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# Generalised Adaptive Fuzzy Rule Interpolation

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**Abstract**—As a substantial extension to fuzzy rule interpolation that works based on two neighbouring rules flanking an observation, adaptive fuzzy rule interpolation is able to restore system consistency when contradictory results are reached during interpolation. The approach first identifies the exhaustive sets of candidates, with each candidate consisting of a set of interpolation procedures which may jointly be responsible for the system inconsistency. Then, individual candidates are modified such that all contradictions are removed and thus interpolation consistency is restored. It has been developed on the assumption that contradictions may only be resulted from the underlying interpolation mechanism, and that all the identified candidates are not distinguishable in terms of their likelihood to be the real culprit. However, this assumption may not hold for real world situations. This paper therefore further develops the adaptive method by taking into account observations, rules and interpolation procedures, all as diagnosable and modifiable system components. Also, given the common practice in fuzzy systems that observations and rules are often associated with certainty degrees, the identified candidates are ranked by examining the certainty degrees of its components and their derivatives. From this, the candidate modification is carried out based on such ranking. This work significantly improves the efficacy of the existing adaptive system by exploiting more information during both the diagnosis and modification processes.

**Index Terms**—Fuzzy inference, adaptive fuzzy rule interpolation, ATMS, GDE.

## I. INTRODUCTION

Fuzzy inference systems have been successfully applied to many real world applications, but the systems may suffer from either too sparse or too complex rule bases. Fuzzy rule interpolation (FRI) alleviates this by supporting inference with incomplete sparse rule bases, or by simplifying complex fuzzy systems that involve very dense rule bases through approximating certain parts of the model with their neighbouring rules [1], [2]. Many important FRI methods and their analysis or variations have been presented in the literature, including [1]–[22]. What is common to most of these techniques is that multiple values may be derived for a single variable. This implies that inconsistencies have been generated in the interpolated results.

*Adaptive fuzzy rule interpolation* (AFRI) was proposed in an effort to address this problem [23], [24]. It was developed upon FRI approaches by which two neighbouring rules that flank an observation are utilised for interpolation. The approach

efficiently detects inconsistencies, directly locates possible sets of fault components (namely, candidates), and effectively modifies the candidates in order to restore consistency, by removing detected inconsistencies. The approach artificially treats a fuzzy rule interpolation system as a component-based mechanism where system components are defined as interpolation procedures. An assumption-based truth maintenance system (ATMS) [25]–[27] is employed to record the depending relationships between interpolated results and their dependent system components (i.e., its proceeding interpolation procedures). Then, the classical general diagnostic engine (GDE) [28] is utilised to hypothesise a set of candidates that each may have led to all the system contradictions. Finally, the system consistency is restored by modifying an identified single candidate.

The adaptive approach outlined above assumes that all the contradictory interpolated results are caused by the underpinning interpolation procedures. This assumption restricts the applications of AFRI to problems with defective fuzzy interpolation procedures only, but observations and rules in a fuzzy inference system may also be ill-specified (to a certain extent). Thankfully this limitation is not a fundamental restriction of the idea underlying the adaptive approach. Supported by the initial preliminary investigations of [29], this paper further develops the work of [24], to allow the diagnosis and modification of observations and rules. This significantly enhances the robustness of the original method as one consistent inference result may still be derived when the original fails, often with intuitively more reasonable interpolated results.

Due to the introduction of more complex and uncertain information to the underlying information and knowledge representation scheme, the number of generated candidates may increase dramatically. However, these candidates can be discriminated as: i) two different values derived for a given variable that have led to a contradiction may not be equally reliable (besides, one may be correct and the other wrong); and ii) all the elements which jointly support one of the two contradictory values may not be equally reliable. A candidate prioritisation mechanism is therefore introduced here to reinforce the present work, starting from the initial report of [30], such that only the most important candidates are considered during the modification stage. Firstly, the classical ATMS is extended to record dependencies and also, to log the extent to which such dependencies are deemed reliable. The candidates are then prioritised using a modified GDE by taking the reliability information into consideration. Thanks to the prioritisation of candidates, a consistent solution can be rapidly derived with saved computational cost.

The remainder of this paper is organised as follows. A brief review of the theoretical underpinnings of AFRI is presented in Sec. II. An extension of the candidate generation procedure

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is reported in Sec. III, by which a candidate element can be an observation, rule, or fuzzy interpolation procedure. A generalisation of the candidate modification procedure is discussed in Sec. IV, which allows the modification of all types of diagnosable candidate component. To facilitate comparison, the application problem considered in [24] is reinvestigated in Sec. V where the proposed approach is employed. The paper is concluded in Sec. VI with important future directions of improvements pointed out.

## II. ADAPTIVE FUZZY RULE INTERPOLATION

AFRI ensures that interpolated results remain consistent to a certain degree throughout the entire interpolation process [24]. In this paper, given two fuzzy sets  $A_i$  and  $A_j$  with respect to the same variable  $x$  within the domain  $D_x$ , the degree of consistency between them is represented as the degree of matching as follows:

$$M(A_i, A_j) = \sup_{x \in D_x} [\min(\mu_{A_i}(x), \mu_{A_j}(x))]. \quad (1)$$

Based on this, the degree  $\beta$  of a contradiction regarding two propositions  $P(x \text{ is } A_i)$  and  $P'(x \text{ is } A_j)$  is defined as:

$$\beta = 1 - M(A_i, A_j). \quad (2)$$

This work adopts a predefined threshold  $\beta_0$  ( $0 \leq \beta_0 \leq 1$ ) to examine whether a pair of values associated with a common variable is unacceptably contradictory. A  $\beta_0$ -contradiction appears if the corresponding contradictory degree between the two concerned propositions is greater than  $\beta_0$ .

As with [24], each pair of neighbouring rules, which may be utilised together for interpolation, is termed as a *fuzzy interpolation component* (FIC). The input of such a component is a vector of observations and/or previous inferred results, which is hereafter referred to an interpolation input for simplicity. The output is the consequence of the interpolated rule which takes such an input as its antecedent. The working process of AFRI is illustrated in Fig. 1. Given a fuzzy inference problem with a sparse rule base, the interpolator performs inference through fuzzy rule interpolation, and the ATMS records the dependencies of contradictions upon the preceding FICs. Then, the GDE diagnoses the cause of the contradictions and generates candidates for modification, and finally the modifier revises the candidates to remove contradictions and restore system consistency.

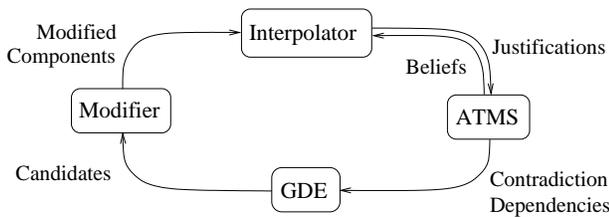


Fig. 1. Adaptive fuzzy interpolation

### A. Rule Interpolation by the Interpolator

Suppose that the interpolation input is

$$O: x_1 = A_{1x}^* \text{ and } \dots \text{ and } x_m = A_{mx}^*, \quad (3)$$

and that rules

$$\begin{aligned} R_i: & \text{ IF } x_1 = A_{1i} \text{ and } \dots \text{ and } x_m = A_{mi}, \text{ THEN } y = B_i, \\ R_j: & \text{ IF } x_1 = A_{1j} \text{ and } \dots \text{ and } x_m = A_{mj}, \text{ THEN } y = B_j, \end{aligned} \quad (4)$$

are the neighbouring ones used for interpolation regarding the input  $O$ . The scale and move transformation-based FRI, upon which AFRI has been introduced, is outlined in Fig. 2. Further details of this approach can be found in [12], [13], but this is out of the scope of this paper.

In this figure, there are  $m$  repeated sub-components, each of which takes  $A_{kx}^*$ ,  $A_{ki}$  and  $A_{kj}$  ( $1 \leq k \leq m$ ) as inputs and produces a relative placement factor  $\lambda_k$ , an intermediate fuzzy set  $A_{kx}^{* \prime}$  and a number of similarity measurements between  $A_{kx}$  and  $A_{kx}^*$ . Each sub-component first uses the so-called *representative values*  $a_{ki}$ ,  $a_{kj}$  and  $a_{kx}^*$  to express the overall positions of  $A_{ki}$ ,  $A_{kj}$  and  $A_{kx}^*$  respectively, computed using the function  $f_1$ . The relation regarding the relative locations between the interpolation input term  $A_{kx}^*$  and the corresponding antecedents terms ( $A_{ki}$  and  $A_{kj}$ ) of a pair of neighbouring rules is computed next, resulting in the required  $\lambda_k$  which is computed by the real function  $f_2$ . From this, an antecedent term of the intermediate rule  $A_{kx}^{* \prime}$  is calculated by applying real function  $f_3$  with a parameter  $\lambda_k$  to  $A_{ki}$  and  $A_{kj}$ . Next, a set of similarity degrees between  $A_{kx}^*$  and  $A_{kx}^{* \prime}$ , expressed as the scale rate  $s_k$ , scale ratio  $\mathbb{S}_k$  and move rate  $\mathbb{M}_k$ , is obtained by applying the function  $f_4$  (which stands for a predefined similarity metric). Function  $f_6$  is then introduced to combine all the resultant  $\lambda_k$  ( $k \in \{1, 2, \dots, m\}$ ) to an overall single scale  $\lambda$ , as is  $f_7$  to combine all the similarity rates ( $s_k, \mathbb{S}_k, \mathbb{M}_k$ ) to  $(s, \mathbb{S}, \mathbb{M})$ . The conclusion  $B^*$  can finally be approximated by transforming the consequent  $B^{* \prime}$  of the intermediate rule. This is implemented by applying the combined single scale similarity rates, through the transformation function  $f_5$ :

$$T(B^*, B^{* \prime}) = T((A_{1x}, \dots, A_{mx}), (A_{1x}^{* \prime}, \dots, A_{mx}^{* \prime})). \quad (5)$$

### B. Truth Maintenance by the ATMS

In implementing AFRI, ATMS is utilised to record the dependency of interpolated results and that of contradictions, upon the FICs from which they are inferred. Using ATMS' terminology, observations, interpolated results, contradictions and FICs can all be represented as ATMS nodes, each of which is formed by a name (standing for its logical or physical meaning), a set of justifications and a label.

A *justification* expresses a logical implication through which a node may be derived from other relevant nodes. An inferred proposition represented as an ATMS node is of the following justification:

$$M_1, M_2, \dots, M_n, R_i R_j \Rightarrow C, \quad (6)$$

where  $R_i R_j$  denotes the FIC formed by the two neighbouring rules  $R_i$  and  $R_j$  ( $i \neq j$ ) which infers the interpolated result  $C$  from  $n$  other nodes  $M_1, M_2, \dots, M_n$  (that are observations and/or interpolated results). Based on the definition of contradiction, a  $\beta_0$ -contradiction is reached if the contradiction degree  $\beta$  between any two propositions  $P(x \text{ is } A_i)$  and  $P'(x$

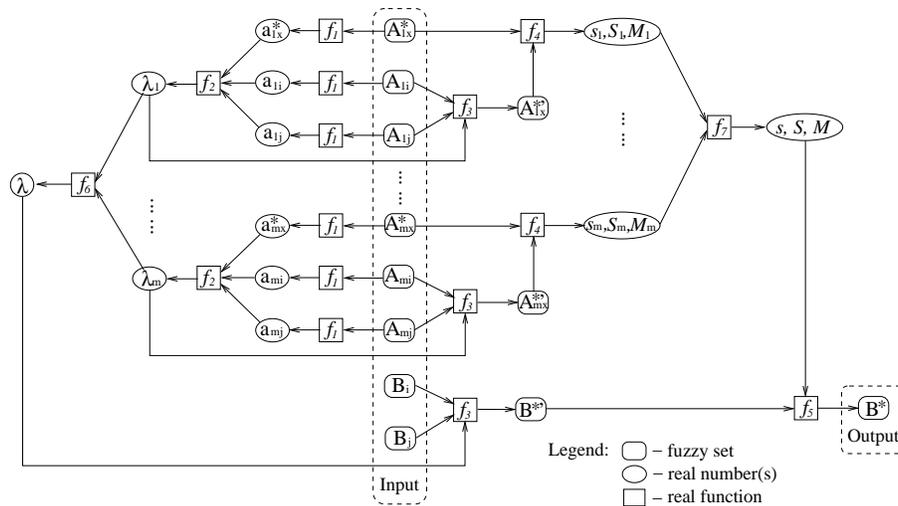


Fig. 2. An outline of transformation-based FRI

is  $A_j$ ) is greater than a predefined threshold  $\beta_0$ , which is expressed in the format of proposition as:

$$P, P' \Rightarrow_{\beta_0} \perp. \quad (7)$$

A *label* is a set of environments, each of which is a minimal set of FICs that jointly entail the supported node. An environment is said to be  $\beta_0$ -inconsistent if  $\beta_0$ -contradiction is logically derivable by the environment and a given justification; otherwise, the environment is  $(1 - \beta_0)$ -consistent. The ATMS label updating algorithm ensures that the label of each node is  $(1 - \beta_0)$ -consistent, sound, minimal and complete, except that the label of the special ‘false’ node is  $\beta_0$ -inconsistent rather than  $(1 - \beta_0)$ -consistent. Whenever a  $\beta_0$ -contradiction is detected, each environment in its label is added into the label of the ‘false’ node and all such environments and their supersets are removed from the label of every other node. Also, any such environment which is a superset of another is removed from the label of the ‘false’ node. Therefore, the label of the special “false” node collectively holds the minimal, complete set of environments each of which leads to a  $\beta_0$ -contradiction.

### C. Candidate Generation by the GDE

A set of minimal candidates for modification can be generated by GDE [28] from the label of the ‘false’ node. A candidate is a set of FICs that may have led to all detected contradictions. Since a  $\beta_0$ -inconsistent environment contained in the label of the ‘false’ node indicates that at least one of its elements is inconsistent (or faulty), a candidate must have a non-empty intersection with each  $\beta_0$ -inconsistent environment. Based on this observation, each candidate is constructed by taking just one FIC from each environment that supports the ‘false’ node. The candidates are guaranteed to be minimal by removing all the supersets of others. As a result of this, the successful correction of any single candidate will remove all contradictions.

### D. Candidate Modification by the Modifier

AFRI always modifies the candidate with the smallest cardinality first. With respect to a given queue of candidates  $\mathbb{Q}$ , the overall modification procedure is outlined in Alg. 1. The main sub-procedure MODIFY(C) takes a single candidate (C) as input and returns a Boolean value to indicate whether the modification succeeds or not.

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#### Algorithm 1 The CONSISTENCYRESTORING procedure

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CONSISTENCYRESTORING( $\mathbb{Q}$ )

**Input:**  $\mathbb{Q}$ , a queue of candidates, each of which is a set of FICs.

**Output:** **True**, if succeeds; **False**, otherwise.

- 1) *modified*  $\leftarrow$  **False**
  - 2) **do**
  - 3)      $C \leftarrow \text{Dequeue}(\mathbb{Q})$
  - 4)     *modified*  $\leftarrow$  MODIFY(C)
  - 5) **while** ((*modified* == **False**) && ( $\mathbb{Q} \neq \emptyset$ ))
  - 6) **return** *modified*
- 

To illustrate the basic ideas embedded in this sub-procedure, suppose that the defective FIC is formed by the pair of neighbouring rules as given in Eq. 4, which flanks the interpolation inputs  $O_x(x \in \{1, 2, \dots, n\})$  in the form of Eq. 3. The implementation of the modification procedure for a candidate consisting of a single FRI can then be summarised in the following steps:

**Step 1.** Find the interpolated rule ‘IF  $x_1 = A_{1k}^*$  and  $\dots$  and  $x_m = A_{mk}^*$ , THEN  $y = B_k^*$ ’ whose antecedent is located in the middle most of the neighbourhood of the antecedents of the two rules used for interpolation, in terms of their representative values that are calculated using a particular integration formula [24]. Suppose that the *relative placement factor* of its consequence  $\lambda_k$  is modified to  $\hat{\lambda}_k$ . The *correction rate pair* can then be calculated as:

$$\begin{cases} c^- = \frac{\hat{\lambda}_k}{\lambda_k} \\ c^+ = \frac{1 - \hat{\lambda}_k}{1 - \lambda_k} \end{cases} \quad (8)$$

**Step 2.** Obtain the modified *relative placement factors* of the consequences of all other interpolated rules, which have been created with respect to the same defective FIC in the same way as that used to compute the *correction rate pair* above, where  $p \in \{1, 2, \dots, k-1\}$  and  $q \in \{k+1, k+2, \dots, n\}$ :

$$\begin{cases} \widehat{\lambda}_p = \lambda_p \cdot c^- \\ 1 - \widehat{\lambda}_q = (1 - \lambda_q) \cdot c^+. \end{cases} \quad (9)$$

**Step 3.** Compute the modified consequences of the intermediate rules corresponding to all interpolated rules that have been generated from the same defective FIC in accordance with their modified *relative placement factors*. Suppose that the intermediate rule corresponding to defective rule ‘IF  $x_1 = A_{1x}^*$  and  $\dots$  and  $x_m = A_{mx}^*$ , THEN  $y = B_x^*$ ’ is ‘IF  $x_1 = A_{1x}'$  and  $\dots$  and  $x_m = A_{mx}'$ , THEN  $y = B_x'$ ’. From this, the modified consequence of the intermediate rule  $\widehat{B}_x'$  is:

$$\widehat{B}_x' = (1 - \widehat{\lambda}_x)B_i + \widehat{\lambda}_x B_j, \quad (10)$$

where  $x \in \{1, 2, \dots, n\}$ . That is, the modified intermediate rule becomes ‘IF  $x_1 = A_{1x}'$  and  $\dots$  and  $x_m = A_{mx}'$ , THEN  $y = \widehat{B}_x'$ ’.

**Step 4.** Compute the modified consequences of all interpolated rules from the consequences of the modified intermediate rules through scale and move transformations:

$$T((A_{1x}', \dots, A_{mx}'), (A_{1x}^*, \dots, A_{mx}^*)) = T(\widehat{B}_x', \widehat{B}_x^*), \quad (11)$$

where  $x \in \{1, 2, \dots, n\}$ , and  $T(\cdot, \cdot)$  represents the transformations based on the scale and move measures [12], [13].

**Step 5.** Impose restriction over the modified consequence such that it becomes consistent with the interpolation context. Suppose that  $m$  object values  $B_1, B_2, \dots, B_m$  are obtained for the variable  $y$ . If they are  $(1 - \beta_0)$ -consistent, they must satisfy:

$$\bigcap_{j=1}^m (B_j)_{\beta_0} \neq \emptyset, \quad (12)$$

where  $(B_j)_{\beta_0}$  denotes the  $\beta_0$ -cut of fuzzy set  $B_j$ .

**Step 6.** Constrain the propagations of all modified consequences so that they are consistent with the rest. Propagate the modified result through the entire reasoning network. For a given variable  $z$ , suppose that  $m$  object values of the variable  $z$  have been modified via the propagation, resulting in modified values  $\widehat{C}_i, i \in \{1, 2, \dots, m\}$ , and that  $n$  object values  $C_j, j \in \{1, 2, \dots, n\}$ , of  $z$  are not affected by the propagation. These modified consequences must satisfy the following such that they are all  $(1 - \beta_0)$ -consistent:

$$\left( \bigcap_{i=1}^m (\widehat{C}_i)_{\beta_0} \right) \cap \left( \bigcap_{l=1}^n (C_l)_{\beta_0} \right) \neq \emptyset. \quad (13)$$

**Step 7.** Solve the set of simultaneous equalities and inequalities as posed above. The solutions imply successfully modified results which guarantee the system reasoning consistency.

### III. GENERALISING CANDIDATE GENERATION

Only FICs are regarded as diagnosable and modifiable candidate elements in the original AFRI approach outlined

above. However, observations and rules may also be faulty to a certain extent. This section extends the existing AFRI such that observations and rules can also be diagnosed and modified. To facilitate this, the certainty degrees of observations, rules and FICs are discussed first.

#### A. Certainty Degrees of Observations and Rules

There are generally four categories of inexact information [31]: 1) vagueness, 2) uncertainty, 3) both vagueness and uncertainty with the latter represented as real numbers, and 4) both vagueness and uncertainty with the latter also defined as fuzzy sets. The existing FRI [24] only considers type 1 information, which is extended in this work by introducing type 2 information into the system, thereby resulting in the exploitation of type 3 information overall.

With the extra information, an observation is represented as:

$$O: x_i = A_{ij}^*(c_O), \quad (14)$$

where  $0 \leq c_O \leq 1$  expresses the certainty degree of the observation  $O$ . Conceptually, the vagueness of an object value can be modelled as a fuzzy set due to the lack of a precise boundary between given bits of information. Here, the certainty degree of an observation is represented as a crisp number, which is either assigned subjectively [32] or estimated from other mechanisms such as statistical data analysis. It indicates the confident level at which the current description of the object value may be regarded as of confidence or being reliable.

Denote the certainty degree of an observation  $O$  as  $c_O$ . Then, the uncertainty degree of the same piece of information is naturally expressed as  $1 - c_O$ . Thus, the modifiable range of the object value  $O$  is intuitively bounded to the proportion of  $1 - c_O$  in reference to the entire variable domain. This means that the factual object value of  $O$  can be obtained by shifting the fuzzy set representation of the defective observation towards either side of the variable domain to a maximal distance of  $\frac{1-c_O}{2}(\max_i - \min_i)$ , where the domain of the variable  $x_i$  is  $D_{x_i} = [\min_i, \max_i]$ . Given that the shifting of a vague term is restricted from changing the shape and area of the underlying fuzzy set, the shifting process is equivalent to adding a real number to the original fuzzy set [33]. Formally, the factual value of  $A_{ij}^*$ , denoted as  $\widehat{A}_{ij}^*$ , of the observation  $O$  as given in Eq. 14 must satisfy:

$$\begin{cases} \widehat{A}_{ij}^* \geq A_{ij} - \frac{1-c_O}{2}(\max_i - \min_i) \\ \widehat{A}_{ij}^* \leq A_{ij} + \frac{1-c_O}{2}(\max_i - \min_i). \end{cases} \quad (15)$$

It is possible that the shifting may be out of the variable domain due to the inaccuracy of the uncertainty information. Therefore, to ensure the final shifting result is within the value range of the variable, the following must be satisfied :

$$\begin{cases} \min(\text{supp}(\widehat{A}_{ij}^*)) \geq \min_i \\ \max(\text{supp}(\widehat{A}_{ij}^*)) \leq \max_i, \end{cases} \quad (16)$$

where  $\text{supp}(\widehat{A}_{ij}^*)$  represents the support of  $\widehat{A}_{ij}^*$ .

Similarly, with the uncertainty information, rules given in Eq. 4 are then extended to be of the following form:

$$\begin{aligned} R_i &: \text{IF } x_1 = A_{1i} \text{ and } \dots \text{ and } x_m = A_{mi}, \\ &\text{THEN } y = B_i (c_{R_i}); \\ R_j &: \text{IF } x_1 = A_{1j} \text{ and } \dots \text{ and } x_m = A_{mj}, \\ &\text{THEN } y = B_j (c_{R_j}). \end{aligned} \quad (17)$$

This means that rules  $R_i$  and  $R_j$  are certain to the degree of  $c_{R_i}$  and  $c_{R_j}$ , respectively. As with the certainty degrees associated with observations, certainty degrees attached to the rules are either subjectively provided or objectively learned.

### B. Certainty Degrees of FICs

A FIC consisted of two neighbouring rules is utilised in this work to represent the fuzzy interpolation mechanism. Essentially, this mechanism is an extension of classical linear interpolation on fuzzy rules. Thus, intuitively, if a FIC is defined on a pair of neighbouring rules that are more certain to derive correct interpolated results, such an artificially created component is deemed to be more reliable, under the linearity assumption. Suppose that the FIC  $R_i R_j$  consists of the following two single-antecedent rules:

$$\begin{aligned} R_i &: \text{IF } x = A_i, \text{ THEN } y = B_i (c_{R_i}); \\ R_j &: \text{IF } x = A_j, \text{ THEN } y = B_j (c_{R_j}). \end{aligned} \quad (18)$$

Then, reflecting this intuition, the certainty degree  $c_{R_i R_j}$  of the component  $R_i R_j$  can be defined by:

$$c_{R_i R_j} = 1 - \left| \frac{d(A_i, A_j)}{\max_x - \min_x} - \frac{d(B_i, B_j)}{\max_y - \min_y} \right|, \quad (19)$$

where  $d(A, A')$  is the distance between  $A$  and  $A'$  (given a certain distance metric);  $\max_z$  and  $\min_z$  are the maximum and minimum of the domain values of the variable  $z$  ( $z = x, y$ ), respectively. Note that  $c_{R_i R_j} \in [0, 1]$ .

For the more general cases where the FIC  $R_i R_j$  is composed by two multi-antecedent rules as given in Eq. 17, the calculation of the certainty degree can be readily extended. The result is given as follows:

$$c_{R_i R_j} = 1 - \left| \frac{\sum_{k=1}^m \frac{d(A_{ki}, A_{kj})}{\max_{x_k} - \min_{x_k}}}{m} - \frac{d(B_i, B_j)}{\max_y - \min_y} \right|. \quad (20)$$

In this equation, the distance between the two sets of antecedents of two multi-antecedent fuzzy rules is defined as the average of the distances between all pairs of corresponding antecedent terms regarding each corresponding variable. This is again, to reflect the underlying linearity assumption.

### C. Certainty Degrees of Interpolated Results

Given an interpolation input  $M_1, M_2, \dots, M_n$ , two neighbouring rules  $R_i$  and  $R_j$  that flank the given interpolation input, and the corresponding FIC  $R_i R_j$ , a logical consequence  $C$  can be generated by applying FRI. Then, the certainty degree  $c_C$  of the conclusion  $C$  can be derived from the certainty degrees of the input terms, the certainty degree

of the neighbouring rules and the certainty degree of the corresponding FIC, which is calculated by:

$$c_C = c_{M_1} \otimes c_{M_2} \otimes \dots \otimes c_{M_n} \otimes c_{R_i} \otimes c_{R_j} \otimes c_{R_i R_j}, \quad (21)$$

where the composition operator  $\otimes$  is a t-norm operator, such as minimum and algebraic product. Note that multiple applications of different interpolation procedures may lead to the same interpolated result  $C$ . However, they may be associated with different certainty degrees, say  $c_{C_1}, c_{C_2}, \dots, c_{C_n}$ . Then the overall certainty degree  $c$  of the interpolated result  $C$  is revised as:

$$c = c_{C_1} \oplus c_{C_2} \oplus \dots \oplus c_{C_n}, \quad (22)$$

where  $\oplus$  is an s-norm operator, such as maximum.

### D. Dependency Recording with Extended ATMS

In the previous work of [24], ATMS records the dependencies of the contradictions (or interpolated results) upon FICs. However, in general, such contradictions may also depend upon the observations and rules used to perform FRI. Therefore, observations, interpolated results, contradictions, FICs, and rules are all represented as ATMS nodes in the present work, which are originally assumed to be true and which may be established to be false (of a certain degree) subsequently. Recall that a justification describes how a node is derivable from other nodes. In general, any ATMS node with an interpolated result  $C$  from an interpolation input  $M_1, M_2, \dots, M_n$  based on neighbouring rules  $R_i$  and  $R_j$  may now be verified by the following ATMS justification:

$$M_1, M_2, \dots, M_n, R_i, R_j, R_i R_j \Rightarrow C. \quad (23)$$

Eq. 23 degenerates to Eq. 6 when rules  $R_i$  and  $R_j$  ( $i \neq j$ ) are fixed and true, and hence not needed to be kept in the dependency records.

The above justification not only explicitly describes how the consequence  $C$  is logically derived from other nodes, but also implicitly expresses to what extent  $C$  can be derived from the nodes  $M_1, M_2, \dots, M_n, R_i, R_j$  and  $R_i R_j$ , with the support of their certainty values. This implicit information is explicitly held in extended ATMS nodes. The certainty degrees of primitive ATMS nodes, including observations, rules and FICs have been discussed in the previous sections, which can be directly used here to extend the corresponding ATMS nodes. The certainty degree of an interpolated result can be derived from its entire set of label environments, based on Eq. 22, whilst the extent to which each individual environment entails the concerned interpolated result can be computed on the basis of Eq. 21. The process of calculating and updating of the certainty degrees of interpolated results is effectively managed by an extended ATMS label-updating mechanism. As a result, an extended ATMS node not only expresses how it is entailed by its label environments, but also indicates to what extent the node is derivable from the label environments.

### E. Candidate Generation with Extended GDE

A  $\beta_0$ -contradiction occurs if two object values are observed and/or derived for a common variable that differ to the extent

of at least  $\beta_0$  and therefore, one or both of the two values is faulty. Due to lack of differentiating information, both contradictory values are supposed to be equally faulty in [24]. With the support of additional information of certainty degrees as recorded in the extended ATMS, two values for a common variable can be distinguished in response to the extent to which each of them is derivable. In addition, for any one of the two ATMS nodes representing the two observations/interpolated results, the elements in its label environments are also distinguishable as some of the elements are of higher certainty degrees than others. Within the label environment of either of the two contradictory values, those elements with the smallest certainty degree are intuitively regarded as the most likely to be the real culprit. Based on these observations, the candidates generated by GED can be prioritised. In order to do so, all the elements in the label environments of the ‘false’ node are ranked first.

Suppose that  $E_{\perp}$  is one of the label environments of the ‘false’ node which is deduced by two contradictory proposition  $P$  and  $P'$ . Then there must exist environments  $E = \{e_1, e_2, \dots, e_m\}$  and  $E' = \{e'_1, e'_2, \dots, e'_n\}$  which entail the corresponding propositions such that  $E \cup E' = E_{\perp}$ . Suppose that the certainty degrees associated with the propositions  $P$  and  $P'$  are  $c$  and  $c'$ , respectively. The procedure of prioritising the label elements of  $E_{\perp}$ , by assigning a ranking value to each element, is shown in Alg. 2. Assuming that  $c \leq c'$ , this algorithm guarantees that  $e_i \leq e'_j, i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, n\}$ , and vice versa.

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**Algorithm 2** The ELEMENTRANKING procedure

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ELEMENTRANKING( $E, E', c, c'$ )

- 1)  $E_{\perp} = E \cup E'$
  - 2) **foreach**  $e \in E_{\perp}$
  - 3)     **if**  $(c \leq c' \ \&\& \ e \in E')$   $|(c' \leq c \ \&\& \ e \in E)$
  - 4)          $r_e = c_e + 1$
  - 5)     **else**
  - 6)          $r_e = c_e$
- 

Recall that each label environment of the ‘false’ node entails a contradiction. Thus, by taking one element from each environment of the ‘false’ node, a candidate is constructed. Repeating this will generate all possible candidates. If all the duplications are deliberately kept, all the originally generated candidates will have the same cardinality, equalling to the number of label environments in the ‘false’ node. From this, all candidates can be prioritised according to the ranking values of their members. Alg. 3 shows a two-step sorting method for this. After the ranking, duplications of candidate elements are removed, and all those candidates which are a superset of one other candidate are also removed to guarantee the candidate set is minimal. Obviously, such removals do not alter the ranking order of the remaining candidates.

Note that a number of extensions to the classic ATMS and GDE have been proposed in the literature. A possibilistic ATMS was proposed in [34], where all the assumptions and justifications are associated with possibility values and handled in the framework of possibility theory [35]. A credibilistic

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**Algorithm 3** The CANDIDATESORTING procedure

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CANDIDATESORTING( $\mathbb{S}$ )

- Input:**  $\mathbb{S}$ , a set of candidates with the same cardinality.
- 1) **foreach**  $C \in \mathbb{S}$
  - 2)     SORT ( $C$ ) // Sort all the members of  $C$  in ascending order by their ranking values
  - 3) **foreach**  $i = |C| : 1$
  - 4)     STABLESORT( $\mathbb{S}, i$ ) // Sort all the candidates in ascending order by the ranking values of their  $i^{th}$  members
- 

ATMS was proposed in [36], which is developed on the basis of credibility theory [37]. The approach of [38] and [39] generalised the classical ATMS to work with reasoning systems using multi-valued logic. The present work differs from these extensions as reliability values are used to reflect certainty degrees. Note too that classical GDE has also been extended from other perspectives, such as for reducing search spaces [40], and for modelling in situations where connections may also be faulty [40]. All these extensions to ATMS and GDE are interesting in further generalising the present study, but are beyond the scope of this paper.

*F. Illustrative Example - Part 1*

The running example in the original work on adaptive fuzzy rule interpolation [24] is reconsidered herein, but all the rules and observations are now associated with the information of certainty degrees. For completeness, the rule base is provided below:

- $R_1$ : IF  $x_1 = A_1$ , THEN  $x_2 = B_1$  (0.80);  
 $R_2$ : IF  $x_1 = A_2$ , THEN  $x_2 = B_2$  (0.90);  
 $R_3$ : IF  $x_2 = B_3$ , THEN  $x_3 = C_3$  (0.60);  
 $R_4$ : IF  $x_2 = B_4$ , THEN  $x_3 = C_4$  (0.70);  
 $R_5$ : IF  $x_3 = C_5$ , THEN  $x_6 = F_5$  (0.70);  
 $R_6$ : IF  $x_3 = C_6$ , THEN  $x_6 = F_6$  (0.80);  
 $R_7$ : IF  $x_3 = C_7$  and  $x_4 = D_7$ , THEN  $x_5 = E_7$  (0.90);  
 $R_8$ : IF  $x_3 = C_8$  and  $x_4 = D_8$ , THEN  $x_5 = E_8$  (0.60);  
 $R_9$ : IF  $x_6 = F_9$ , THEN  $x_7 = G_9$  (0.90);  
 $R_{10}$ : IF  $x_6 = F_{10}$ , THEN  $x_7 = G_{10}$  (0.80);  
 $R_{11}$ : IF  $x_5 = E_{11}$ , THEN  $x_7 = G_{11}$  (0.70);  
 $R_{12}$ : IF  $x_5 = E_{12}$ , THEN  $x_7 = G_{12}$  (0.90).

The parameter set and representation schemes used in [24] are also utilised in this work and thus the details are omitted. With the support of extra information, suppose that the four observations are now:  $O_1 : x_1 = A^* = (9.0, 9.5, 10.0, 10.5)$  (0.70),  $O_2 : x_2 = B^* = (7.0, 7.5, 8.0, 8.5)$  (0.60),  $O_3 : x_4 = D^* = (5.5, 6.0, 6.5, 7.0)$  (0.90) and  $O_4 : x_6 = F^* = (11.0, 11.5, 12.0, 12.5)$  (0.80). By applying the classical scale and move transformation-based FRI, multiple pairs of contradictions result (e.g.,  $F^*$  and  $F_2^*$ ), which are summarised in Fig. 3.

The interpolation procedures are outlined as a component-based diagram, as illustrated in Fig. 4. In this figure,

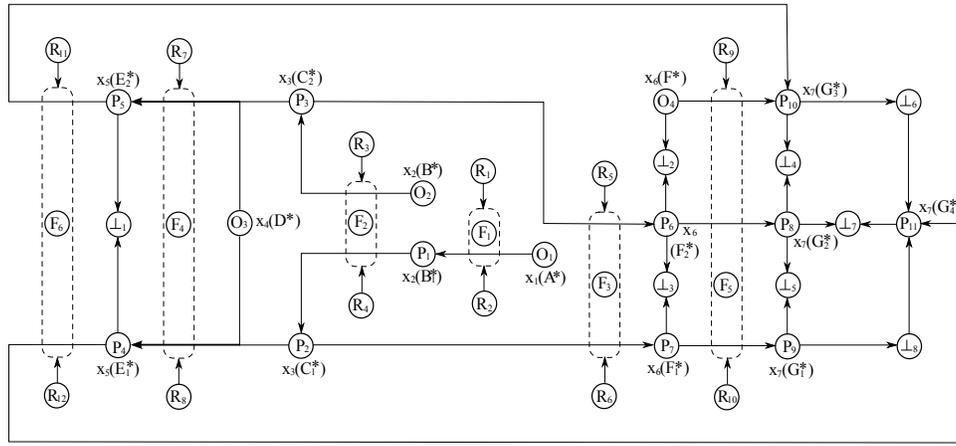


Fig. 4. Discrepancy records in ATMS

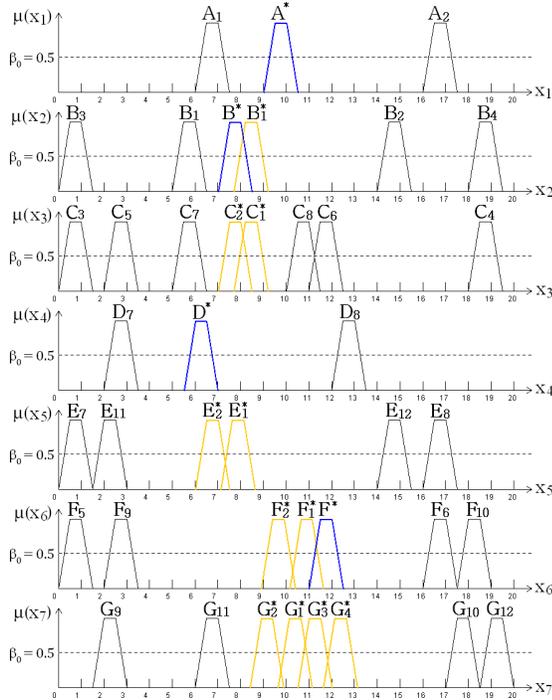


Fig. 3. Fuzzy sets and contradictions involved in the example

all the ATMS nodes and contradictions are shown as circles. Take node  $P_5$  as an example. This node is inferred from the nodes  $P_3$  and  $O_3$  by the FIC  $F_4$  which uses the rules  $R_7$  and  $R_8$ , whose justification is therefore  $P_3, O_3, R_7, R_8, F_4 \Rightarrow P_5$ , where  $O_3$  is an observation and  $P_3$  is a previously interpolated result. By running the label-updating algorithm of the extended ATMS, the label of the node  $P_5$  ( $\{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4\}\}$ ) can be derived from the labels of: the observation  $O_3$  ( $\{\{O_3\}\}$ ), the interpolated result  $P_3$  ( $\{\{O_2, R_3, R_4, F_2\}\}$ ), the rules  $R_7$  ( $\{\{F_7\}\}$ ) and  $R_8$  ( $\{\{F_8\}\}$ ), and the FIC  $F_4$  ( $\{\{F_4\}\}$ ).

The certainty degrees of all FICs can be obtained by applying the approach introduced in Sec. III-B. For instance,

the certainty degree of the FIC  $F_1$  is calculated as follows:

$$\begin{aligned} c_{F_1} &= 1 - \left| \frac{d(A_1, A_2)}{\max_{x_1} - \min_{x_1}} - \frac{d(B_1, B_2)}{\max_{x_2} - \min_{x_2}} \right| \\ &= 1 - \left| \frac{\text{Rep}(A_2) - \text{Rep}(A_1)}{\max_{x_1} - \min_{x_1}} - \frac{\text{Rep}(B_2) - \text{Rep}(B_1)}{\max_{x_2} - \min_{x_2}} \right| \\ &= 1 - \left| \frac{16.75 - 6.75}{20 - 0} - \frac{14.75 - 5.75}{20 - 0} \right| \\ &= 0.05, \end{aligned}$$

where  $\text{Rep}(A)$  denotes the representative value of the fuzzy set  $A$  [12]. The certainty degrees of derived nodes can be computed by following Eq. 22. As an example, the certainty degree of the derived node  $P_{10}$  is computed as follows:

$$\begin{aligned} c_{P_{10}} &= (c_{O_2} \otimes c_{O_3} \otimes c_{R_3} \otimes c_{R_4} \otimes c_{F_2} \otimes c_{R_7} \otimes c_{R_8} \otimes c_{F_4} \\ &\quad \otimes c_{R_{11}} \otimes c_{R_{12}} \otimes c_{F_6}) \oplus (c_{O_4} \otimes c_{R_9} \otimes c_{R_{10}} \otimes c_{F_5}) \\ &= \max(0.60 * 0.90 * 0.60 * 0.70 * 1.00 * 0.90 * 0.60 * \\ &\quad 0.75 * 0.70 * 0.90 * 1.00, 0.80 * 0.90 * 0.80 * 1.00) \\ &= 0.58. \end{aligned}$$

The certainty degrees of all other derived nodes can be calculated in the same manner. All the ATMS nodes (i.e., observations, rules, FICs) and contradictions are summarised below:

- $R_1 : \langle x_1 = A_1 \Rightarrow x_2 = B_1, 0.80, \{\{R_1\}\} \rangle;$
- $R_2 : \langle x_1 = A_2 \Rightarrow x_2 = B_2, 0.90, \{\{R_2\}\} \rangle;$
- $R_3 : \langle x_2 = B_3 \Rightarrow x_3 = C_3, 0.60, \{\{R_3\}\} \rangle;$
- $R_4 : \langle x_2 = B_4 \Rightarrow x_3 = C_4, 0.70, \{\{R_4\}\} \rangle;$
- $R_5 : \langle x_3 = C_5 \Rightarrow x_6 = F_5, 0.70, \{\{R_5\}\} \rangle;$
- $R_6 : \langle x_3 = C_6 \Rightarrow x_6 = F_6, 0.80, \{\{R_6\}\} \rangle;$
- $R_7 : \langle x_3 = C_7, x_4 = D_7 \Rightarrow x_5 = E_7, 0.90, \{\{R_7\}\} \rangle;$
- $R_8 : \langle x_3 = C_8, x_4 = D_8 \Rightarrow x_5 = E_8, 0.60, \{\{R_8\}\} \rangle;$
- $R_9 : \langle x_6 = F_9 \Rightarrow x_7 = G_9, 0.90, \{\{R_9\}\} \rangle;$
- $R_{10} : \langle x_6 = F_{10} \Rightarrow x_7 = G_{10}, 0.80, \{\{R_{10}\}\} \rangle;$
- $R_{11} : \langle x_5 = E_{11} \Rightarrow x_7 = G_{11}, 0.70, \{\{R_{11}\}\} \rangle;$
- $R_{12} : \langle x_5 = E_{12} \Rightarrow x_7 = G_{12}, 0.90, \{\{R_{12}\}\} \rangle;$
- $F_1 : \langle R_1 R_2, 0.95, \{\{F_1\}\} \rangle;$
- $F_2 : \langle R_3 R_4, 1.00, \{\{F_2\}\} \rangle;$
- $F_3 : \langle R_5 R_6, 0.65, \{\{F_3\}\} \rangle;$
- $F_4 : \langle R_7 R_8, 0.75, \{\{F_4\}\} \rangle;$
- $F_5 : \langle R_9 R_{10}, 1.00, \{\{F_5\}\} \rangle;$
- $F_6 : \langle R_{11} R_{12}, 1.00, \{\{F_6\}\} \rangle;$
- $O_1 : \langle x_1 = A^*, 0.70, \{\{O_1\}\} \rangle;$
- $O_2 : \langle x_1 = B^*, 0.60, \{\{O_2\}\} \rangle;$

$O_3 : \langle x_4 = D^*, 0.90, \{\{O_3\}\} \rangle;$   
 $O_4 : \langle x_6 = F^*, 0.80, \{\{O_4\}\} \rangle;$   
 $P_1 : \langle x_2 = B_1^*, 0.48, \{\{O_1, R_1, R_2, F_1\}\} \rangle;$   
 $P_2 : \langle x_3 = C_1^*, 0.20, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2\}\} \rangle;$   
 $P_3 : \langle x_3 = C_2^*, 0.25, \{\{O_2, R_3, R_4, F_2\}\} \rangle;$   
 $P_4 : \langle x_5 = E_1^*, 0.07, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}\} \rangle;$   
 $P_5 : \langle x_5 = E_2^*, 0.09, \{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4\}\} \rangle;$   
 $P_6 : \langle x_6 = F_2^*, 0.09, \{\{O_2, R_3, R_4, F_2, R_5, R_6, F_3\}\} \rangle;$   
 $P_7 : \langle x_6 = F_1^*, 0.07, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}\} \rangle;$   
 $P_8 : \langle x_7 = G_2^*, 0.06, \{\{O_2, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\} \rangle;$   
 $P_9 : \langle x_7 = G_1^*, 0.05, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\} \rangle;$   
 $P_{10} : \langle x_7 = G_3^*, 0.58, \{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}, \{O_4, R_9, R_{10}, F_5\}\} \rangle;$   
 $P_{11} : \langle x_7 = G_4^*, 0.05, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}\} \rangle;$   
 $\perp_1 : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}\} \rangle;$   
 $\perp_2 : \langle \perp, \{\{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3\}\} \rangle;$   
 $\perp_3 : \langle \perp, \{\{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}\} \rangle;$   
 $\perp_4 : \langle \perp, \{\{O_2, O_3, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\} \rangle;$   
 $\perp_5 : \langle \perp, \{\{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\} \rangle;$   
 $\perp_6 : \langle \perp, \{\{O_1, O_3, O_4, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}\} \rangle;$   
 $\perp_7 : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\} \rangle;$   
 $\perp_8 : \langle \perp, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\} \rangle.$

The ‘false’ node, denoted by  $P_{\perp}$ , collectively represents all the contradictions  $\perp_1, \perp_2, \dots, \perp_8$  by only containing a minimal set of label environments, which is given as follows:  
 $P_{\perp} : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}, \{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3\}, \{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}, \{O_2, O_3, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_3, O_4, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\} \rangle.$

Applying the extended GDE as introduced in Sec. III-D, a ranked list of minimal candidates (including 85 candidates) are generated as follows:

$C_1 = [R3, 0.6], C_2 = [O2, 0.6; R8, 0.6],$   
 $C_3 = [R8, 0.6; F3, 0.65], C_4 = [O2, 0.6; F3, 0.65; O4, 0.8],$   
 $C_5 = [O2, 0.6; O1, 0.7], C_6 = [R8, 0.6; R5, 0.7],$   
 $C_7 = [O2, 0.6; R11, 0.7], C_8 = [O2, 0.6; R5, 0.7; O4, 0.8],$   
 $C_9 = [R8, 0.6; O1, 0.7; O4, 0.8], C_{10} = [O2, 0.6; F4, 0.75],$   
 $C_{11} = [O2, 0.6; R1, 0.8], C_{12} = [O2, 0.6; O4, 0.8; R6, 0.8],$   
 $C_{13} = [R8, 0.6; O4, 0.8; R1, 0.8],$   
 $C_{14} = [R8, 0.6; R6, 0.8], C_{15} = [O2, 0.6; R10, 0.8],$   
 $C_{16} = [R8, 0.6; O4, 0.8; R2, 0.9],$   
 $C_{17} = [R8, 0.6; O4, 0.8; F1, 0.95],$   
 $C_{18} = [O2, 0.6; O3, 0.9], C_{19} = [O2, 0.6; R2, 0.9],$

$C_{20} = [O2, 0.6; R7, 0.9], C_{21} = [O2, 0.6; R9, 0.9],$   
 $C_{22} = [O2, 0.6; R12, 0.9], C_{23} = [O2, 0.6; F1, 0.95],$   
 $C_{24} = [O2, 0.6; F5, 1.0], C_{25} = [O2, 0.6; F6, 1.0],$   
 $C_{26} = [F3, 0.65; O1, 0.7], C_{27} = [F3, 0.65; F4, 0.75],$   
 $C_{28} = [F3, 0.65; R1, 0.8], C_{29} = [F3, 0.65; O3, 0.9],$   
 $C_{30} = [F3, 0.65; R2, 0.9], C_{31} = [F3, 0.65; R7, 0.9],$   
 $C_{32} = [F3, 0.65; F1, 0.95],$   
 $C_{33} = [O1, 0.7; R5, 0.7; O1, 0.7], C_{34} = [R4, 0.7],$   
 $C_{35} = [O1, 0.7; R11, 0.7; O4, 0.8],$   
 $C_{36} = [R5, 0.7; F4, 0.75],$   
 $C_{37} = [O1, 0.7; F4, 0.75; O4, 0.8],$   
 $C_{38} = [O1, 0.7; R6, 0.8],$   
 $C_{39} = [O1, 0.7; O4, 0.8; R10, 0.8],$   
 $C_{40} = [R5, 0.7; R1, 0.8], C_{41} = [O1, 0.7; O4, 0.8; O3, 0.9],$   
 $C_{42} = [O1, 0.7; O4, 0.8; R7, 0.9],$   
 $C_{43} = [O1, 0.7; O4, 0.8; R9, 0.9],$   
 $C_{44} = [O1, 0.7; O4, 0.8; R12, 0.9],$   
 $C_{45} = [O1, 0.7; O4, 0.8; F5, 1.0],$   
 $C_{46} = [O1, 0.7; O4, 0.8; F6, 1.0],$   
 $C_{47} = [R5, 0.7; O3, 0.9], C_{48} = [R5, 0.7; R2, 0.9],$   
 $C_{49} = [R5, 0.7; R7, 0.9], C_{50} = [R5, 0.7; F1, 0.95],$   
 $C_{51} = [R11, 0.7; R1, 0.8; O4, 0.8; R1, 0.8],$   
 $C_{52} = [R11, 0.7; R1, 0.8; R6, 0.8],$   
 $C_{53} = [R11, 0.7; O4, 0.8; R2, 0.9],$   
 $C_{54} = [R11, 0.7; O4, 0.8; F1, 0.95],$   
 $C_{55} = [F4, 0.75; O4, 0.8; R1, 0.8],$   
 $C_{56} = [F4, 0.75; R6, 0.8],$   
 $C_{57} = [F4, 0.75; O4, 0.8; R2, 0.9],$   
 $C_{58} = [F4, 0.75; O4, 0.8; F1, 0.95],$   
 $C_{59} = [R1, 0.8; R6, 0.8; R1, 0.8],$   
 $C_{60} = [R1, 0.8; O4, 0.8; R1, 0.8; R10, 0.8],$   
 $C_{61} = [R1, 0.8; O4, 0.8; R1, 0.8; R9, 0.9],$   
 $C_{62} = [R1, 0.8; O4, 0.8; R1, 0.8; R12, 0.9],$   
 $C_{63} = [R1, 0.8; O4, 0.8; R1, 0.8; F5, 1.0],$   
 $C_{64} = [R1, 0.8; O4, 0.8; R1, 0.8; F6, 1.0],$   
 $C_{65} = [O4, 0.8; R1, 0.8; O3, 0.9],$   
 $C_{66} = [O4, 0.8; R1, 0.8; R7, 0.9], C_{67} = [R6, 0.8; O3, 0.9],$   
 $C_{68} = [R6, 0.8; R2, 0.9], C_{69} = [R6, 0.8; R7, 0.9],$   
 $C_{70} = [O4, 0.8; R10, 0.8; R2, 0.9],$   
 $C_{71} = [R6, 0.8; F1, 0.95],$   
 $C_{72} = [O4, 0.8; R10, 0.8; F1, 0.95],$   
 $C_{73} = [O4, 0.8; R2, 0.9; O3, 0.9],$   
 $C_{74} = [O4, 0.8; R2, 0.9; R7, 0.9],$   
 $C_{75} = [O4, 0.8; R2, 0.9; R9, 0.9],$   
 $C_{76} = [O4, 0.8; R2, 0.9; R12, 0.9],$   
 $C_{77} = [O4, 0.8; O3, 0.9; F1, 0.95],$   
 $C_{78} = [O4, 0.8; R7, 0.9; F1, 0.95],$   
 $C_{79} = [O4, 0.8; R2, 0.9; F5, 1.0],$   
 $C_{80} = [O4, 0.8; R2, 0.9; F6, 1.0],$   
 $C_{81} = [O4, 0.8; R9, 0.9; F1, 0.95],$   
 $C_{82} = [O4, 0.8; R12, 0.9; F1, 0.95],$   
 $C_{83} = [O4, 0.8; F1, 0.95; F5, 1.0],$   
 $C_{84} = [O4, 0.8; F1, 0.95; F6, 1.0], C_{85} = [F2, 1.0].$

From this, the reasoning consistency can be restored by successfully modifying one of the above candidates, which is detailed in Sec. IV.

### G. Discussion on Generated Candidates

In order to effectively modify a candidate, it is necessary to examine if multiple related diagnosable ATMS nodes regarding a single interpolation step can be included in one candidate. If this is the case, the modifications of the related components must be considered jointly; otherwise, the modification of the candidate can be decomposed into that of its individual members.

Given a step of interpolation  $M_1, M_2, \dots, M_n, R_i, R_j, R_i R_j \Rightarrow C$ , for notational simplicity, let  $N_{M_1}, N_{M_2}, \dots, N_{M_n}, N_{R_i}, N_{R_j}, N_{R_i R_j}$  and  $N_C$  denote the following nodes:  $M_1, M_2, \dots, M_n, R_i, R_j, R_i R_j$  and the consequence  $C$ , respectively. Recall that the environment of each primitive ATMS node, which may be an observation, a rule or a FIC, contains only one node which represents itself [25]–[27]. Based on the label updating algorithm, every combination of one label environment from each node  $N_{M_i}, i \in \{1, 2, \dots, n\}$ , and those label environments of nodes  $\{N_{R_i}, N_{R_j}, N_{R_i R_j}\}$  jointly form a label environment of the node  $N_C$ . Assume that  $N_C$  contributes to a certain contradiction. Then, if any of its label environments contains  $N_{R_i R_j}$ , it must also contain  $N_{R_i}$  and  $N_{R_j}$ , and vice versa. Since a candidate is generated by taking one element from every label environment of each contradiction and any candidate which is a superset of another is removed, it is impossible that  $\{N_{R_i}, N_{R_i R_j}\}$  or  $\{N_{R_j}, N_{R_i R_j}\}$  is contained within a minimal candidate. Similarly, suppose that the node  $N$  is any element in the label environments of the nodes  $N_{M_1}, N_{M_2}, \dots$ , and  $N_{M_n}$ , then  $\{N, N_{R_i}\}$ ,  $\{N, N_{R_j}\}$ , or  $\{N, N_{R_i R_j}\}$  cannot jointly appear in any single minimal candidate.

Note that  $N_{R_i}$  may also be used in conjunction with another rule rather than  $N_{R_j}$  to perform interpolation, and vice versa. Thus, it is possible that one label environment of the ‘false’ node only contains  $N_{R_i}$  but not  $N_{R_j}$  while another only contains  $N_{R_j}$  but not  $N_{R_i}$ . Therefore, a minimal candidate may contain both  $N_{R_i}$  and  $N_{R_j}$ . In this situation, the modification of related candidate elements  $N_{R_i}$  and  $N_{R_j}$  needs to be considered jointly.

## IV. GENERALISING CANDIDATE MODIFICATION

Having generated and prioritised all the candidates, one (and only one) of them needs to be modified in order to restore system consistency. This process naturally starts from the highest prioritised candidate. The principle underlying the consistency-restoring algorithm as given in Alg. 1 is extended here by treating all observations, rules, and FICs as modifiable candidate elements. Recall that a candidate in general consists of a number of elements. Given a candidate, the modification of each of its elements will lead to a set of constraints in the format of equalities and inequalities. A satisfied solution of all joint equalities and inequalities imposed by the modifications of all the elements within a candidate will guarantee the modified result to be  $\beta_0$ -contradiction-free. The modification of FICs has been briefed in Sec. II-D, and thus omitted here. The modification processes regarding observations, individual rules, and pairs of rules corresponding to a single interpolation step, are discussed below.

### A. Observation Modification

It has an intuitive appeal to amend an observation based on the uncertainty value without changing the vagueness level associated with the relevant piece of information, which is reflected by the shape and area of the underlying fuzzy set. Such amendment may help maintain the interpretability of the fuzzy sets whilst offering an opportunity of removing inconsistencies in interpolation during the process of inference. Thus, the modification of a defective observation associated with a certainty degree of  $c$  is to shift the fuzzy set within its value range while keeping its shape and area unchanged. The shifting is required to satisfy the following:

- 1) The range of the shifting is bounded by Eqs. 15 and 16, regarding the given  $c$ .
- 2) The shifted result should not cause disruption regarding the definitions of the other object values of the same variable, maintaining consistency in the specification of that variable’s value domain. This is a similar constraint as that imposed in Step 5 for the modification of a FIC as described in Sec. II-D.
- 3) The propagation of the shifted result should maintain mutual consistency with that of any other object value of the same variable. This is a similar constraint as that imposed in Step 6 for the modification of a FIC, again as described in Sec. II-D.

All three constraints listed above can be satisfied by constructing and then solving a set of simultaneous equalities and inequalities. The modification of observations can then be readily propagated by applying the modified results as interpolation inputs within the process of fuzzy rule interpolation. Note that as indicated above, constraints 2 and 3 are enforced in a way similar to those required over the case of modifying a FIC, whilst the computation implementing such modification has been generally presented in detail in [24]. Therefore, such common sub-procedures of modification are omitted here; they are also omitted from the description of the modifications of interpolation rules that is to be described next.

### B. Single Rule Modification

The problem considered here is for situations where only one of a given pair of neighbouring rules is identified as defective. Following the scale and move transformation-based FRI (which AFRI is developed upon), the interpolated result in response to a given input (that may be an observation or a previously inferred value) is derived from the consequent of an artificially created intermediate rule through the process outlined in Sec. II-A. This process involves the use of a pair of neighbouring rules regarding the given input. Whilst the antecedent of the intermediate rule and the input share the same overall location, the interpolated value is achieved by transferring the consequence of the intermediate rule with the same proportion of the area and shape differences between them. Therefore, in order to maintain interpretability, the single defective rule should be modified while keeping the shape and area of its consequence unchanged. The present work follows on this intuition.

Similar to the process of modifying an observation, the modification of a defective rule is to shift the consequence of the rule within its value range by satisfying the three constraints listed in the last sub-section. However, all the interpolated results that have been generated by applying this defective rule also need to be modified accordingly, as the defective rule has been utilised for their interpolation.

Although AFRI is applicable to fuzzy inference problems with multiple-antecedent rules, for illustrative simplicity, rules with two antecedents are taken in this work as an example to show the underlying approach. The method can be extended to rules with more than two antecedent variables in a straightforward manner. Given an input  $(A_k^*, B_k^*)$ , suppose that the (closest) neighbouring rules  $A_i, B_i \Rightarrow C_i$  and  $A_j, B_j \Rightarrow C_j$  flank this input. Without losing generality, assume that the second rule is defective and is included in the candidate to be modified, and that  $(A_i, B_i)$  is less than  $(A_j, B_j)$  in accordance with the integration of their representative values (for a given integration method). Based on the location of the antecedent of this defective rule, in reference to the other rule that was jointly fired with it to derive the detected contradictory interpolated result, two mirrored cases need to be addressed.

First, consider the case where the location relation between the input  $(A_k^*, B_k^*)$ , and its corresponding interpolated consequence  $C_k^*$  is mapped by the line  $P_1P_3$  within the assumed three dimensional space, as shown in Fig. 5. This line is determined by the locations of the two neighbouring rules used for interpolation. Suppose that the defective rule consequence is modified from  $C_j$  to  $\hat{C}_j$ , then the original mapping line  $P_1P_3$  is accordingly shifted to the line  $P_1P_5$ . To quantitatively measure the extent of such shifting, the following *correction rate*  $c^-$  is introduced:

$$c^- = \frac{d(C_i, \hat{C}_j)}{d(C_i, C_j)}, \quad (24)$$

where  $d(C, C')$  stands for the distance between the fuzzy sets  $C$  and  $C'$ , computed as the distance between the representative values of these two fuzzy sets. Suppose that the modified result of  $C_k^*$  is denoted as  $\hat{C}_k^*$ . Then, by applying the correction rate  $c^-$  to the distance between  $C_i$  and  $C_k^*$ , the distance from  $C_i$  to  $\hat{C}_k^*$  can be determined. Having known the locations of  $C_i$  and  $C_k^*$ , the location of  $\hat{C}_k^*$  can be computed, resulting in the modified interpolated value.

The case discussed above covers the case where an input which has invoked the defective rule for interpolation is less than the integrated antecedent of the rule. For the case where an input is greater than the antecedent, a mirrored procedure is followed to perform the modification, with a different *correction rate*  $c^+$ . Assume that the input  $(A_k^*, B_k^*)$  is flanked by the defective rule  $A_i, B_i \Rightarrow C_i$  and the other neighbouring rule,  $A_j, B_j \Rightarrow C_j$ , then  $c^+$  is defined as:

$$c^+ = \frac{d(\hat{C}_i, C_j)}{d(C_i, C_j)}. \quad (25)$$

The modified result of  $(A_k^*, B_k^*)$  can then be calculated using this correction rate, in a way similar to that utilised in the first case.

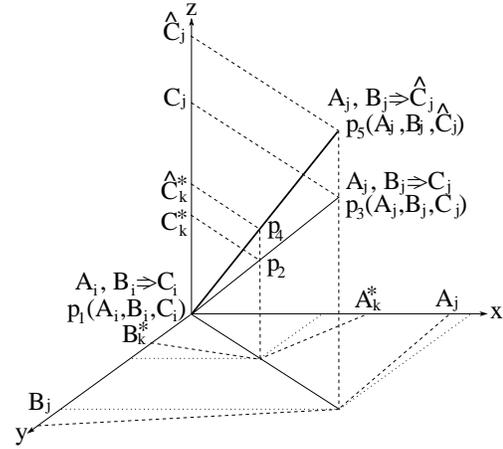


Fig. 5. Propagation of rule modification

### C. Modification of Both Neighbouring Rules

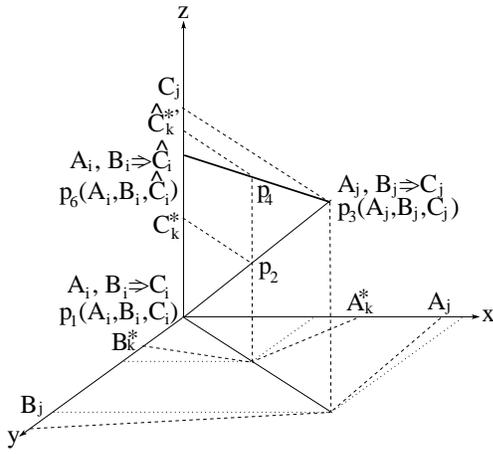
Having addressed the situations where only one of the two neighbouring rules appears in a candidate for modification, this sub-section discusses the modification of both neighbouring rules which are defective (i.e., both are included in a given candidate).

Suppose that the two defective neighbouring rules are  $A_i, B_i \Rightarrow C_i$  and  $A_j, B_j \Rightarrow C_j$ , and denote the (to be) modified consequences of them as  $\hat{C}_i$  and  $\hat{C}_j$ , respectively. For easy reference, call the defective rule whose integrated antecedent is less than the input the left rule and the other the right. If the left rule is modified first as illustrated in Fig. 6(a), then the right defective rule will be modified using the result of modifying the left rule, as shown in Fig. 6(b). Then, the final modification can be represented by shifting the original defective location mapping line  $P_1P_3$  to the line  $P_6P_5$  as also illustrated in Fig. 6(b). If, however, the modification begins with the right defective rule, the modification will be performed as illustrated in Fig. 7, which also results in the final result that is the same as the one represented by the line  $P_6P_5$  in Fig. 6(b). From this, due to the generality in the expression of the two rules, it can be concluded that the revised result is independent of the order of modifications. Therefore, the modification of both neighbouring rules in a single candidate can be done by revising the two individual defective rules separately in either order.

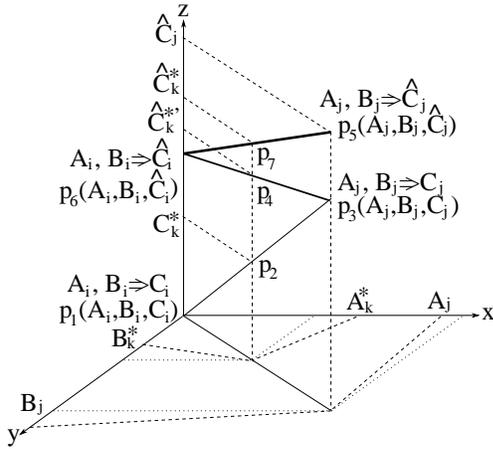
### D. Illustrative Example - Part 2

Continue the example given in Sec. III-F, the candidate  $C_1$ , which is of the highest priority, will be modified first. As only one modifiable element  $R_3$  (If  $x_2 = B_3$ , THEN  $x_3 = C_3$ ) is contained in this candidate, the modification procedure given in Section IV-B is applied. With respect to Eqs. 15 and 16, the modification of the defective rule,  $R_3$  needs to satisfy:

$$\begin{cases} \hat{C}_3 \geq C_3 - \frac{1-0.6}{2}(20-0) \\ \hat{C}_3 \leq C_3 + \frac{1-0.6}{2}(20-0) \\ \min(\text{supp}(\hat{C}_3)) \geq 0 \\ \max(\text{supp}(\hat{C}_3)) \leq 20. \end{cases}$$

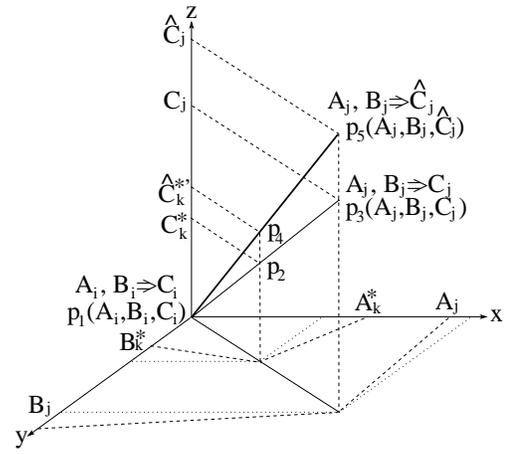


(a) Left rule modification first

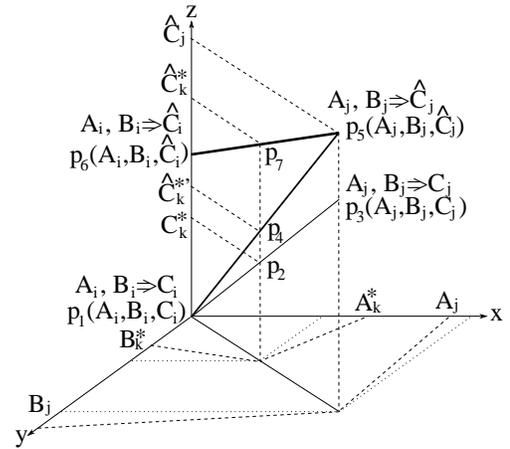


(b) Right rule modification second

Fig. 6. Rule modification starting from left defective rule



(a) Right rule modification first



(b) Left rule modification second

Fig. 7. Rule modification starting from right defective rule

Running interpolation with the two neighbouring rules consisting of the rule  $R_4$  and the defective one  $R_3$  leads to the following two interpolated rules:

$$\begin{aligned} IR_1 &: \text{IF } x_2 \text{ is } B^*, \text{ THEN } x_3 \text{ is } C_2^* \\ IR_2 &: \text{IF } x_2 \text{ is } B_1^*, \text{ THEN } x_3 \text{ is } C_1^*. \end{aligned}$$

Since both antecedents of  $IR_1$  and  $IR_2$  are greater than the antecedent of the defective rule,  $C^+$  is applied:

$$c^+ = \frac{d(\hat{C}_3, C_4)}{d(C_3, C_4)}.$$

From this, the overall location of the modified results will then satisfy:

$$\begin{cases} d(\hat{C}_1^*, C_4) = d(C_1^*, C_4) \cdot c^+ \\ d(\hat{C}_2^*, C_4) = d(C_2^*, C_4) \cdot c^+. \end{cases}$$

These results are then utilised to further constrain the modified interpolated values such that

$$\begin{cases} \hat{C}_1^* = C_1^* + (d(\hat{C}_1^*, C_4) - d(C_1^*, C_4)) \\ \hat{C}_2^* = C_2^* + (d(\hat{C}_2^*, C_4) - d(C_2^*, C_4)). \end{cases}$$

The remaining process of the modification is to ensure that the modified results and their propagations are consistent with

the rest. This sub-process is again, the same as that of the modification of a FIC as previously reported [24]. However, by solving all the simultaneous equalities and inequalities as listed above, including those imposed by the consistency-ensuring sub-process, there is no solution found. Therefore, the candidate with the second highest priority, that is  $C_2$  in this example, is modified next.

The candidate  $C_2$  includes two elements, the observation  $O_2$  and the rule  $R_8$ , both of which need to be modified simultaneously in order to remove inconsistency. The modifications of  $O_2$  and  $R_8$  are carried out based on the procedures given in Sections IV-A and IV-B, respectively. In particular, according to constraint number 1 of the observation modification process, the modified value of  $O_2$  must satisfy:

$$\begin{cases} \hat{B}^* \geq B^* - \frac{1-0.6}{2}(20-0) \\ \hat{B}^* \leq B^* + \frac{1-0.6}{2}(20-0) \\ \min(\text{supp}(\hat{B}^*)) \geq 0 \\ \max(\text{supp}(\hat{B}^*)) \leq 20. \end{cases}$$

Similar constraints are also applied to the modified result of the consequence of  $R_8$ . As the modification procedure of  $R_8$  is the same as that of  $R_3$ , as described above, the computational details are omitted here. By solving the equalities and

inequalities, including those posed for consistency-ensuring, one solution is obtained as illustrated in Fig. 8. With the consistency restored, this concludes this illustrative example.

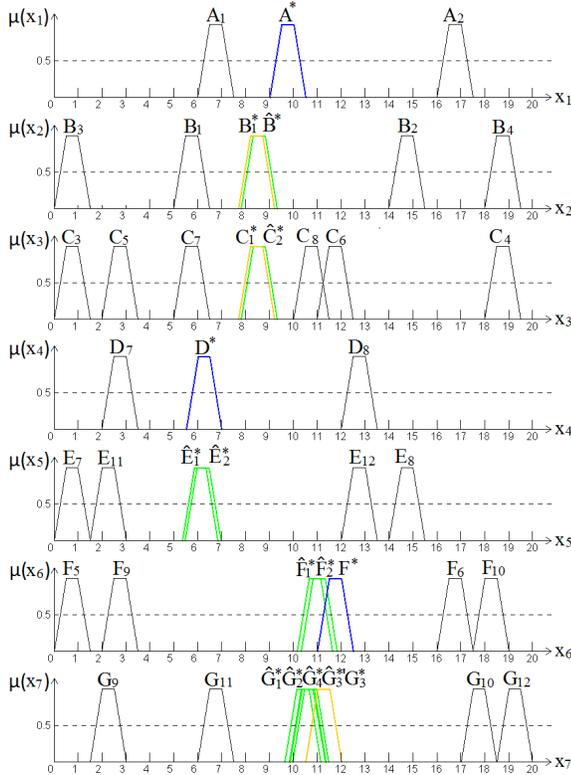


Fig. 8. One solution of the running example

### E. Computational Complexity

As the generalisation of AFRI, it may be expected that the generalised AFRI will involve more computation than its original. In particular, as compared to the computational complexity of AFRI, that of the generalised version can be considered from the following two viewpoints:

- Impact of adding rules and observations as diagnosable candidate elements during candidate generation;
- Impact of the constraints led by these extra candidate elements during candidate modification.

The computational complexity of candidate generation mainly depends on the complexity of the ATMS. It is well known that the standard ATMS has a computational complexity of exponential order in the worst case [41], but the average-case complexity can be greatly improved during practice use [42], [43]. The introduction of observations and rules as diagnosable candidate elements certainly increases the processing time because of a more sophisticated problem is being addressed. However, this does not affect the general time complexity of the underlying ATMS. The complexity of the candidate modification stage is mainly determined by the constraint satisfaction mechanism which for the problem of FRI in general, can be resolved in polynomial time complexity [24]. Although the introduction of additional constraints may increase the

absolute computing time, the general time complexity will not be affected as the constraints introduced by the extra modifiable candidate elements are of the same type with those used in AFRI. Putting both aspects together, at the system level, the overall computational complexity of the generalised version does not deteriorate from that of the original AFRI approach.

## V. APPLICATION AND DISCUSSION

Disease burden may result from environmental changes [44]–[46]. An example study of this concerns how a previously roadless area in northern coastal Ecuador may be affected by the construction of a new road or railway in term of epidemiology of infectious diseases [47]. The causal relationship between the key factors driven by road construction has been established in the work of [47], which has been further quantitatively investigated using AFRI in [24]. As the theoretical development reported in this paper carries a substantial extension of [24], the application problem is reconsidered in this paper to facilitate direct comparison. For completeness, the sparse rule base is given below and the fuzzy values included in the rules are listed in Table I.

- $R_1$ : IF  $x_1 = A_1$  and  $x_2 = B_1$ , THEN  $x_3 = C_1$  (0.9);
- $R_2$ : IF  $x_1 = A_2$  and  $x_2 = B_2$ , THEN  $x_3 = C_2$  (0.9);
- $R_3$ : IF  $x_3 = C_3$  and  $x_4 = D_3$ , THEN  $x_5 = E_3$  (0.7);
- $R_4$ : IF  $x_3 = C_4$  and  $x_4 = D_4$ , THEN  $x_5 = E_4$  (0.8);
- $R_5$ : IF  $x_5 = E_5$ , THEN  $x_6 = F_5$  (0.8);
- $R_6$ : IF  $x_5 = E_6$ , THEN  $x_6 = F_6$  (0.6);
- $R_7$ : IF  $x_6 = F_7$ , THEN  $x_7 = G_7$  (0.7);
- $R_8$ : IF  $x_6 = F_8$ , THEN  $x_7 = G_8$  (0.7);
- $R_9$ : IF  $x_5 = E_9$ , THEN  $x_8 = H_9$  (0.8);
- $R_{10}$ : IF  $x_5 = E_{10}$ , THEN  $x_8 = H_{10}$  (0.6);
- $R_{11}$ : IF  $x_8 = H_{11}$ , THEN  $x_9 = I_{11}$  (0.7);
- $R_{12}$ : IF  $x_8 = H_{12}$ , THEN  $x_9 = I_{12}$  (0.9);
- $R_{13}$ : IF  $x_9 = I_{13}$ , THEN  $x_{10} = J_{13}$  (0.7);
- $R_{14}$ : IF  $x_9 = I_{14}$ , THEN  $x_{10} = J_{14}$  (0.8);
- $R_{15}$ : IF  $x_7 = G_{15}$  and  $x_{10} = J_{15}$ , THEN  $x_{11} = K_{15}$  (0.6);
- $R_{16}$ : IF  $x_7 = G_{16}$  and  $x_{10} = J_{16}$ , THEN  $x_{11} = K_{16}$  (0.8).

Suppose that four pieces of uncertain information are observed:  $O_1 : x_1 = A^* = (0.16, 0.18, 0.20, 0.22)(0.7)$ ,  $O_2 : x_2 = B^* = (0.34, 0.36, 0.38, 0.40)(0.9)$ ,  $O_4 : x_4 = D^* = (0.65, 0.67, 0.69, 0.71)(0.6)$ , and  $O_8 : x_8 = H^* = (0.54, 0.56, 0.58, 0.60)(0.7)$ . These observations do not invoke any rule in the rule base (with only  $B^*$  overlapping with the second antecedent attribute  $B_2$  of the rule  $R_2$ ). Thus, traditional fuzzy system techniques that are based on the use of compositional rule of inference cannot be employed to address the problem. However, fuzzy rule interpolation may help.

Assume that the set-theory-based similarity measure is utilised to compute the degree of contradiction, and let  $\beta_0 = 0.5$ .  $\beta_0$ -contradictions will result from most of the existing interpolation methods [24]. In particular, the interpolated result using the scale and move transformation-based FRI, which the proposed work is built upon, leads to multiple (indeterminate)  $\beta_0$ -inconsistencies as shown in Fig. 9.

To obtain a consistent solution, the proposed adaptive fuzzy interpolation approach is applied. From the modifiable components (i.e., observations, rules and FICs) upon which the

TABLE I  
FUZZY VARIABLES AND THEIR NORMALIZED OBJECT VALUES

Var	Meaning	Object value
$x_1$	Railway station proximity	$A_1 = \{0.02, 0.04, 0.06, 0.08\}$ ; $A_2 = \{0.28, 0.30, 0.32, 0.34\}$
$x_2$	Road proximity	$B_1 = \{0.18, 0.20, 0.22, 0.24\}$ ; $B_2 = \{0.39, 0.41, 0.43, 0.45\}$
$x_3$	Connectivity to transportation systems	$C_1 = \{0.46, 0.48, 0.50, 0.52\}$ ; $C_2 = \{0.62, 0.64, 0.66, 0.68\}$ $C_3 = \{0.52, 0.54, 0.56, 0.58\}$ ; $C_4 = \{0.85, 0.87, 0.89, 0.91\}$
$x_4$	Distance to the closest town	$D_3 = \{0.52, 0.54, 0.56, 0.58\}$ ; $D_4 = \{0.82, 0.84, 0.86, 0.88\}$
$x_5$	Remoteness	$E_3 = \{0.41, 0.43, 0.45, 0.47\}$ ; $E_4 = \{0.72, 0.74, 0.76, 0.78\}$ $E_5 = \{0.27, 0.29, 0.31, 0.33\}$ ; $E_6 = \{0.58, 0.60, 0.62, 0.64\}$ $E_9 = \{0.39, 0.41, 0.43, 0.45\}$ ; $E_{10} = \{0.62, 0.64, 0.66, 0.68\}$
$x_6$	Contact outside of the community	$F_5 = \{0.62, 0.64, 0.66, 0.68\}$ ; $F_6 = \{0.30, 0.32, 0.34, 0.36\}$ $F_7 = \{0.38, 0.40, 0.42, 0.44\}$ ; $F_8 = \{0.70, 0.72, 0.74, 0.76\}$
$x_7$	Reintroduction of pathogenic strains	$G_7 = \{0.46, 0.48, 0.50, 0.52\}$ ; $G_8 = \{0.65, 0.67, 0.69, 0.71\}$ $G_{15} = \{0.30, 0.32, 0.34, 0.36\}$ ; $G_{16} = \{0.60, 0.62, 0.64, 0.66\}$
$x_8$	Demographic changes	$H_9 = \{0.60, 0.62, 0.64, 0.66\}$ ; $H_{10} = \{0.30, 0.32, 0.34, 0.36\}$ $H_{11} = \{0.46, 0.48, 0.50, 0.52\}$ ; $H_{12} = \{0.68, 0.70, 0.72, 0.74\}$
$x_9$	Social connectedness	$I_{11} = \{0.52, 0.54, 0.56, 0.58\}$ ; $I_{12} = \{0.20, 0.22, 0.24, 0.26\}$ $I_{13} = \{0.28, 0.30, 0.32, 0.34\}$ ; $I_{14} = \{0.55, 0.57, 0.59, 0.61\}$
$x_{10}$	Hygiene and sanitation infrastructure	$J_{13} = \{0.26, 0.28, 0.30, 0.32\}$ ; $J_{14} = \{0.61, 0.63, 0.65, 0.67\}$ $J_{15} = \{0.36, 0.38, 0.40, 0.42\}$ ; $J_{16} = \{0.58, 0.60, 0.62, 0.64\}$
$x_{11}$	Infectious disease rates	$K_{15} = \{0.18, 0.20, 0.22, 0.24\}$ ; $K_{16} = \{0.68, 0.70, 0.72, 0.74\}$

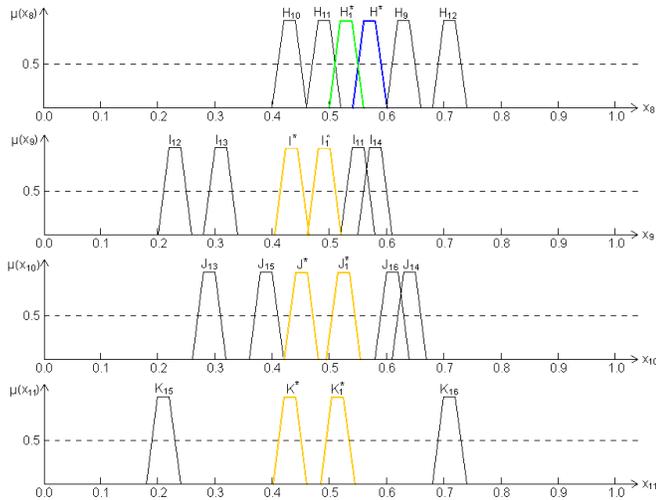


Fig. 9. Interpolated result by the HS method

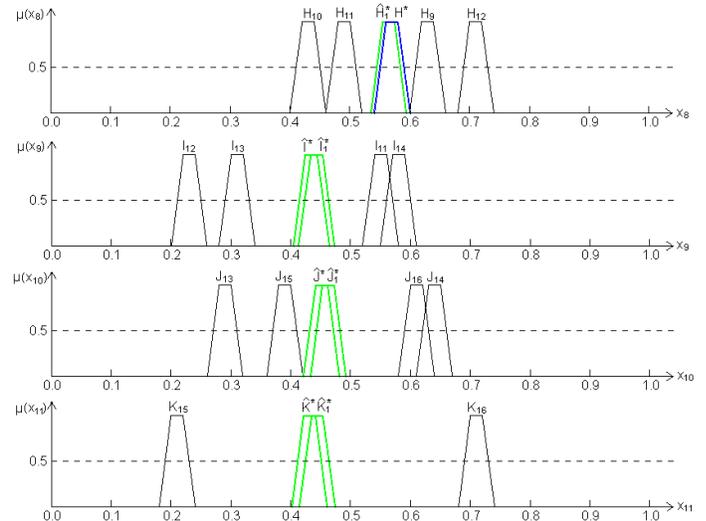


Fig. 10. Interpolated result by the adaptive approach (based on the HS method)

detected contradictions depend, GDE generates 16 minimal candidates:  $C_1 = [R_{10}, 0.6]$ ,  $C_2 = [O_1, 0.7]$ ,  $C_3 = [R_3, 0.7]$ ,  $C_4 = [R_{11}, 0.7]$ ,  $C_5 = [O_3, 0.8]$ ,  $C_6 = [R_4, 0.8]$ ,  $C_7 = [R_9, 0.8]$ ,  $C_8 = [O_2, 0.9]$ ,  $C_9 = [R_1, 0.9]$ ,  $C_{10} = [R_2, 0.9]$ ,  $C_{11} = [R_{12}, 0.9]$ ,  $C_{12} = [F_6, 0.92]$ ,  $C_{13} = [F_5, 0.93]$ ,  $C_{14} = [F_1, 0.94]$ ,  $C_{15} = [F_2, 0.99]$ , and  $C_{16} = [O_4, 1.6]$ . One solution resulted from the modification of the first prioritised candidate  $C_1$  is shown in Fig. 10.

From this figure, it can be seen that there is no  $\beta_0$ -contradiction any more and thus consistency has been success-

fully restored. That is, the original inconsistent interpolated result has been successfully removed, demonstrating the effectiveness of the present work. Interestingly, different from the problem-solving process of the previous work reported in [24], this solution has resulted from the modification of the very first candidate  $C_1$ . This is due to the employment of the proposed candidate prioritisation method. By this method, the priority of each candidate is calculated from their reliability rather than from the informal intuition as used previously.

## VI. CONCLUSIONS

This paper has presented a generalised framework for adaptive fuzzy rule interpolation. The generalisation allows the identification and modification of observations and rules, in addition to that of interpolation procedures that were addressed in the previous work. This is supported by introducing extra information of certainty degrees associated such basic elements of fuzzy rule interpolation. The work also allows for all candidates for modification to be prioritised, based on the extent to which a candidate is likely to lead to all detected contradictions, by extending the classic ATMS and GDE. The working of the extended approach is illustrated with a running example throughout Secs. III and IV, and also demonstrated by a realistic application in Sec. V.

This research can be further improved in several directions. At the present, it works with interpolation involving just two multiple-antecedent rules. It is worthwhile to investigate how this work may be generalised to perform interpolation and extrapolation with multiple multi-antecedent rules. Note that the FRI approach proposed in [48] also deals with inconsistency problems, but in a different way by considering the relevant degrees of rules relevant to a given observation. In particular, the relevant degree of a certain rule is determined by the reciprocal distance from the observation to the rule. An interesting piece of further work is therefore to compare these two approaches. In addition, the proposed adaptive approach is developed on the HS method only. It is desirable to apply the adaptive approach to other FRI methods, such as those implemented in Matlab FRI toolbox [49], and to compare the generated results. Finally, it is of great interest to study how the classical ATMS and GDE can be utilised to support traditional fuzzy inference systems, and to develop an integrated inconsistency detection and fault-correction platform that supports both standard fuzzy inference and fuzzy rule interpolation.

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