + \varkappa_{j,2} \right), \end{aligned} \quad (18)$$

$$|j| = 1, \dots, N,$$

and the carrier components $\tilde{k}_m, \tilde{\omega}_m$ (8):

$$\begin{aligned} \tilde{k}_j + \sum_{|m|=1}^N k_m \text{Re} \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= -4\pi \text{Im} \varkappa_{j,1}, \\ \tilde{\omega}_j + \sum_{|m|=1}^N \omega_m \text{Re} \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= -8\pi \text{Im} \left(\varkappa_{j,1} \sum_{k=1}^{2N+1} \text{Re} \alpha_k + \varkappa_{j,2} \right), \end{aligned} \quad (19)$$

$$|j| = 1, \dots, N$$

of the wave number and frequency vectors. In particular, for $N = 1$ one can show that Eqs. (18) and (19) imply that $\omega_1/k_1 \rightarrow c_g$ and $\tilde{\omega}_1/\tilde{k}_1 \rightarrow c_p$ in the breather limit $\delta_1 \rightarrow 0$, where c_g, c_p are defined in (5).

We are now in a position to establish the key property (11) for the wave numbers and frequencies in the soliton limit. Indeed, one can see that relations (18) together with (A10) imply that, for fixed N , the solitonic wave number and frequency k_j, ω_j go to zero as the j th bandwidth $|\delta_j| \rightarrow 0$. Indeed, in this case, the integral $\oint_{B_j} \frac{P_j}{R} dz$ behaves like $\ln \delta_j$ due to the contour B_j crossing the shrinking band γ_j (see Fig. 1), whereas the remaining integrals (the coefficients of the linear system) remain bounded. Thus, to keep a balance of terms in (18), k_j and ω_j must go to zero together with δ_j , whereas the carrier wave numbers \tilde{k}_j and frequencies $\tilde{\omega}_j$ given by system (19) generally remain $O(1)$.

We note that the relations similar to Eqs. (18) for the solitonic components of the wave number and frequency vectors also arise in the finite-gap KdV theory [52], where they follow from the relations between real periods and imaginary quasiperiods of the finite-gap KdV solution, with the mapping between the two being realized by the Riemann period matrix. Equations (19) do not have a KdV counterpart.

III. THERMODYNAMIC SPECTRAL SCALINGS

A. 1D case

We shall refer to the configuration described in the previous section, when the spectrum of the finite-gap potential is located on a Schwarz symmetric curve $\Gamma \subset \mathbb{C}$, as the 1D case. While being quite restrictive, this configuration provides major insight into the spectral properties of breather and soliton gases in the more physically realistic 2D case, where the bands γ_j are located in some (Schwarz symmetric) region $\Lambda \subset \mathbb{C}$. It also covers the case of a bound-state soliton or breather gas, when Γ lies on a vertical line (so that all the solitons in the gas have the same velocity).

Recall that we assumed an even genus, $n = 2N$. Due to the symmetry of the curve Γ (which may consist of several arcs), it is sufficient to consider only the upper complex half-plane (\mathbb{C}^+) part of it, which we denote Γ^+ (so that $\Gamma^+ = \Gamma \cap \mathbb{C}^+$). We shall be interested in a special, large- N limit of the nonlinear dispersion relations (18). The main requirements of this limit are that all the gaps cannot shrink faster than $O(N^{-1})$, and all but finitely many of them are of the order $O(N^{-1})$. At the same time, all but finitely many bands are of order much smaller than $O(N^{-1})$. We assume that the only bands with the width $O(1)$ are the exceptional bands, i.e., the Stokes bands crossing the real axis. In what follows, we shall be assuming that there is at most one exceptional band γ_0 , although our results could be readily extended to the case of any finite number of exceptional bands.

We assume that the shrinking bands fill all the curve Γ except the exceptional band γ_0 and the gaps adjacent to it with some density $\varphi > 0$. In a more general setting, one can consider the bands $\gamma_j, j \neq 0$, filling only certain (Schwarz symmetrical) parts of Γ , separated from each other by exceptional order 1 gaps. With slight abuse of notations, we will keep the notation Γ for the locus of accumulation of the bands γ_j on the original curve Γ . We also assume $\varphi(\mu)|d\mu|$ to be a probability measure on Γ^+ . Since collapsing a Schwarz-symmetric pair of (nonexceptional) bands into a pair of complex conjugate double points corresponds to the appearance of a soliton, and the finite exceptional band, considered in isolation, corresponds to the plane wave, it is natural to associate the finite-band potential with $N \gg 1$ and all but one band being close to ‘collapse’ with the gas of solitons on a finite background, i.e., breather gas. In the case of no exceptional bands, we will have the soliton gas limit, and in the case of more than one exceptional band, we will have the generalized breather gas limit. The generalized breather gas can be viewed as a gas of solitons on the background of n -gap fNLS solution with $n \geq 1$. Indeed, multiple exceptional bands, considered in isolation, would represent the corresponding finite-gap solutions. Considered together

with collapsing (in the appropriate $n \rightarrow \infty$ limit) bands, one obtains soliton gas on the finite-gap background.

We now assume that for $N \gg 1$, the centers η_j of the bands γ_j , $j = 1, \dots, N$, are distributed along Γ^+ with some limiting density $\varphi(\mu) > 0$, $\mu \in \Gamma^+$, that is smooth on Γ^+ . It then follows that $|\eta_j - \eta_{j+1}| \sim 1/N$.

As for the scaling of the bandwidths, we consider the following options:

(i) *Exponential spectral scaling*: the bandwidths $2|\delta_j|$ of γ_j are exponentially narrow in N :

$$|\delta_j| \sim e^{-N\tau(\eta_j)}, \quad |j| = 1, \dots, N, \quad (20)$$

where $\tau(\mu)$ is a smooth positive function on Γ^+ having the meaning of the normalised logarithmic bandwidth [$\tau(\eta_j) \sim -\ln|\delta_j|/N$]. The limit obtained in this scaling will be referred to as a (regular) soliton or a breather gas limit depending on the size of the exceptional band (in the soliton gas limit $\delta_0 \rightarrow 0$).

(ii) *Subexponential spectral scaling*: for any $a > 0$,

$$e^{-aN} \ll |\delta_j| \ll \frac{1}{N}, \quad |j| = 1, \dots, N. \quad (21)$$

We shall refer to the $N \rightarrow \infty$ limit obtained in this scaling as a soliton (breather) condensate limit (the reasons for this name will become clear later). It is clear that in this limit, $\tau(\eta) \rightarrow 0$.

(iii) *Superexponential spectral scaling*: for any $a > 0$,

$$e^{-aN} \gg |\delta_j|, \quad |j| = 1, \dots, N. \quad (22)$$

The limit obtained in this scaling will be referred to as an ideal breather or soliton gas limit. In this limit, $\tau(\eta) \rightarrow \infty$.

Of course, one can also consider the case of simultaneous different scalings on different parts Γ^+ (a ‘‘mixed’’ scaling). Even though such cases can be very interesting, we will not discuss them in any detail here.

Note that in all three scalings, $|\delta_j| \ll |\eta_j - \eta_{j+1}|$ so the width of gaps $|c_j| \sim N^{-1}$ and so $|\delta_j|/|c_j| \rightarrow 0$ as $N \rightarrow \infty$. We then say that in the limit, each collapsed band $\gamma_j \rightarrow \eta_j$ corresponds to a soliton (breather) state within a soliton (breather) gas. Invoking the interpretation of classical solitons as elastically interacting wave particles, these states can be viewed as quasiparticles that do not necessarily manifest as localized entities in physical space, except in the case of a rarefied gas. We also note that the exponential and subexponential scalings have the ‘‘thermodynamic’’ property in the sense that they preserve finiteness of the total density of waves $K_N = \sum_{j=1}^N k_j$ in the limit $N \rightarrow \infty$ so that $\lim_{N \rightarrow \infty} K_N = K_\infty$, where $0 < K_\infty < \infty$. Note that for the superexponential scaling, $K_\infty = 0$.

B. 2D case

In the case when the shrinking bands γ_j , $j > 0$ fill a compact 2D region Λ^+ of the upper complex half-plane, the counterpart of the exponential scaling (20)

$$|\delta_j| \sim e^{-N^2\tau(\eta_j)}. \quad (23)$$

Here $\tau(\eta)$ is a positive smooth function on Λ^+ . The scaling of the gaps remains $O(1/N)$, where by the gap width we understand the closest distance between the bands. In this case, $\varphi(\eta)$ is the 2D density of bands (and we also distinguish the cases of

exponential, subexponential, and superexponential scalings of bands, similarly to the 1D case). We assume $\varphi(\eta) > 0$ on Λ^+ . We shall call such scalings the 2D thermodynamic scalings. In what follows, we shall be using the unified notations Γ^+ for both 1D and 2D configurations, explicitly distinguishing between these cases when necessary.

IV. NONLINEAR DISPERSION RELATIONS FOR BREATHER AND SOLITON GAS

We now proceed with the characterization of breather and soliton gases as thermodynamic limits of finite-gap solutions of fNLS equation. For convenience, we will sometimes be using the term ‘‘breather gas’’ in a generalized sense, assuming that it includes soliton gas as well, the transition to soliton gas being achieved by letting $\delta_0 \rightarrow 0$. We also note that the full construction of a breather gas [which ultimately includes the determination of the random wave field $\psi(x, t)$] implies, along with the thermodynamic spectral scaling, the random phase approximation, i.e., the uniform distribution of the phase vector $\theta^{(0)} \in \mathbb{T}^n$ [24,45,46,53], but in this paper we shall be concerned only with the spectral characterization of breather gases. The description of the associated fNLS solutions [the random process $\psi(x, t)$ generated by the thermodynamic spectral scaling and the uniform phase distribution] will be the subject of a separate work. We only mention here that the uniform phase distribution on \mathbb{T}^n gets replaced in the thermodynamic limit by the appropriately normalized Poisson distribution on \mathbb{R} as shown in [53] for the KdV equation.

A. 1D case

We now apply the 1D thermodynamic spectral scalings to the nonlinear dispersion relations (18). Without much loss of generality, from now on we shall be assuming that $\gamma_0 \subset i\mathbb{R}$, i.e., that the exceptional band γ_0 lies on the imaginary axis.

We scale the solitonic wave numbers and frequencies as

$$k_j = \frac{\kappa_j}{N}, \quad \omega_j = -\frac{\nu_j}{N}, \quad N \gg 1, \quad (24)$$

so that $\kappa_j = \kappa(\eta_j)$ and $\nu_j = \nu(\eta_j)$, where the functions $\kappa(\eta) \geq 0$ and $\nu(\eta)$ are smooth interpolations of κ_j, ν_j . We note that the existence of the interpolating functions $\kappa(\eta)$, $\nu(\eta)$, and the non-negativity of $\kappa(\eta)$ are physically reasonable assumptions that need to be justified mathematically. A clarifying comment will be presented later, when the (integral) equations (28) and (29) for $\kappa(\eta)$ and $\nu(\eta)$ are derived. We also note that the sign of $\nu(\eta)$ is not fixed.

The $1/N$ scaling (24) for the wave numbers and frequencies follows from the requirement that the diagonal and off-diagonal terms of systems (18) contribute at the same order; this scaling is consistent with the exponential rate of shrinking of the bands given in Eq. (20) since $k_j \sim -\ln^{-1}|\delta_j|$ for $|\delta_j| \ll 1$. The scalings different from $1/N$ for k_j, ω_j are possible in the cases of subexponential or superexponential spectral distributions given by Eqs. (21) and (22), respectively. The functions $\kappa(\eta)$ and $\nu(\eta)$ in (24) are determined from $\tau(\eta)$ and $\varphi(\eta)$, and the geometry of Γ^+ , as we will now explain.

We denote $R_0(z) = \sqrt{z^2 - \delta_0^2}$ with the branch cut $[-\delta_0, \delta_0]$, where $\delta_0 \in i\mathbb{R}^+$ and the branch of the radical is defined by $R_0(z) \rightarrow z$ as $z \rightarrow \infty$. Then, applying the limit $N \rightarrow \infty$ to the nonlinear dispersion relations (18) augmented by an

exponential spectral scaling (20) leads to the following relations (see Appendix D for details of the derivation):

$$i \int_{\eta_\infty}^{\eta_1} \left[\ln \frac{\bar{\mu} - \eta}{\mu - \eta} + \ln \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\eta)R_0(\bar{\mu}) + \eta\bar{\mu} - \delta_0^2} + i\pi \chi_\eta(\mu) \right] \times u(\mu)|d\mu| + i\sigma(\eta)u(\eta) = R_0(\eta) + \tilde{u}(\eta), \quad (25)$$

$$i \int_{\eta_\infty}^{\eta_1} \left[\ln \frac{\bar{\mu} - \eta}{\mu - \eta} + \ln \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\eta)R_0(\bar{\mu}) + \eta\bar{\mu} - \delta_0^2} + i\pi \chi_\eta(\mu) \right] \times v(\mu)|d\mu| + i\sigma(\eta)v(\eta) = -2\eta R_0(\eta) + \tilde{v}(\eta), \quad (26)$$

where \tilde{u} and \tilde{v} are some smooth functions on Γ^+ interpolating $\tilde{k}_j, \tilde{\omega}_j$, specifically, $2\tilde{u}(\eta_j) = \tilde{k}_j$, $-2\tilde{v}(\eta_j) = \tilde{\omega}_j$, $j = 1, \dots, N$, the integration is performed between $\eta_\infty = \lim_{N \rightarrow \infty} \eta_N$ and η_1 along Γ^+ , χ_η denotes the characteristic (indicator) function of the arc (η_∞, η) of Γ^+ , and

$$u(\eta) = \frac{1}{2\pi} \kappa(\eta)\varphi(\eta), \quad v(\eta) = \frac{1}{2\pi} v(\eta)\varphi(\eta), \quad \sigma(\eta) = \frac{2\tau(\eta)}{\varphi(\eta)}. \quad (27)$$

Note that $\ln \frac{\bar{\mu} - \eta}{\mu - \eta} + i\pi \chi_\eta(\mu)$ varies smoothly as μ traverses the contour Γ^+ bypassing $\eta \in \Gamma^+$ from the left (positive) side of Γ^+ . Equations (25) and (26) represent general complex nonlinear dispersion relations for breather gas. They specify four unknown functions $u(\eta)$, $v(\eta)$, $\tilde{u}(\eta)$, and $\tilde{v}(\eta)$ in terms of a single non-negative smooth function $\sigma(\eta)$ and a contour Γ^+ characterizing the Riemann surface \mathcal{R} of (4) in the thermodynamic limit. The function $u(\eta) \geq 0$ defined in Eq. (27) has the meaning of the *density of states* [12,13] as $u(\eta_*)|d\eta|$ is the number of localized (soliton or breather) states located in the spectral interval $[\eta_*, \eta_* + d\eta] \subset \Gamma^+$ and c.c. per unit interval of space (provided the gas parameters do not depend on x). At the intuitive level, one can think of the density of states in a soliton gas as of the spectral distribution of solitons ‘‘contained in a portion of gas of unit length’’ by first assuming the zero boundary conditions at $x = \pm L$, where $L \gg 1$, and then normalizing the obtained distribution by L . This should be modified for breather gas by replacing zero boundary conditions with the plane-wave boundary conditions. The integral $\int_{\eta_\infty}^{\eta_1} u(\eta)|d\eta|$ gives the total density of waves K_∞ introduced earlier. The function $v(\eta)$ then represents the temporal counterpart of the density of states. The functions $\tilde{u}(\eta), \tilde{v}(\eta)$ can be interpreted as the carrier wave number and carrier frequency spectral functions of a breather gas. Note that the integral term in (25) and (26) corresponds to the off-diagonal terms of the linear systems (15), whereas the nonintegral (secular) terms on the left-hand sides of (25) and (26) correspond to the diagonal terms of (15).

Taking the imaginary part of Eqs. (25) and (26), we obtain the solitonic component of the breather gas nonlinear dispersion relations

$$\int_{\Gamma^+} \left[\ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| + \ln \left| \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\bar{\eta})R_0(\mu) + \bar{\eta}\mu - \delta_0^2} \right| \right] \times u(\mu)|d\mu| + \sigma(\eta)u(\eta) = \text{Im}R_0(\eta), \quad (28)$$

$$\int_{\Gamma^+} \left[\ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| + \ln \left| \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\bar{\eta})R_0(\mu) + \bar{\eta}\mu - \delta_0^2} \right| \right] \times v(\mu)|d\mu| + \sigma(\eta)v(\eta) = -2 \text{Im}[\eta R_0(\eta)], \quad (29)$$

where, with a slight abuse of notation, we denoted $\int_{\eta_\infty}^{\eta_1} \dots |d\mu| \equiv \int_{\Gamma^+} \dots |d\mu|$. We recall here that the interpolation function $\kappa(\eta)$ was assumed to be non-negative, which, together with positivity of $\varphi(\eta)$, implies non-negativity of the function $u(\eta) = \kappa(\eta)\varphi(\eta)$ defined by Eq. (28). This is actually a nontrivial fact that, generally speaking, needs to be proven. Moreover, the very existence of solutions to the integral equations (28) and (29), justifying the existence of the interpolating functions $\kappa(\eta)$ and $v(\eta)$ for the scaled wave numbers and the frequencies (24), is not obvious. In this paper, we shall present a number of physically relevant explicit solutions to these equations, relegating the general mathematical proofs to an upcoming publication.

Taking now the real parts of Eqs. (25) and (26), we obtain the carrier component of the breather gas dispersion relations

$$\int_{\Gamma^+} \left[\arg \frac{\mu - \eta}{\bar{\mu} - \eta} - \arg \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\eta)R_0(\bar{\mu}) + \eta\bar{\mu} - \delta_0^2} - \pi \chi_\eta(\mu) \right] \times u(\mu)|d\mu| = \text{Re}R_0(\eta) + \tilde{u}(\eta), \quad (30)$$

$$\int_{\Gamma^+} \left[\arg \frac{\mu - \eta}{\bar{\mu} - \eta} - \arg \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\eta)R_0(\bar{\mu}) + \eta\bar{\mu} - \delta_0^2} - \pi \chi_\eta(\mu) \right] \times v(\mu)|d\mu| = -2 \text{Re}[\eta R_0(\eta)] + \tilde{v}(\eta), \quad (31)$$

which relate the unknown functions $\tilde{u}(\eta)$ and $\tilde{v}(\eta)$ with the density of states $u(\eta)$ and its temporal analog $v(\eta)$.

In the case $\delta_0 \rightarrow 0$, which corresponds to the transition from a breather gas to a soliton gas, the solitonic dispersion relations (28) and (29) simplify to

$$\int_{\Gamma^+} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| u(\mu)|d\mu| + \sigma(\eta)u(\eta) = \text{Im}\eta, \quad (32)$$

$$\int_{\Gamma^+} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| v(\mu)|d\mu| + \sigma(\eta)v(\eta) = -4 \text{Im}\eta \text{Re}\eta. \quad (33)$$

These are analogs of similar equations obtained in [11] for soliton gas of the KdV equation.

The corresponding dispersion relations for the carrier components become

$$\int_{\Gamma^+} \left[\arg \frac{\mu - \eta}{\bar{\mu} - \eta} - 2 \arg \mu - \pi \chi_\eta(\mu) \right] u(\mu)|d\mu| = \text{Re}\eta + \tilde{u}(\eta),$$

$$\int_{\Gamma^+} \left[\arg \frac{\mu - \eta}{\bar{\mu} - \eta} - 2 \arg \mu - \pi \chi_\eta(\mu) \right] v(\mu)|d\mu| = -2 \text{Re}\eta^2 + \tilde{v}(\eta). \quad (34)$$

These equations do not have a KdV counterpart as the KdV solitons do not have a carrier component. Note also that in the (ideal gas) limit $u, v \rightarrow 0$ the expressions for $\tilde{k}_j = 2\tilde{u}(\eta_j) = -2 \text{Re}\eta$ and for $\tilde{\omega}_j = -2\tilde{v}(\eta_j) = -4 \text{Re}\eta_j^2$ coincide with the respective coefficients in front of x and t of the phase ϕ of the oscillatory factor $e^{i\phi}$ in the fundamental soliton formula (6).

The nonlinear dispersion relations (28) describe the gas of generic moving breathers, which can also be called the TW breather gas. By considering the particular spectral limits motivated by the limiting cases of the TW breather [Eqs. (12)–(14)], one can derive the reductions of nonlinear dispersion relations describing gases of Akhmediev, Kuznetsov-Ma, and

Peregrine breathers. A special case of a soliton gas—the bound-state soliton gas—when all the solitons in the gas move with the same velocity, will be considered in detail later on.

B. 2D case

We now relax the basic restriction imposed on the spectrum locus that was used for the derivation of the dispersion relations and the equation of state of the breather and soliton gases, namely the requirement that $\eta \in \Gamma$, where Γ is a 1D Schwarz-symmetric curve in the complex plane. We now assume one of the 2D spectral thermodynamic spectral scalings (exponential, subexponential, and superexponential) when the shrinking bands γ_j fill a 2D region Λ of the complex plane; see Sec. III B. The exponential 2D spectral scaling is given by Eq. (23) while the gaps are scaled as $O(N^{-1})$ (see Sec. III B). For the wave numbers and frequencies instead of (24) we introduce

$$k_j = \frac{\kappa_j}{N^2}, \quad \omega_j = -\frac{\nu_j}{N^2}, \quad N \gg 1, \quad (35)$$

where $\kappa_j = \kappa(\eta_j)$ and $\nu_j = \nu(\eta_j)$, and the interpolating functions $\kappa(\eta) \geq 0$, $\nu(\eta) \geq 0$ are assumed to be smooth on Λ^+ .

Then the 2D thermodynamic limit of the nonlinear dispersion relations (18) leads to the same integral equations (28)–(37) but with the line integration along Γ^+ replaced by the integration over a 2D compact domain Λ^+ :

$$\int_{\Gamma^+} \cdots |d\mu| \rightarrow \iint_{\Lambda^+} \cdots d\xi d\zeta, \quad (36)$$

where $\mu = \xi + i\zeta$. The density of states $u(\eta)$ in the 2D case is defined in such a way that $u(\mu^*)d\xi d\zeta$ gives the number of localized (breather or soliton) states per the element $[\xi^*, \xi^* + d\xi] \times [\zeta^*, \zeta^* + d\zeta]$ of the spectral complex plane and per unit interval of space, assuming spatially uniform gas.

For convenience of the exposition, in what follows we shall be using the 1D notation $\int_{\Gamma^+} \cdots |d\mu|$ in both the 1D and 2D cases, keeping in mind that in the 2D case the meaning of the integral is given by Eq. (36).

V. EQUATION OF STATE

We now look closer at the solitonic component of dispersion relations for breather and soliton gases [Eqs. (28) and (32), respectively]. In both cases, elimination of $\sigma(\eta)$ yields a single relation:

$$s(\eta) = s_0(\eta) + \int_{\Gamma^+} \Delta(\eta, \mu) [s(\eta) - s(\mu)] u(\mu) |d\mu|, \quad (37)$$

where $s(\eta) = \nu(\eta)/u(\eta)$, and $s_0(\eta)$ and $\Delta(\eta, \mu)$ are defined as follows.

For breather gas,

$$s_0(\eta) = -2 \frac{\text{Im}[\eta R_0(\eta)]}{\text{Im}R_0(\eta)},$$

$$\Delta(\eta, \mu) = \frac{1}{\text{Im}R_0(\eta)} \left[\ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| + \ln \left| \frac{R_0(\eta)R_0(\mu) + \eta\mu - \delta_0^2}{R_0(\bar{\eta})R_0(\mu) + \bar{\eta}\mu - \delta_0^2} \right| \right], \quad (38)$$

and for soliton gas [obtained from (38) by letting $\delta_0 \rightarrow 0$],

$$s_0(\eta) = -4 \text{Re}\eta, \quad \Delta(\eta, \mu) = \frac{1}{\text{Im}\eta} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right|. \quad (39)$$

The relation (37) complemented by (38) or (39) can be viewed as the *equation of state* of a breather (soliton) gas.

Since $\kappa(\eta)$ and $\nu(\eta)$ are the scaled wave number and frequency, respectively, in the thermodynamic limit [see (24)], the quantity $s(\eta) = \nu(\eta)/u(\eta) = \nu(\eta)/\kappa(\eta)$ in (37) has a clear physical meaning of the mean velocity of a “tracer” breather (soliton) in a breather (soliton) gas. As we shall see, in a spatially inhomogeneous breather gas, the function $s(\eta) \equiv s(\eta, x, t)$ defined by Eq. (37) with $u(\eta) \equiv u(\eta; x, t)$ has the meaning of the gas’ transport velocity; see Eq. (87) below. For the fNLS equation, the soliton gas equation of state (37), (39) was originally proposed in [14] using physical reasoning while its KdV analog had been derived in [11] using the exponential spectral scaling analogous to (20) (as a matter of fact, the KdV spectral scaling occurs along the real axis). The derivation presented here provides, along with mathematical justification of the fNLS results of [14], their major generalization to the case of breather gas with a number of novel physical implications.

The equation of state (37) has a transparent physical interpretation. The first term, $s_0(\eta)$, has the meaning of the speed of a “free” (isolated) breather or soliton with the spectral parameter η . Indeed, in Eq. (39) $s_0(\eta) = -4 \text{Re}\eta$, which coincides with the group velocity c_g of the fNLS fundamental soliton (6), while in Eq. (38) $s_0(\eta) = -2 \frac{\text{Im}[\eta R_0(\eta)]}{\text{Im}R_0(\eta)}$ is the group velocity of the TW breather (5) [one sets $\delta_0 = iq$ in Eq. (38)]. The second (integral) term in (37) describes the modification of the “tracer” breather (soliton) mean velocity due to its interaction with other breathers (solitons) in the gas. The interaction kernel $\Delta(\eta, \mu)$ for the soliton gas [cf. the second expression in Eq. (39)] coincides with the well-known expression for the position shift in the two-soliton interactions [14,27]. We then conclude that Eq. (38) describes the position shift in a two-breather interaction. The expressions for the position shifts in the interaction of two TW breathers were recently obtained in Refs. [49,54] in a different, less explicit, form. While we could not see an obvious way to verify the equivalence between the two representations analytically, we have undertaken a numerical comparison with the representation obtained in [49] for a range of parameters, which convincingly confirmed full agreement between the two.

The fact that the pairwise interaction kernels (38) and (39) show up in the equations of state (37) without assuming any dilute nature of the gas implies that the properties of breather and soliton gases are fully determined by the “fundamental” two-particle interactions for the whole range of admissible densities. We also note that the spectral thermodynamic limit only yields the soliton-soliton (breather-breather) interaction kernel but not the kernel related to the interaction of solitons (breathers) with the continuous spectrum component, confirming thus the original premise of our paper that the thermodynamic limit of finite-gap potentials corresponds to a “genuine” soliton (breather) gas.

Finally, we note that the inequality $\sigma(\eta) \geq 0$ in Eq. (28) imposes a fundamental constraint

$$\int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu| \leq 1 \quad (40)$$

on the function $u(\eta)$.

Example: Multicomponent breather gas

Consider a breather gas characterized by the density of states in the form of a linear combination of Dirac δ -functions centered at different spectral points $\eta^{(j)}$ (we use an upper index to distinguish these spectral points from the centers of spectral bands η_j , used earlier in the thermodynamic scaling construction),

$$u(\eta) = \sum_{j=1}^M w_j \delta(\eta - \eta^{(j)}), \quad (41)$$

where $w_j > 0$ are the given components' weights and $\text{Im}\eta^{(j)} > 0 \forall j$. The structure of the multicomponent reduction of the equation of state for generalized soliton gas of the KdV type [i.e., when the integration in (37) occurs along the interval of the real line] has been studied in [55]. For a particular case of a two-component ($M = 2$) fNLS soliton gas, such a reduction has been considered in [14]. Here we present a straightforward extension of the results of [14,55] to breather gas.

Substituting (41) into the equation of state (37), we obtain a linear system for the gas' component velocities $s^{(j)} \equiv s(\eta^{(j)})$,

$$s^{(j)} = s_0^{(j)} + \sum_{m=1, m \neq j}^M \Delta_{jm} w_m (s^{(j)} - s^{(m)}), \quad j = 1, 2, \dots, M, \quad (42)$$

where $s_0^{(j)} \equiv s_0(\eta^{(j)}) = s_{\text{TW}}(\eta^{(j)})$, $\Delta_{jm} = \Delta(\eta^{(j)}, \eta^{(m)})$ [cf. (38) for $\Delta(\eta, \mu)$ in breather gas]. For $M = 2$, system (42) can be solved explicitly to give

$$\begin{aligned} s^{(1)} &= s_0^{(1)} + \frac{\Delta_{12} w_2 (s_0^{(1)} - s_0^{(2)})}{1 - (\Delta_{12} w_2 + \Delta_{21} w_1)}, \\ s^{(2)} &= s_0^{(2)} - \frac{\Delta_{21} w_1 (s_0^{(1)} - s_0^{(2)})}{1 - (\Delta_{12} w_2 + \Delta_{21} w_1)}. \end{aligned} \quad (43)$$

An important remark is in order on the meaning of the δ -function ansatz (41) for the density of states $u(\eta)$. As a matter of fact, the representation (41) is a mathematical idealization, which has a formal sense in the context of the integral equation of state (37), but it cannot be applied to the original dispersion relations (28) and (29) where it appears in both the integral and the secular terms. In a physically realistic description, the δ -functions in (41) should be replaced by some narrow distributions around the spectral points $\eta^{(j)}$, i.e., we first take the limit $N \rightarrow \infty$ and then later allow the distributions to become sharply peaked. As a result, Eq. (28) would impose a constraint (40) on $u(\eta)$ which, among other things, implies that the denominators in (43) must be positive. Other constraints could arise due to the requirements of non-negativity for some statistical parameters of the fNLS field

(see [56] for the KdV soliton gas consideration), but we do not consider them here.

VI. PROPAGATION OF AN ISOLATED SOLITON OR BREATHER THROUGH A GAS

The equation of state (37) can be used to describe the propagation of an isolated soliton (or breather) with the spectral parameter $\eta \notin \Gamma^+$ through a soliton (breather) gas with known density of states $u(\mu)$ [and the corresponding velocity $s(\mu)$ by (37)], where $\mu \in \Gamma^+$. We shall call such an isolated soliton (breather) a *trial* soliton (breather) [not to be confused with the *tracer* soliton (breather) with $\eta \in \Gamma^+$]. Apart from being a convenient tool for the numerical verification of the developed theory (see the relevant comparisons [57] for the KdV soliton gas case), the known dependence of the trial soliton (breather) velocity on η [obtained, e.g., from a series of measurements of the mean velocity of trial solitons (breathers) of different amplitudes propagating through the same gas] can be used for posing the inverse problem: recover the density of states $u(\mu)$, $\mu \in \Gamma^+$ of a breather or soliton gas from a given function $s(\eta)$, $\eta \notin \Gamma^+$, i.e., determine the gas by irradiating it with the trial breathers or solitons and measuring velocities of their propagation through the gas.

Expressing $s(\eta)$ from (37), we obtain the expression for the mean velocity of such a trial breather or soliton propagating through a respective gas:

$$s(\eta) = \frac{s_0(\eta) - \int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) s(\mu) |d\mu|}{1 - \int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu|}. \quad (44)$$

We note that technically, Eq. (37) is valid only for $\eta \in \Gamma^+$, but we can always assume that an isolated point $\eta \in \mathbb{C}^+$ can be added to Γ^+ with $u(\mu) = w\delta(\mu - \eta)$ near that point. Then, substituting this "extended" $u(\mu)$ in (37) and taking the limit $w \rightarrow 0^+$, we obtain (44).

We would like to stress an important subtlety associated with Eq. (44) that can be easily overlooked. For $\eta \in \Gamma^+$, Eq. (44) is equivalent to the equation of state (37) and thus it represents an integral equation for $s(\eta)$. However, for $\eta \notin \Gamma^+$, the right-hand side of (44) is assumed to be already known [from the solution of Eq. (37)] so in that case Eq. (44) represents an expression for $s(\eta)$ rather than an equation to be solved.

Example: Propagation of a trial breather through a one-component breather gas

Consider a trial breather with the spectral parameter $\eta = \eta^{(1)}$, where $\text{Im}\eta^{(1)} > |\delta_0|$, propagating through a one-component breather gas with $u(\eta) = w_2\delta(\eta - \eta^{(2)})$, $s(\eta^{(2)}) = s_0(\eta^{(2)}) \equiv s_0^{(2)}$ [it is clear that in a one-component gas due to the absence of interactions, the velocity of the gas coincides with the free-soliton velocity; see Eq. (44) with the interaction terms removed]. From Eq. (44) we obtain

$$s^{(1)} = \frac{s_0^{(1)} - \Delta_{12} w_2 s_0^{(2)}}{1 - \Delta_{12} w_2}, \quad (45)$$

which is consistent with Eq. (43) in the limit $w_1 \rightarrow 0^+$.

Peregrine gas. Consider now the case when $\eta^{(2)} = \delta_0 = iq$, where $q \in \mathbb{R}^+$, i.e., the "mass" w of the δ -function is at

the end of the exceptional band $\gamma_0 = [-\delta_0, \delta_0]$. This is the “spectral portrait” of the Peregrine gas. Then the logarithmic kernel in (28), evaluated at $\mu = \delta_0$, is

$$-\ln \left| \frac{\delta_0 - \eta}{\delta_0 - \bar{\eta}} \right| + \ln \left| \frac{\eta \delta_0 - \delta_0^2}{\bar{\eta} \delta_0 - \delta_0^2} \right| = 0, \quad (46)$$

where η is any point in the upper half-plane that has a branch cut $[0, \delta_0]$. Thus, by (38), $\Delta_{12} = 0$ and Eq. (45) yields $s^{(1)}(\eta) = s_0^{(1)}(\eta)$, i.e., the group velocity of the (free) TW breather. Thus, the propagation of the trial TW breather through a Peregrine gas is *ballistic*.

Kuznetsov-Ma breather gas. For the KM breather gas with $\eta^{(2)} = ip$, where $p > q$, we have $s_0^{(2)} = 0$ [see (5)], so the propagation speed of a trial breather through a KM gas is found from Eq. (45) to be $s^{(1)} = s_0^{(1)}/(1 - \Delta_{\text{KM}} w_2)$, where $\Delta_{\text{KM}} = \Delta_{12}$ with $\mu = ip$ and $p > |\delta_0|$.

Akhmediev breather gas. Finally, for the AB gas with $\eta^{(2)} = ip \pm \epsilon$, where $p \in (0, |\delta_0|)$, $\epsilon \rightarrow 0$, we have $s_0^{(2)} \rightarrow \pm \infty$ [see (5)]. Denote the interaction kernel $\Delta_{12} \equiv \Delta_{\text{AB}}$. Then the velocity of a trial breather propagating through an AB gas is $s^{(1)} \sim -\frac{\Delta_{\text{AB}} w_2}{1 - \Delta_{\text{AB}} w_2} s_0^{(2)}$ as $\epsilon \rightarrow 0$. Note that due to the infinite spatial extent of the AB, one should require that the density of the AB gas $w_2 \rightarrow 0$ so that $w_2 s_0^{(2)} = O(1)$ in the latter expression, thus ensuring finite velocity of the test breather. These are preliminary considerations as the case of AB gas requires further careful study.

The breather or soliton interactions in a breather or soliton gas not only modify the “particle” group velocity c_g of a tracer breather (soliton), but they also change the “wave” phase velocity of its carrier c_p . The phase velocity distribution in the carrier of a breather gas can be naturally defined as $\tilde{s}(\eta) = \tilde{v}/\tilde{u}$, where $\tilde{u}(\eta)$ and $\tilde{v}(\eta)$ are the continuous carrier wave number and frequency functions, respectively, satisfying the dispersion relations (30) (breather gas) and (34) (soliton gas). As a result, we obtain

$$\tilde{s}(\eta) = \frac{\tilde{s}_0 - \int_{\Gamma^+} \tilde{\Delta}(\eta, \mu) u(\mu) s(\mu) |d\mu|}{1 - \int_{\Gamma^+} \tilde{\Delta}(\eta, \mu) u(\mu) |d\mu|}, \quad (47)$$

where for breather gas,

$$\begin{aligned} \tilde{s}_0 &= -\frac{2 \operatorname{Re}[\eta R_0(\eta)]}{\operatorname{Re}[R_0(\eta)]}, \\ \tilde{\Delta}(\eta, \mu) &= \frac{1}{\operatorname{Re}[R_0(\eta)]} \left[\arg \frac{\eta - \mu}{\eta - \bar{\mu}} \right. \\ &\quad \left. - \arg \frac{R_0(\eta) R_0(\mu) + \eta \mu - \delta_0^2}{R_0(\eta) R_0(\bar{\mu}) + \eta \bar{\mu} - \delta_0^2} - \pi \chi_\eta(\mu) \right], \quad (48) \end{aligned}$$

and for soliton gas,

$$\begin{aligned} \tilde{s}_0 &= -\frac{2 \operatorname{Re}[\eta^2]}{\operatorname{Re} \eta}, \\ \tilde{\Delta}(\eta, \mu) &= \frac{1}{\operatorname{Re} \eta} \left[\arg \frac{\eta - \mu}{\eta - \bar{\mu}} - 2 \arg \mu - \pi \chi_\eta(\mu) \right] \quad (49) \end{aligned}$$

The expressions for $\tilde{s}_0(\eta)$ in (48) and (49) coincide with phase velocities c_p of a carrier in a TW breather [cf. (5)] and fundamental soliton [cf. (6)], respectively. The interaction kernels $\tilde{\Delta}$ are clearly related to (but do not coincide with) the

expressions of the phase shifts in two-breather (two-soliton) interactions (see [27,49,54]).

One cannot help noticing the similarity between Eqs. (44) and (47) defining, respectively, the group and phase velocity of a “trial” or “tracer” breather (soliton) in a breather (soliton) gas. We need to stress, however, that despite the apparent similarity, these expressions have very different structure. Indeed, Eq. (44) contains only one type of velocities (the group velocities) while Eq. (47) connects two types of velocities. Also, Eq. (44) has a different meaning depending on whether $\eta \in \Gamma^+$ (a “tracer” breather or soliton) or $\eta \notin \Gamma^+$ (a “trial” breather or soliton) [see the discussion after Eq. (44)], while Eq. (47) does not distinguish between these two types of propagating breathers or solitons.

VII. RAREFIED BREATHER OR SOLITON GAS AND SOLITON CONDENSATE

The nonlinear dispersion relations (28) and (32) for breather and soliton gas, respectively, were derived assuming the general, exponential spectral scaling (20) implying that generically, the integral and the secular terms in these relations are of the same order. In the other two scalings considered in Sec. III, one of these two terms must be subdominant: this will be the integral term in the superexponential scaling and the nonintegral term in the subexponential scaling.

It is convenient to characterize the spectral scalings and the corresponding breather or soliton gases in terms of the function $\sigma(\eta)$ parametrizing the dispersion relations (28) and (32). From Eq. (28) we have for breather gas

$$\sigma(\eta) = \frac{\operatorname{Im}[R_0(\eta)][1 - \int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu|]}{u(\eta)} \geq 0, \quad (50)$$

where the interaction kernel $\Delta(\eta, \mu)$ is given by Eq. (38). The expression for $\sigma(\eta)$ in a soliton gas is obtained from Eq. (50) by replacing $\operatorname{Im} R_0(\eta)$ with $\operatorname{Im} \eta$ and using Eq. (39) for $\Delta(\eta, \mu)$. For the exponential scaling $\sigma(\eta) = O(1)$, while the limiting cases $\sigma \rightarrow \infty$ and $\sigma \rightarrow 0$ correspond to the super- and subexponential spectral scalings, respectively.

A. Rarefied breather and soliton gases

Rarefied breather or soliton gas represents an infinite random ensemble of weakly interacting breathers or solitons characterized by small density of states, $u \ll 1$, and therefore $\sigma \gg 1$ by (50). We shall refer to the limit $u \rightarrow 0$, $\sigma \rightarrow \infty$, $u\sigma = O(1)$ as the *ideal gas limit* as it corresponds to the gas of noninteracting breathers (solitons). Spectrally this limit corresponds to the superexponential spectral scaling (22). For a rarefied gas, the interaction (integral) term in the equation of state (37) is subdominant so the leading-order term $s(\eta) = s_0(\eta)$ describes the group velocity distribution in an ideal breather (soliton) gas. Then the first correction to the ideal gas velocity $s_0(\eta)$ is readily computed to give

$$s(\eta) \approx s_0(\eta) + \int_{\Gamma^+} \Delta(\eta, \mu) [s_0(\eta) - s_0(\mu)] u(\mu) |d\mu|. \quad (51)$$

Equation (51) represents a fNLS breather or soliton gas counterpart of the equation for the soliton velocity in a rarefied KdV soliton gas obtained in [10]. Similarly, the carrier phase

velocities in the ideal gas are determined by the leading-order term $\tilde{s}(\eta) = \tilde{s}_0(\eta)$ in (47), while the correction due to weak interactions in a rarefied gas yields

$$\tilde{s}(\eta) \approx \tilde{s}_0(\eta) + \int_{\Gamma^+} \tilde{\Delta}(\eta, \mu) [\tilde{s}_0(\eta) - s_0(\mu)] u(\mu) |d\mu|. \quad (52)$$

[Note that, unlike in (51), the integral term in (52) involves the difference between the phase and group velocities of a free breather or soliton.]

In the ideal gas limit, the interaction terms in the complex nonlinear dispersion relations (25) for breather gas are subdominant, leading to

$$\tilde{u}(\eta) - i\sigma(\eta)u(\eta) = -R_0(\eta), \quad \tilde{v}(\eta) - i\sigma(\eta)v(\eta) = 2\eta R_0(\eta). \quad (53)$$

Taking the real and imaginary parts of (53), we recover the expressions (38) and (39) for the group velocity $s_0(\eta)$ and (48) and (49) for the carrier phase velocity \tilde{s}_0 in the ideal breather gas (soliton gas in the limit $\delta_0 \rightarrow 0$) as expected. We also observe that the ratio

$$\frac{\tilde{v}(\eta) - i\sigma(\eta)v(\eta)}{\tilde{u}(\eta) - i\sigma(\eta)u(\eta)} = -2\eta \quad (54)$$

is the same for breather and soliton gas in this regime.

We note that the discrete ($2N$ -gap solution) counterpart of (54),

$$\frac{\tilde{\omega}_j + \frac{2i \ln |\delta_j|}{\pi} \omega_j}{\tilde{k}_j + \frac{2i \ln |\delta_j|}{\pi} k_j} = -2\eta_j, \quad j = 1, \dots, N, \quad (55)$$

suggests the following wave-number–frequency scaling:

$$k_j \sim \omega_j \sim \ln^{-1} |\delta_j|, \quad (56)$$

which is generally different from the one given by Eq. (24) (1D) and (35) (2D), as it does not involve N . In particular, the wave-number–frequency scaling (56), together with superexponential spectral scaling (22), covers the case of the transition from the $2N$ -gap solution to the N -soliton solution for N fixed.

B. Soliton and breather condensates

It has already been mentioned that the inequality $\sigma(\eta) \geq 0$ in Eq. (50) imposes a fundamental constraint on the density of states $u(\eta)$. As discussed in Sec. III A, the critical value $\sigma(\eta) = 0$ corresponds to the subexponential spectral scaling (21). One can see from the nonlinear dispersion relations (28) and (29) that in this case the gas properties are fully determined by the interaction (integral) terms, while the information about the individual quasiparticles (described by the secular terms) is completely lost. By analogy with Bose-Einstein condensation, we shall call the breather (soliton) gas at $\sigma = 0$ the *breather (soliton) condensate*. From (50) we obtain the criticality condition

$$\int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu| = 1, \quad (57)$$

which is the limiting case of the constraint (40). For a given interaction kernel $\Delta(\eta, \mu)$ [Eq. (38) for breather gas and Eq. (39) for soliton gas], Eq. (57) represents an integral

equation (the Fredholm integral equation of the first kind) for the critical density of states $u = u_c(\eta)$.

Equation (57) admits a lucid physical interpretation. Indeed, introducing the total density of states in the condensate, $\rho_c = \int_{\Gamma^+} u_c(\mu) |d\mu|$, we rewrite (57) as

$$\langle \Delta \rangle = \rho_c^{-1}, \quad (58)$$

where $\langle \dots \rangle$ denotes averaging over the spectral measure $\rho_c^{-1} u_c(\eta)$. Equation (58) then implies that in a soliton (breather) condensate the average position shift due to collisions is equal to the average distance between quasiparticles.

If the spectral locus curve Γ belongs to a vertical line (a bound-state gas), Eq. (57) with the soliton gas interaction kernel (39) can be solved explicitly using the inversion formula for the Hilbert transform. Another explicit solution for the density of states in soliton condensate can be obtained for the special case when the curve Γ represents a circle or a circular arc in the complex plane. Below we consider these two important examples.

1. Example 1. Bound-state soliton condensate

Let $\delta_0 = 0$ (soliton gas case) and $\Gamma = [-iq, iq]$ for some $q > 0$, which we shall call the soliton condensate intensity. Then $v(\eta) \equiv 0$ solves (33). The remaining equation (32) with $\sigma \equiv 0$ and $\eta \in \Gamma$ can be rewritten as

$$\int_{-iq}^{iq} \ln |\mu - \eta| u(\mu) i d\mu = \text{Im}\eta, \quad (59)$$

where we assume an odd (anti-Schwarz symmetric) extension of $u(\mu)$ onto $[-iq, 0]$. Indeed,

$$\ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| u(\mu) = -[\ln |\mu - \eta| u(\mu) + \ln |\bar{\mu} - \eta| u(\bar{\mu})], \quad (60)$$

where we have assumed that $u(\bar{\mu}) = -u(\mu)$ for all $\mu \in [0, iq]$. Introducing new variables $\xi = \text{Im}\eta$, $y = \text{Im}\mu$, we obtain

$$- \int_{-q}^q \ln |\xi - y| \hat{u}(y) dy = \xi, \quad (61)$$

where $\hat{u}(y) = u(iy)$. Differentiating (61), we obtain

$$\pi H[\hat{u}](\xi) := \int_{-q}^q \frac{\hat{u}(y) dy}{y - \xi} = 1, \quad (62)$$

where $H[\hat{u}]$ denotes the finite Hilbert transform (FHT) of \hat{u} over $[-q, q]$ [58]. Inverting the FHT, H subject to an additional constraint $H[\hat{u}](0) = 0$ (see, e.g., [59]), we obtain the density of states for the bound-state soliton condensate

$$u(\eta) = \frac{-i\eta}{\pi \sqrt{\eta^2 + q^2}}, \quad \eta \in (-iq, iq). \quad (63)$$

One can observe that the density of states (63) of the bound-state soliton condensate coincides with the appropriately normalized semiclassical distribution of the discrete ZS spectrum for a rectangular barrier, obtained as a derivative of the corresponding Weyl's law following from the Bohr-Sommerfeld quantization rule for the ZS operator (see [27,31] and references therein).

We now use Eqs. (34) to evaluate the bound-state soliton condensate carrier wave parameters \tilde{u} , \tilde{v} . Since $\arg \mu = \frac{\pi}{2}$ and

$$\arg \frac{\eta - \mu}{\eta - \bar{\mu}} - \pi \chi_\eta(\mu) = -\pi, \quad (64)$$

where χ denotes the characteristic function and $\mu, \eta \in [0, iq]$, we calculate

$$\tilde{u}(\eta) = -2q, \quad \tilde{v}(\eta) = 2 \operatorname{Re}(\eta^2), \quad (65)$$

where we used the fact that the integral term in (34) is zero due to $v(\eta) = 0$ when $\eta \in [-iq, iq]$. The carrier phase velocity in the bound-state soliton condensate is then

$$\tilde{s}(\eta) = \frac{\tilde{v}(\eta)}{\tilde{u}(\eta)} = -\frac{\operatorname{Re}(\eta^2)}{q}. \quad (66)$$

To evaluate the speed of a trial soliton with $\eta \notin [-iq, iq]$ propagating through the bound-state soliton condensate, we substitute (63) in (44) and arrive at

$$s(\eta) = \frac{-4 \operatorname{Im} \eta \operatorname{Re} \eta}{\operatorname{Im} \eta - \frac{1}{\pi} \operatorname{Re} \int_{-iq}^{iq} \ln(\mu - \eta) \frac{\mu d\mu}{\sqrt{\mu^2 + q^2}}}. \quad (67)$$

Applying integration by parts and evaluating residues, we obtain

$$s(\eta) = -\frac{4 \operatorname{Im} \eta \operatorname{Re} \eta}{\operatorname{Im} \sqrt{\eta^2 + q^2}}. \quad (68)$$

It is interesting to compare the group velocity (68) of the trial soliton propagating through a soliton condensate of intensity q with the group velocity c_g of the TW breather with the same soliton eigenvalue η and the same background intensity q [see Eq. (5)]. One can see that, although the expressions for these two velocities are different, they exhibit the same asymptotic behavior up to the second-order terms: $s(\eta) = -4 \operatorname{Re} \eta (1 - d^2/2) + O(d^4)$, where $d = q/\operatorname{Im} \eta \ll 1$ and $\operatorname{Re} \eta = O(1)$.

Concluding this important example, we note that the very recent numerical study [19] has shown that statistical characteristics (the probability density function, the Fourier power spectrum, and the autocorrelation function) of the bound-state soliton gas modeled by an n -soliton solution with $n \gg 1$ and the ‘‘Weyl’’ density of states (63) agree with remarkable accuracy with the counterpart characteristics of the stationary integrable turbulence describing the asymptotic, long-time behavior of the spontaneous modulational instability (i.e., the modulational instability of a plane wave of intensity q perturbed by a small noise) studied numerically [2] and experimentally [60].

2. Example 2. Non-bound-state ‘‘circular’’ soliton condensate

We now present an example of a soliton condensate that is not a bound state, i.e., a dynamic soliton condensate. Consider the spectral locus curve Γ in the form of an arc of the circle $|\eta| = \rho > 0$ connecting, counterclockwise, the points $\bar{\eta}_1$ and η_1 of Γ , where $|\eta_1| = \rho$.

To solve Eq. (57) in this case we introduce the change of variables $\mu = \rho e^{i\theta}$, $\eta = \rho e^{i\xi}$. Then (57) becomes

$$\rho \int_0^{\xi_1} \ln \left| \frac{\sin \frac{\xi - \theta}{2}}{\sin \frac{\xi + \theta}{2}} \right| \hat{u}(\theta) d\theta = -\rho \sin \xi, \quad (69)$$

where $\hat{u}(\theta) = u(\rho e^{i\theta})$, $\eta_1 = \rho e^{i\xi_1}$. We now differentiate both sides in ξ to obtain

$$\frac{1}{\pi} \int_b^1 \frac{f(q)}{q-p} dq = -\frac{p}{\pi}, \quad (70)$$

where we introduced the new variables $p = \cos \xi$, $q = \cos \theta$, and notations $b = \cos \xi_1$ and $f(p) = \hat{u}(\cos^{-1} p)$. The integral on the left-hand part of Eq. (70) represents the FHT $H[f](p)$ of the function f over $[b, 1]$; see [58,59]. To ensure uniqueness of the FHT inversion in (70) we impose a constraint $f(1) = 0$, which is equivalent to a physically natural condition $\hat{u}(0) = 0$. Omitting the calculations, we present the result

$$f(p) = -\frac{(p-1)(p+\frac{1-b}{2})}{\pi R_+(p)}, \quad (71)$$

$$R_+(p) = \sqrt{(1-p)(p-b)},$$

assuming the positive value of the radical. Note that $f(p) > 0$ for $p \in [b, 1)$. Thus, we obtain the density of states in the circular soliton condensate

$$u(\eta) = \frac{1}{\pi} \sqrt{\frac{1 - \cos \xi}{\cos \xi - \cos \xi_1}} \left(\cos \xi + \frac{1-b}{2} \right)$$

$$= \frac{1}{\pi} \left(\frac{\operatorname{Re} \eta}{|\eta|} + \frac{1-b}{2} \right) \sqrt{\frac{|\eta| - \operatorname{Re} \eta}{\operatorname{Re} \eta - \operatorname{Re} \eta_1}}. \quad (72)$$

Equation (33) with $\sigma = 0$ for $v(\eta)$ for this condensate is solved in a similar fashion. As a result, the soliton group velocity $s(\eta) = v(\eta)/u(\eta)$ is obtained in the form

$$s(\eta) = -8\rho \frac{\cos^2 \xi + \frac{1-b}{2} \cos \xi - \frac{(b+1)^2}{8}}{\cos \xi + \frac{1-b}{2}}$$

$$= -\frac{8(\operatorname{Re} \eta)^2 + 4(1-b)|\eta| \operatorname{Re} \eta - (b+1)^2 |\eta|^2}{\operatorname{Re} \eta + \frac{1-b}{2} |\eta|}. \quad (73)$$

In particular, for Γ being a circle, i.e., $b = -1$, we have

$$u(\eta) = \frac{\operatorname{Im} \eta}{\pi \rho}, \quad v(\eta) = -8 \frac{\operatorname{Re} \eta \operatorname{Im} \eta}{\pi \rho}, \quad s(\eta) = -8 \operatorname{Re} \eta. \quad (74)$$

The latter expression means that the soliton speed within the ‘‘circular’’ soliton condensate is exactly twice the speed $s_0(\eta) = -4 \operatorname{Re} \eta$ of the free soliton with the same spectral parameter η . The opposite limit $b \rightarrow 1^-$ (a small circular arc near the origin with $\cos \xi \rightarrow 1^-$) yields $s(\eta) \approx -8\rho \frac{\cos^2 \xi - \frac{1}{2}}{\cos \xi} \approx -4 \operatorname{Re} \eta$, which is consistent with the speed of free (noninteracting) fundamental soliton (6).

Using Eqs. (72) and (73) for $u(\eta)$, $v(\eta) = u(\eta)s(\eta)$, the wave number $\tilde{u}(\eta)$ and frequency $\tilde{v}(\eta)$ of the carrier wave in the ‘‘circular’’ condensate can be evaluated from Eqs. (34).

C. From ideal soliton gas to soliton condensate

The explicit results for soliton condensates obtained in the previous subsection can be readily generalized to some ‘‘genuine’’ (noncondensate) soliton gases, enabling one to span a continuum of states of increasing density between an ideal soliton gas and a soliton condensate. We shall obtain the corresponding results for the bound-state soliton gas and

the dynamic circular soliton gas following Examples 1 and 2 above.

We use the bound-state soliton condensate solution (63) of the integral Eq. (59) [Eq. (32) with $\sigma \equiv 0$] to derive a particular solution to the full Eq. (32) with $\sigma \neq 0$. Let $u = u_c(\eta)$ be the ‘‘Weyl’’ distribution (63) and assume that $\sigma(\eta)u_c(\eta) = m \operatorname{Im}\eta$ for some $m \geq 0$. Then $u(\eta) = u_c(\eta)/(m+1)$ is the solution of Eq. (32) with $\sigma(\eta) = m\pi\sqrt{\eta^2 + q^2}$. One can see that in the limit $m \rightarrow 0^+$ one has $u \rightarrow u_c$, i.e., the soliton gas approaches the state of condensate [cf. Eq. (63)], whereas in the limit $m \rightarrow +\infty$ it approaches the state of ideal gas with $u(\eta) \rightarrow 0^+$, $\sigma(\eta) \rightarrow +\infty$, $u\sigma \rightarrow \pi u_c \sqrt{\eta^2 + q^2} = -i\eta$.

We now derive the speed of a trial soliton with spectral parameter η , moving through the bound-state soliton gas on $[-iq, iq]$. We substitute the density of states $u = u_c/(m+1)$ in the trial soliton speed formula Eq. (44) to obtain [cf. Eqs. (67) and (68)]

$$\begin{aligned} s(\eta) &= \frac{-4 \operatorname{Im}\eta \operatorname{Re}\eta}{\operatorname{Im}\eta - \frac{1}{\pi(m+1)} \operatorname{Re} \int_{-iq}^{iq} \ln(\mu - \eta) \frac{\mu d\mu}{\sqrt{\mu^2 + q^2}}} \\ &= \frac{-4 \operatorname{Im}\eta \operatorname{Re}\eta}{\frac{m}{m+1} \operatorname{Im}\eta - \frac{1}{m+1} \operatorname{Im}\sqrt{\eta^2 + q^2}}, \end{aligned} \quad (75)$$

where, as earlier, we have used integration by parts and residues to obtain the final expression.

In particular, for a ‘‘trial’’ soliton propagating through a bound-state soliton condensate ($m = 0$) we recover Eq. (68). In the opposite limit $m \rightarrow +\infty$ (a trial soliton propagating through an ideal soliton gas) we recover the free soliton speed $-4 \operatorname{Re}\eta$.

Similar to the condensate case, we use (30) and (31) to evaluate \tilde{u} , \tilde{v} to obtain [cf. (65)]

$$\tilde{u}(\eta) = -\frac{2q}{m+1}, \quad \tilde{v}(\eta) = 2 \operatorname{Re}(\eta^2). \quad (76)$$

When $m \rightarrow \infty$ (ideal gas) we have $\tilde{u} = 0$, $\tilde{v}(\eta) = 2 \operatorname{Re}(\eta^2)$, in full agreement with the parameters of an isolated soliton (6).

The extension of the circular condensate results to the full range of regimes proceeds in the similar way. For simplicity, we only present the results for the circular soliton gas when the spectral locus curve Γ is a complete circle $|\eta| = \rho$. To study the range of regimes between an ideal soliton gas and the corresponding soliton condensate, we take the ‘‘seed’’ condensate solution $u_c = \operatorname{Im}\eta/\pi\rho$ [Eq. (72) with $b = -1$] for the density of states and choose σ in the integral equation (32) so that $\sigma(\eta)u_c(\eta) = m \operatorname{Im}\eta$, where $m \geq 0$. Now, invoking the same arguments that we used for the description of the bound-state soliton gas, we obtain the density of states and the group velocity in the circular gas as

$$\begin{aligned} u(\eta) &= \frac{\operatorname{Im}\eta}{\pi\rho(m+1)}, \quad v(\eta) = -8 \frac{\operatorname{Re}\eta \operatorname{Im}\eta}{\pi\rho(2m+1)}, \\ s(\eta) &= -8 \operatorname{Re}\eta \frac{m+1}{2m+1}. \end{aligned} \quad (77)$$

As expected, Eq. (77) shows the growing density of states u for the circular soliton gas as m varies from $m \rightarrow +\infty$ (ideal gas) to $m = 0$ (condensate). In this range of m , the speed $s(\eta)$ of the tracer soliton interpolates between the speed of a free

soliton $-4 \operatorname{Re}\eta$ at $m = \infty$ and the condensate speed (74) at $m = 0$.

Let us now use (30) to calculate \tilde{u} , \tilde{v} for the circular gas. Using elementary geometry, we obtain

$$\arg \frac{\eta - \mu}{\eta - \bar{\mu}} - \pi \chi_\eta(\mu) = \mu - \pi, \quad (78)$$

where χ_η denotes the characteristic function of the arc $(1, \eta)$ of our circle. Then

$$\int_\rho^{-\rho} \left[\arg \frac{\mu - \eta}{\mu - \bar{\eta}} - 2 \arg \mu - \pi \chi_\eta(\mu) \right] u(\mu) |d\mu| = \frac{-3\rho}{m+1}, \quad (79)$$

so that

$$\begin{aligned} \tilde{u}(\eta) &= -\operatorname{Re}\eta - \frac{3\rho}{m+1}, \\ \tilde{v}(\eta) &= 2 \operatorname{Re}(\eta^2) + \frac{6\rho^2}{2m+1} \end{aligned} \quad (80)$$

where the integral for \tilde{v} was calculated similarly to (79). One can now obtain the phase speed

$$\tilde{s}(\eta) = -2\rho \frac{\cos(2 \arg \eta) + \frac{3}{2m+1}}{\cos(\arg \eta) + \frac{3}{m+1}}, \quad (81)$$

which exhibits the expected limit $-2 \frac{\operatorname{Re}\eta^2}{\operatorname{Re}\eta}$ as $m \rightarrow \infty$; compare with the phase speed of free soliton (6). In the condensate regime, (81) yields

$$\tilde{s}(\eta) = -2\rho \frac{\cos(2 \arg \eta) + 3}{\cos(\arg \eta) + 3}. \quad (82)$$

VIII. KINETIC EQUATIONS FOR BREATHER AND SOLITON GAS

A. General construction

So far we have assumed that the spectral characteristics $u(\eta)$, $v(\eta)$, $\tilde{u}(\eta)$, and $\tilde{v}(\eta)$ of a breather or soliton gas do not depend on x, t . We call such breather or soliton gases uniform, or homogeneous, gases. For a nonhomogeneous breather or soliton gas, we introduce $u \equiv u(\eta, x, t)$, $v \equiv v(\eta, x, t)$, $\tilde{u} \equiv \tilde{u}(\eta, x, t)$, $\tilde{v} \equiv \tilde{v}(\eta, x, t)$ assuming that variations of the gas’ parameters occur on much larger spatiotemporal scales $\Delta x, \Delta t$ than the typical scales $\Delta x \sim \Delta t = O(1)$ of the oscillations corresponding to individual solitons or breathers.

To derive evolution equations for a breather or soliton gas, we go back to the original, discrete wave number and frequency components $k_j(\boldsymbol{\alpha})$, $\omega_j(\boldsymbol{\alpha})$, $\tilde{k}_j(\boldsymbol{\alpha})$, and $\tilde{\omega}_j(\boldsymbol{\alpha})$ of the finite-gap potential, defined in terms of the fixed branch points $\boldsymbol{\alpha}$ of the Riemann surface \mathcal{R} of (4). Let us now consider a slowly modulated finite-gap potential, so that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(x, t)$. The equations describing the evolution of the $\boldsymbol{\alpha} = \boldsymbol{\alpha}(x, t)$ —the so-called Whitham modulation equations [29]—for the fNLS equation have been derived in [61] for $n = 1$ and in [33] for an arbitrary genus n . The resulting system of $2n$ quasilinear modulation equations for the branch points $\alpha_j(x, t)$ has an infinite number of conservation laws, which include a finite subset of n ‘‘wave conservation’’ laws (2), which we write in

terms of the special wave-number–frequency set satisfying the nonlinear dispersion relations (15),

$$\partial_t k_j(\boldsymbol{\alpha}) = \partial_x \omega_j(\boldsymbol{\alpha}), \quad j = 1, \dots, N, \quad (83)$$

$$\partial_t \tilde{k}_j(\boldsymbol{\alpha}) = \partial_x \tilde{\omega}_j(\boldsymbol{\alpha}), \quad j = 1, \dots, N \quad (84)$$

(as earlier, we assume even genus $n = 2N$). We fix the exceptional band γ_0 , which is consistent with the Whitham equations as the branch points α_j are the analogs of Riemann invariants [33].

We now apply the thermodynamic wave-number–frequency scaling (24) to the conservation equations (83) and (84). Let $K_M = \sum_{j=1}^M k_j$, $W_M = \sum_{j=1}^M \omega_j$, where $1 \leq M \leq N$. For convenience we shall be using a 1D spectral configuration while the result will remain valid for 2D. Invoking the 1D scaling (24), we obtain

$$K_M = \sum_{j=1}^M \frac{\kappa(\eta_j)}{N} \rightarrow \int_{\eta_\infty}^{\eta} \kappa(\mu) \varphi(\mu) |d\mu| \equiv \mathcal{K}(\eta), \quad (85)$$

where we made a replacement $\eta_M \rightarrow \eta$ in the continuous limit. Now, using (27) we see that $\mathcal{K}'(\eta) = u(\eta)$, where the prime denotes differentiation along Γ^+ . Thus, $\mathcal{K}(\eta)$ has the meaning of the integrated density of states in a breather or soliton gas. Similarly, we have

$$W_M = - \sum_{j=1}^M \frac{v(\eta_j)}{N} \rightarrow - \int_{\eta_\infty}^{\eta} v(\mu) \varphi(\mu) |d\mu| \equiv \mathcal{V}(\eta), \quad (86)$$

so that $\mathcal{V}'(\eta) = -v(\eta)$. Now, the thermodynamic limit of (83) yields the continuity equation for the density of states $u(\eta, x, t)$,

$$\partial_t u + \partial_x (us) = 0, \quad (87)$$

where $u = u(\eta, x, t)$, $s = v(\eta, x, t)/u(\eta, x, t)$, and the dependence $s[u]$ is given by the equation of state (37). One can see from (87) that the breather or soliton group velocity $s(\eta, x, t)$ has the meaning of the transport velocity of the corresponding gas. Equation (87), together with the equation of state (37), form the kinetic equation for a breather gas [using Eq. (48)] or soliton gas [using Eq. (49)].

Similarly, for the carrier wave components we have $\tilde{k}_j \rightarrow 2\tilde{u}(\eta, x, t)$, $\tilde{\omega}_j \rightarrow -2\tilde{v}(\eta, x, t) = \tilde{u}(\eta, x, t)\tilde{s}(\eta, x, t)$ in the thermodynamic limit, and Eqs. (84) transform into a single equation for the carrier wave number \tilde{u} :

$$\partial_t \tilde{u} + \partial_x (\tilde{u}\tilde{s}) = 0, \quad (88)$$

where the phase velocity \tilde{s} also plays the role of the transport velocity for the carrier wave number, and the dependencies $\tilde{u}[u]$, $\tilde{s}[u]$ are given by (30) and (47). Together with the carrier wave nonlinear dispersion relations (30), Eq. (88) yields a “satellite” kinetic equation for the carrier wave, reflecting the dual, “wave-particle” nature of breathers and solitons. We stress that the previous works on soliton gas kinetic equations (see, e.g., [14,15,55] and references therein) have been concerned only with the “particle” equation (87).

One may note that the application of the thermodynamic limit to the Whitham modulation equations is in apparent conflict with the original premise of the Whitham theory [29],

which is based on the scale separation, in which spatiotemporal modulation scales are large compared to multiperiodic wavelengths and periods. We note, however, that the soliton limit of the Whitham equations, while being formally inconsistent with the original assumptions of the modulation theory, is well known to yield an accurate description of the dynamics of individual solitons and soliton trains in a variety of problems, including the dispersive shock wave theory [62] and the recently introduced theory of soliton-mean flow interactions [63,64] confirmed by both direct numerical simulations and physical experiments. In fact, Whitham in his book showed how to describe solitary wave trains by an appropriate limit of the modulation system; see [29], Chap. 16.16.

B. Multicomponent hydrodynamic reductions

The kinetic equations (87) and (37) can be reduced to a system of quasilinear PDEs if one takes advantage of the multicomponent δ -function ansatz (41) for the density of states u , where one now assumes that the “densities” w_j and the speeds $s^{(j)}$ of the gas components are slow functions of x, t . As we mentioned in Sec. V, there are certain constraints on the admissible values of w_j ; we shall assume that these constraints are satisfied.

The resulting system for $w_j(x, t)$, $s^{(j)}(x, t)$, $j = 1, \dots, M$, has the form of a system of hydrodynamic conservation laws

$$(w_j)_t + (w_j s^{(j)})_x = 0, \quad j = 1, \dots, M \quad (89)$$

with the closure conditions given by the multicomponent equation of state Eq. (42).

The hydrodynamic-type system (89) and (42) was extensively studied in [55], where it was shown that it represents a *hyperbolic integrable linearly degenerate system for any $M \in \mathbb{N}$* . Thus, all the previously obtained general mathematical results for finite-component KdV-type soliton gases [55] can be readily extended to the case of fNLS soliton and breather gases. In particular, as was shown in [14,55] for $M = 2$, the system (89) and (42) reduces to

$$(s^{(1)})_t + s^{(2)}(s^{(1)})_x = 0, \quad (s^{(2)})_t + s^{(1)}(s^{(2)})_x = 0 \quad (90)$$

with the relations between $s^{(1,2)}$ and $w^{1,2}$ given by Eq. (43).

System (90) represents the diagonal form of the so-called Chaplygin gas equations, the system of isentropic gas dynamics with the equation of state $p = -A/\rho$, where p is a pressure, ρ is the gas density, and $A > 0$ is a constant. It occurs in certain theories of cosmology (see, e.g., [65]) and is also equivalent to the 1D Born-Infeld equation [29,66] arising in nonlinear electromagnetic field theory.

We note that hyperbolicity of the hydrodynamic system (89) and (42) might look surprising in the context of the fNLS equation as the fNLS-Whitham system is known to be elliptic for a generic set of modulation parameters and for any genus [33]. The apparent paradox is resolved by noticing that the fNLS-Whitham system exhibits real eigenvalues (characteristic speeds) in the soliton limit. For example, for the genus 1 case, the two pairs of complex-conjugate eigenvalues of the modulation matrix degenerate into a single, quadruply degenerate real eigenvalue, corresponding to the velocity of the fundamental soliton; see, e.g., [67].

Due to the availability of conservation laws (89), one can solve a general Riemann problem for a multicomponent-component breather gas described by (89) and (42). As is known, a weak solution to the Riemann problem for a system of hydrodynamic conservation laws (89) generally consists of $M + 1$ disparate constant states separated by M propagating discontinuities or rarefaction waves (one of each family), where the jumps of w^i and s^i across the discontinuities are determined from the Rankine-Hugoniot conditions [68]. Linear degeneracy of the system (89) and (42) implies that there are no rarefaction waves, and the shocks are contact discontinuities with the speeds coinciding with the speeds of the relevant components [69]. Importantly, contact discontinuities do not require regularization via higher-order mechanisms such as dispersion or dissipation. Following the KdV and fNLS soliton gas results [14,57], we present here the solution to a Riemann problem for a two-component fNLS breather gas.

We consider system (89) and (43) for $M = 2$ and the “shock tube” type initial conditions

$$\begin{aligned} w_1(x, 0) &= w_0^1, \quad w_2(x, 0) = 0, \quad x < 0, \\ w_2(x, 0) &= w_0^2, \quad w_1(x, 0) = 0, \quad x > 0, \end{aligned} \tag{91}$$

where $w_0^1, w_0^2 > 0$ are some constants. We shall also assume that $s_0^{(1)} > s_0^{(2)} > 0$ so that the gases undergo an “overtaking” collision resulting in the formation of an expanding “interaction” region $c^-t < x < c^+t$, where both components are present. Note that, unlike in the classical gas-dynamics shock tube problem, the initial velocity of the breather gases is not zero but is fully determined, via Eq. (43), by the density distribution (91).

The weak solution for w_1 and w_2 has a piecewise constant form:

$$\begin{aligned} w_1(x, t) &= \begin{cases} w_0^1, & x < c^-t, \\ w_c^1, & c^-t < x < c^+t, \\ 0, & x > c^+t, \end{cases} \\ w_2(x, t) &= \begin{cases} 0, & x < c^-t, \\ w_c^2, & c^-t < x < c^+t, \\ w_0^2, & x > c^+t. \end{cases} \end{aligned} \tag{92}$$

Here c^- and c^+ are the velocities of the left and right discontinuity, respectively, and w_c^1, w_c^2 and $s_c^{(1)}, s_c^{(2)}$ are the densities and velocities of the breather gas components in the interaction region $x \in [c^-t, c^+t]$. The velocities $s_c^{(1)}$ and $s_c^{(2)}$ are expressed in terms of w_c^1, w_c^2 by relations (43). The interaction region densities w_c^1, w_c^2 and the contact discontinuities’ velocities c^\pm are found from the Rankine-Hugoniot conditions:

$$\begin{aligned} -c^- [w_0^1 - w_c^1] + [w_0^1 s_0^{(1)} - w_c^1 s_c^{(1)}] &= 0, \\ -c^- [0 - w_c^2] + [0 - w_c^2 s_c^{(2)}] &= 0, \end{aligned} \tag{93}$$

$$\begin{aligned} -c^+ [w_c^1 - 0] + [w_c^1 s_c^{(1)} - 0] &= 0, \\ -c^+ [w_c^2 - w_0^2] + [w_c^2 s_c^{(2)} - w_0^2 s_0^{(2)}] &= 0, \end{aligned} \tag{94}$$

resulting in

$$\begin{aligned} w_c^1 &= \frac{w_0^1(1 - \Delta_{21}w_0^2)}{1 - \Delta_{12}\Delta_{21}w_0^1w_0^2}, \\ w_c^2 &= \frac{w_0^2(1 - \Delta_{12}w_0^1)}{1 - \Delta_{12}\Delta_{21}w_0^1w_0^2}, \end{aligned} \tag{95}$$

$$\begin{aligned} c^- &= s_0^{(2)} - \frac{(s_0^{(1)} - s_0^{(2)})\Delta_{12}w_c^1}{1 - (\Delta_{12}w_c^1 + \Delta_{21}w_c^2)}, \\ c^+ &= s_0^{(1)} + \frac{(s_0^{(1)} - s_0^{(2)})\Delta_{21}w_c^2}{1 - (\Delta_{12}w_c^1 + \Delta_{21}w_c^2)}. \end{aligned} \tag{96}$$

In conclusion, we note that, being an integrable hydrodynamic-type system, Eqs. (89) and (42) are amenable to the generalized hodograph transform, enabling in principle the construction of all nonconstant smooth solutions [34,70]. Indeed, a number of nontrivial exact solutions (such as similarity and quasiperiodic solutions) were obtained in [55]. Their physical interpretation in terms of the fNLS breather and soliton gas is an interesting outstanding problem.

IX. SUMMARY AND OUTLOOK

In this paper, we have constructed nonlinear spectral theory of breather and soliton gases in the fNLS equation. This was done by considering a special, infinite-genus thermodynamic limit of finite-gap potentials and of the associated nonlinear modulation equations.

The core result of the paper is the system of nonlinear dispersion relations (25) and (26) for the spectral parameters of breather gas: the density of states $u(\eta)$ and its temporal counterpart $v(\eta)$, as well as the wave number $\tilde{u}(\eta)$ and the frequency $\tilde{v}(\eta)$ of the carrier wave. These nonlinear dispersion relations yield the integral equation of state [Eqs. (37) and (38)] connecting the velocity $s(\eta)$ of the quasiparticles (breathers) in the gas with the gas’ spectral density of states $u(\eta)$. The respective relations for soliton gas are obtained collapsing the “exceptional” spectral band corresponding to the background Stokes mode in the breather gas.

The equation of state (37), together with the transport equation (87) for the slowly varying density of states $u(\eta, x, t)$, forms the kinetic equation for breather gas. Combining this equation with the “satellite” kinetic equations (88) and (47) for the carrier wave parameters $\tilde{u}(\eta, x, t)$ and $\tilde{v}(\eta, x, t)$, we obtained a full set of equations characterizing the macroscopic spectral dynamics in a spatially nonhomogeneous breather gas. These include the kinetic equation for soliton gas [14], as a particular case. Our consideration also includes the bound-state (nonpropagating) breather and soliton gases not considered previously. One of the immediate implications of our analysis is the prediction of the critical state of breather (soliton) gases, which we term breather (soliton) condensate, and whose properties are fully determined by the interactions between the quasiparticles in the gas, while the individual characterization of these quasiparticles is suppressed. The criticality condition (57) yields the density of states $u_c(\eta)$ in the condensate, and we present two notable examples in which this critical density of states can be found explicitly: the

bound-state soliton condensate and the so-called “circular” soliton condensate with the spectrum located on a circumference in the complex plane.

We now outline some important directions of future physical and mathematical research suggested by our work.

(i) Statistical characterization of the nonlinear random wave field $\psi(x, t)$ in breather and soliton gases: namely, the determination of the probability density function $\mathcal{P}(|\psi|)$, the power spectrum, the correlation function, etc.—in terms of the spectral density of states $u(\eta)$. This is a subject of ongoing research, and the results will be published elsewhere.

(ii) Realization of breather and soliton gases in numerical simulations and laboratory experiments and verification of the predictions of the spectral theory developed here. While the numerical realization of the KdV soliton gas has been reported in a number of works (see, e.g., [57,71]), the challenge of the modeling of a fNLS soliton gas has been successfully addressed only recently in [19,20], where dense statistical ensembles of N -soliton solutions with large N and random, uniformly distributed phases of the so-called norming constants were constructed numerically based on a specific implementation of the dressing method. We note that the above papers employ periodic boundary conditions for the numerical realization of N -soliton solutions, i.e., they essentially realize the finite-gap solutions of genus $2N$ approximating N -soliton solutions. In particular, the numerical simulations in [19] have shown that the bound-state soliton condensate described by the “Weyl” distribution of the density of states (63) represents an accurate model for the developed, nonlinear stage of spontaneous (noise-induced) modulational instability of a plane wave [2,60]. We can add that our preliminary numerical simulations of fNLS soliton gas dynamics show a very good agreement with the results obtained in this paper, in particular with the solution (93) and (94) of the “shock-tube” problem in Sec. VIII B and with the formula (68) for the velocity of a trial soliton propagating through the bound-state soliton condensate. These results will be reported in a separate paper.

(iii) Yet another promising line of research, where the developed theory can find applications, is related to the rogue wave formation (see, e.g., [20] for recent numerical observations of rogue waves in soliton gas). The relative roles of solitons and breathers in the rogue wave statistics of integrable turbulence have been discussed in [72] based on the numerical implementation of the traditional IST method. The finite-gap theory has proved a powerful tool for the description of rogue waves (see, e.g., [24,46,73,74] and references therein), and the application of the spectral theory of soliton and breather gases, in particular of solutions to the kinetic equation, could be the next important step in this direction.

(iv) The subject of this paper is closely related to the study of the semiclassical limit of the fNLS equation for a class of generic potentials with large “solitonic content” of $\sim 1/\epsilon$ solitons, where $\epsilon \ll 1$ is a small dispersion parameter. Indeed, the fNLS evolution of such potentials is known to typically lead to the appearance of coherent structures of increasing complexity that can be locally approximated by genus n finite-gap solutions, with n increasing in time (see [31,67,75–78] and references therein). Our preliminary considerations indicate that for sufficiently large t (and consequently large n), the semiclassical spectrum of these solutions fits into

one of the thermodynamic scaling requirements described in this paper. Taking into account the effective randomization of phases, the large t evolution of semiclassical solutions is expected to provide the dynamical realization of soliton gas construction described in this paper. Indeed, some features of the soliton gas development from an initial rectangular barrier (box) potential predicted by the semiclassical analysis in [67] have been recently observed in an optics experiment [79].

(v) Related to the previous comment, we mention the possibility of an alternative construction of a soliton gas via an appropriate limit as $n \rightarrow \infty$ of n -soliton solutions (rather than n -gap potentials) of the fNLS equation. Indeed, as we already mentioned, this is the way (n -solitons for large n) the dense soliton gas has been realized numerically in [19,20]. In this connection, an extension to the fNLS equation of the theory of the so-called primitive potentials originally developed in the context of the KdV equation [80] could prove useful (see [81] for a recent study, where a particular infinite-soliton solution of the KdV equation was constructed in the form of a primitive potential). Additionally, the Lax-Levermore type approach to the semiclassical limit of N -soliton solutions of the fNLS equation (see [82] and references therein) could provide a complementary tool for the soliton gas description, with possible extension to breather gas.

(vi) This work is concerned with the properties of soliton and breather gases in the *focusing* NLS equation. A similar theory can be constructed for the *defocusing* NLS (dNLS) equation. The kinetic equation for a gas of *dark* (gray) solitons would necessarily require the presence of a nonzero background, and the corresponding spectral theory would represent a “bidirectional” counterpart of the KdV soliton gas theory due to the spectrum of the self-adjoint ZS operator associated with the dNLS equation being located on the real line. Kinetic theory of bidirectional soliton gases is the subject of a separate work [83].

(vii) The effects of small perturbations (e.g., dissipation, higher-order nonlinearity and dispersion, or a trapping potential) on the dynamics of “integrable” soliton gases are of great interest for applications. In particular, the properties of a rarefied gas of dark solitons in quasi-1D repulsive Bose-Einstein condensates (BECs) in a harmonic trapping potential were considered in [84], whereas the methods of the experimental realization of soliton gas in trapped BECs were discussed in [85]. Additionally, the dark soliton gas type structures have been observed in the laminar-turbulent transitions in fiber laser [86]. The examples of relevant mathematical models include the higher-order NLS equations (deep water waves, nonlinear optics) and the Gross-Pitaevskii equation (BECs).

One of the most physically pertinent questions arising in connection with soliton gas dynamics in perturbed systems is that of thermalization and the associated equipartition of energy. This topic is currently under active investigation in the context of propagation of weakly nonlinear random waves in perturbed integrable systems (see, e.g., [87] and references therein). On the other hand, it is known that soliton gases in nonintegrable systems can exhibit a peculiar “soliton attractor” scenario observed numerically in [88]. The development of an analytical approach to the description of soliton or

breather gas in the perturbed fNLS equation via the tools of nonlinear spectral theory represents a major challenge.

(viii) Finally, we mention the intriguing connections of the spectral dynamics of soliton and breather gases with the generalized hydrodynamics of many-body quantum integrable systems, which turn out to be governed by the kinetic equations of the type (87) and (37) [15–17]. Establishing a bridge between these two topical areas is yet another promising avenue for future cross-disciplinary research.

Concluding, we expect that our work will stimulate further theoretical and experimental studies of soliton and breather gases in various physical contexts, including nonlinear optics, water waves, and superfluids, as well as in connection with quantum integrable systems.

ACKNOWLEDGMENTS

The work of GE was partially supported by Engineering and Physical Sciences Research Council (EPSRC) Grant No. EP/R00515X/2. The authors thank M. Bertola, E. Blackstone, T. Congy, A. Kuijlaars, S. Randoux, G. Roberti, M. Sigal, and P. Suret for stimulating discussions. The authors also thank E. Blackstone for the preparation of some figures.

APPENDIX: MATHEMATICAL UNDERPINNINGS

We now outline the rationale behind the derivation of Eqs. (7), (8), and (18) using the Riemann-Hilbert problem approach to finite-gap theory [73,89].

1. Riemann-Hilbert problem

We start with the construction of the real normalized quasimomentum dp and quasienergy dq differentials for the Riemann surface \mathcal{R} . Let $f(z)$ be a polynomial with real coefficients. Consider the Riemann-Hilbert problem (RHP) for the function $g(z)$:

- (i) It is analytic in \mathbb{C} with the exception of the jump discontinuity on all the bands γ_j and all the gaps c_j .
- (ii) It satisfies the jump conditions

$$g_+(z) + g_-(z) = f(z) + W_j \text{ on } \gamma_j, |j| = 0, 1, \dots, N,$$

and

$$g_+(z) - g_-(z) = \Omega_j \text{ on } c_j, |j| = 1, \dots, N, \tag{A1}$$

on the bands and gaps, where $W_0 = 0, W_j = W_{-j}, \Omega_j = \Omega_{-j}$, and the real constants $W_j, \Omega_j, j = 1, \dots, N$, are to be determined.

- (iii) It is analytic at $z = \infty$.

Here $g_{\pm}(z)$ denote the limiting values of $g(z)$ on the oriented (see Fig. 3) bands γ_j and gaps c_j . By the Sokhotsky-Plemelj formula, the solution to this RHP is given by

$$g(z) = \frac{R(z)}{2\pi i} \left[\sum_{|j|=0}^N \int_{\gamma_j} \frac{[f(\zeta) + W_j]d\zeta}{(\zeta - z)R_+(\zeta)} + \sum_{|j|=1}^N \int_{c_j} \frac{\Omega_j d\zeta}{(\zeta - z)R(\zeta)} \right]. \tag{A2}$$

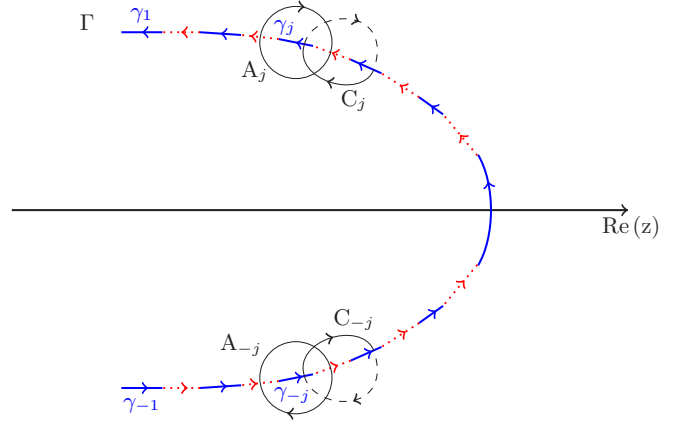


FIG. 3. Spectrum geometry for the breather gas construction. The spectral bands (solid lines) lie on the Schwarz-symmetric contour Γ in the complex plane $z \in \mathbb{C}$. The cycles A_j and C_j surround the bands γ_j and gaps c_j (dotted lines) respectively. The contours on the second sheet of \mathcal{R} are shown by dashed lines.

Since $g(z)$ must be analytic at infinity, the constants W_j, Ω_j must satisfy the linear system

$$\sum_{|j|=1}^N W_j \oint_{A_j} \frac{\zeta^m d\zeta}{R(\zeta)} + \sum_{|j|=1}^N \Omega_j \oint_{C_j} \frac{\zeta^m d\zeta}{R(\zeta)} = - \sum_{|j|=0}^N \oint_{A_j} \frac{f(\zeta)\zeta^m d\zeta}{R(\zeta)}, \tag{A3}$$

where $m = 0, 1, \dots, 2N - 1$, obtained from (A2) by expanding $\frac{1}{\zeta - z}$ in powers of $\frac{1}{z}$ as $z \rightarrow \infty$. Here C_j denotes the negatively oriented loop around the gap $c_j, |j| = 1, \dots, N$; see Fig. 3.

2. Wave numbers and frequencies

The system (A3) for $2N$ real unknowns W_j, Ω_j has a unique solution [76] provided that all the branch points α_k of the Riemann surface \mathcal{R} are distinct. Thus (A2) indeed defines $g(z)$. Note that dg is a meromorphic differential on \mathcal{R} with the only pole at $z = \infty_2$ on the second sheet of \mathcal{R} . Then $dh = 2dg - df$ is a meromorphic differential on \mathcal{R} with the poles at $z = \infty_{1,2}$ on both sheets of \mathcal{R} and with the periods

$$\oint_{A_{\pm j}} dh = \pm 2(\Omega_{j-1} - \Omega_j),$$

$$\oint_{C_{\pm j}} dh = \pm 2(W_j - W_{j+1}), \tag{A4}$$

$|j| = 1, \dots, N$, where $\Omega_0 = W_{N+1} = 0$. Thus, the differential $dh = d(2g - f)$ is a second kind of real normalized meromorphic differential with the asymptotics $dh \sim -df$ as $z \rightarrow \infty$ on the main sheet. Since such differentials are uniquely defined by the polynomial f [90], we obtain the real normalized quasimomentum dp and quasienergy dq differentials on \mathcal{R} by choosing $f(z) = -z$ and $f(z) = -2z^2$, respectively. Introducing now

$$k_j = 2(\Omega_j - \Omega_{j-1}), \quad \tilde{k}_j = 2W_j, \quad |j| = 1, \dots, N \tag{A5}$$

for dp, dq with $f(z) = -z$ and the same expressions for $\omega_j, \tilde{\omega}_j$ with $f(z) = -2z^2$, we obtain (7) and (8). The signs of the integrals in (7) and (8) will be opposite if we replace j by $-j$.

Let us now deform the contours γ_j and c_j of the jump conditions (A1) into $2N + 1$ vertical jump contours $v_j = [\tilde{\alpha}_j, \alpha_j]$, $j = 1, \dots, 2N + 1$. In particular, γ_0 is deformed into v_1 , the jumps on $c_{\pm N}$ after the deformation become distributed between v_1 and v_{2N+1} , etc.; see [76], Sec. 8, for details. Then we obtain a new RHP with vertical jump contours v_j , but it is satisfied by the same function $g(z)$ [to be more precise, by the corresponding analytic continuation of $g(z)$]. If we would choose the new (deformed) A-cycles to be the loops around each vertical contour v_j , $j = 2, \dots, 2N + 1$, then the meromorphic differential dh would have all A-cycles equal to zero, i.e., it would be A-cycles normalized. The wave numbers and frequencies for the fNLS are defined in [33,35] as B periods of the A-cycles' normalized quasimomentum dp and quasienergy dq differentials on the Riemann surface of the radical $R(z)$ from (4) with vertical branch cuts v_j . Any two sets of A and B cycles of the same Riemann surface are integer linear combinations of each other (see, e.g., [91]). As any integer combination of wave numbers (frequencies) is also a wave number (frequency), we arrive at the statement that A- and B-cycles of dp, dq , defined by (9), are wave numbers and frequencies, respectively.

3. Nonlinear dispersion relations for finite-gap potentials

The system (A3) is the starting point for the derivation of the nonlinear dispersion relations (18). First of all, introducing the constants $U_k \in \mathbb{R}$ by

$$\Omega_l = \sum_{j=1}^l U_j, \quad l = 1, \dots, N, \quad (\text{A6})$$

we get

$$\sum_{|j|=1}^N \Omega_j \oint_{C_j} \frac{\zeta^m d\zeta}{R(\zeta)} = \sum_{|j|=1}^N U_j \oint_{B_j} \frac{\zeta^m d\zeta}{R(\zeta)}, \quad m = 0, \dots, 2N - 1, \quad (\text{A7})$$

and thus (A3) is reduced to

$$\begin{aligned} & \sum_{|j|=1}^N W_j \oint_{A_j} \frac{\zeta^m d\zeta}{R(\zeta)} + \sum_{|j|=1}^N U_j \oint_{B_j} \frac{\zeta^m d\zeta}{R(\zeta)} \\ &= - \sum_{|j|=0}^N \oint_{A_j} \frac{f(z)\zeta^m d\zeta}{R(\zeta)}. \end{aligned} \quad (\text{A8})$$

Taking now linear combinations of Eqs. (A8), according to (16) and (17) we obtain

$$W_j + \sum_{|m|=1}^N U_m \oint_{B_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} = - \oint_{\hat{\gamma}} \frac{f(z)P_j(\zeta)d\zeta}{R(\zeta)}, \quad (\text{A9})$$

where $\hat{\gamma}$ is a large clockwise oriented contour containing all γ_j . Substituting ζ and $2\zeta^2$ for $f(\zeta)$, in view of (A5), we obtain (18). Since $W_j = W_{-j}$, $\Omega_j = \Omega_{-j}$, see (A1), we can combine the integrals over the loops B_m, B_{-m} into one integral over contour $\tilde{B}_m = B_m + B_{-m}$, $m = 1, \dots, N$, and consider the system (A9) only for $j = 1, \dots, N$.

4. Thermodynamic limit of nonlinear dispersion relations

In the thermodynamic limit, the leading-order behavior of the coefficients of the linear system (15) is given by

$$\begin{aligned} \oint_{\tilde{B}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= \frac{1}{i\pi} \left[\ln \frac{R_0(\eta_j)R_0(\eta_m) + \eta_j\eta_m - \delta_0^2}{R_0(\eta_j)R_0(\bar{\eta}_m) + \eta_j\bar{\eta}_m - \delta_0^2} \right. \\ &\quad \left. - \ln \frac{\eta_m - \eta_j}{\bar{\eta}_m - \eta_j} \right] + h(m - j) + O(N^2\delta_0^{\frac{2}{3}}) \\ &\text{when } m \neq j \text{ and } \oint_{\tilde{B}_j} \frac{P_j(\zeta)d\zeta}{R(\zeta)} \\ &= i \frac{2 \ln |\delta_j|}{\pi} + O(1), \end{aligned} \quad (\text{A10})$$

where $j, m = 1, \dots, N$ and h denotes the Heaviside function $h(\xi) = \frac{1}{2}(1 + \text{sign } \xi)$. Note that $-\frac{1}{i\pi} \ln \frac{\eta_m - \eta_j}{\bar{\eta}_m - \eta_j} + h(m - j)$ varies smoothly as η_m traverses the contour Γ^+ bypassing $\eta_j \in \Gamma^+$ from the left (positive) side of Γ^+ . The choice of the branch of each logarithm in the first equation in (A10) should be the same for both the numerator and the denominator.

Substituting these estimates together with (24) into the system (15) and replacing the Riemann sum with the corresponding integral, we obtain (25) and (26).

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