

1 **ON THE ELLIPTIC-HYPERBOLIC TRANSITION IN WHITHAM**
2 **MODULATION THEORY**

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4 **Abstract.** The dispersionless Whitham modulation equations in one space dimension and time
5 are generically hyperbolic or elliptic, and breakdown at the transition, which is a curve in the
6 frequency-wavenumber plane. In this paper, the modulation theory is reformulated with a slow
7 phase and different scalings resulting in a phase modulation equation near the singular curves which
8 is a geometric form of the two-way Boussinesq equation. This equation is universal in the same sense
9 as Whitham theory. Moreover, it is dispersive, and it has a wide range of interesting multiperiodic,
10 quasiperiodic and multi-pulse localized solutions. This theory shows that the elliptic-hyperbolic
11 transition is a rich source of complex behaviour in nonlinear wave fields. There are several examples
12 of these transition curves in the literature to which the theory applies. For illustration the theory
13 is applied to the complex nonlinear Klein-Gordon equation which has two singular curves in the
14 manifold of periodic travelling waves.

15 **Key words.** nonlinear waves, modulation, Lagrangian, multisymplectic, traveling waves

16 **AMS subject classifications.** 74J30,70S05,76B15

17 **1. Introduction.** *Modulational instability* is one of the key ways that periodic
18 travelling waves become unstable. The wavelength of the perturbation is slightly
19 longer than the wavelength of the underlying periodic wave. In conservative sys-
20 tems this instability, in the weakly nonlinear case, is most closely associated with
21 the *Benjamin-Feir instability* [4], and in non-conservative systems with the *Eckhaus*
22 *instability* [15]. For weakly nonlinear periodic travelling waves, the simplest way
23 to analyze modulational instability is to derive a nonlinear Schrödinger equation or
24 complex Ginzburg Landau equation [35]. A history of the beginnings of modulation
25 instability is given in [40].

26 For *finite-amplitude* periodic travelling waves in conservative systems modulation
27 instability is captured by the Whitham modulation theory. For a nonlinear periodic
28 travelling wave of frequency ω and wavenumber k , modulation of the form

29 (1) $k \mapsto k + q(X, T, \varepsilon) \quad \text{and} \quad \omega \mapsto \omega + \Omega(X, T, \varepsilon),$

30 where $X = \varepsilon x$, $T = \varepsilon t$, in the Whitham theory, results in

31 (2) $q_T = \Omega_X \quad \text{and} \quad \mathcal{A}_T + \mathcal{B}_X = 0,$

32 to leading order in ε , where $\mathcal{A}(\omega + \Omega, k + q)$ and $\mathcal{B}(\omega + \Omega, k + q)$ are the *wave action*
33 and *wave action flux* respectively, evaluated on the family of periodic travelling waves
34 [37, 38]. The Whitham modulation equations (WMEs) in (2) are a closed nonlinear
35 first order set of PDEs for the functions Ω and q . Generically, the WMEs are either
36 hyperbolic or elliptic. The linearization of these equations about the basic state,
37 represented by ω and k , is

38 (3) $q_T = \Omega_X \quad \text{and} \quad \mathcal{A}_\omega \Omega_T + \mathcal{A}_k q_T + \mathcal{B}_\omega \Omega_X + \mathcal{B}_k q_X = 0,$

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39 or, with the assumption $\mathcal{A}_\omega \neq 0$, they can be written in the standard form,

$$40 \quad (4) \quad \begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

41 where

$$42 \quad (5) \quad \mathbf{A}(\omega, k) = \frac{1}{\mathcal{A}_\omega} \begin{bmatrix} 0 & -\mathcal{A}_\omega \\ \mathcal{B}_k & \mathcal{A}_k + \mathcal{B}_\omega \end{bmatrix}.$$

43 Here, \mathcal{A} and \mathcal{B} are evaluated at $\Omega = q = 0$. The characteristics are

$$44 \quad (6) \quad c^\pm = \frac{\mathcal{A}_\omega + \mathcal{B}_k}{2\mathcal{A}_\omega} \pm \frac{1}{\mathcal{A}_\omega} \sqrt{-\Delta_L}, \quad \Delta_L = \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} = \det \begin{bmatrix} \mathcal{L}_{\omega\omega} & \mathcal{L}_{\omega k} \\ \mathcal{L}_{k\omega} & \mathcal{L}_{kk} \end{bmatrix},$$

45 using the identities, $\mathcal{A} = \mathcal{L}_\omega$ and $\mathcal{B} = \mathcal{L}_k$, from Whitham theory, where \mathcal{L} is the
46 averaged Lagrangian. The modulation instability is recovered by letting

$$47 \quad \begin{pmatrix} q(X, T) \\ \Omega(X, T) \end{pmatrix} = \text{Re} \left\{ \begin{pmatrix} \hat{q} \\ \hat{\Omega} \end{pmatrix} e^{\lambda T + i\nu X} \right\},$$

48 and substituting into (3) giving

$$49 \quad \lambda = ic^\pm \nu,$$

50 and so an unstable exponent (positive real part of λ) with modulation wave number ν
51 exists precisely when $\Delta_L > 0$. While ν is of order one, $\nu X = \varepsilon \nu x$, and so the modu-
52 lation wave number can be interpreted as being of order ε in the original coordinates.
53 Since the WMEs are dispersionless, there is no wavenumber cutoff of the modulation
54 instability.

55 In terms of characteristics, this modulation instability highlights the Lighthill
56 condition [22]: when $\Delta_L > 0$ the linearized WMEs are elliptic and when $\Delta_L < 0$ they
57 are hyperbolic. This criterion, and other features of Whitham modulation theory
58 have been widely studied and there is a vast literature; recent examples are the book
59 [20], the review articles [29, 11], and the special issue on Whitham theory [5].

60 In this paper the interest is in the case when the Lighthill determinant is singular

$$61 \quad (7) \quad \Delta_L := \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} = 0 \quad \text{but} \quad \mathcal{A}_\omega \neq 0 \quad \text{and} \quad \mathcal{A}_k \neq 0.$$

62 The condition $\Delta_L = 0$ defines a curve in the (ω, k) plane locally separating stable and
63 unstable states. The set $\Delta_L^{-1}(0)$, which is not necessarily connected, will be denoted
64 by

$$65 \quad (8) \quad \Sigma^1 = \Delta_L^{-1}(0) = \{(\omega, k) \in U \subset \mathbb{R}^2 : \Delta_L = 0\},$$

66 where U is the open subset of \mathbb{R}^2 for which periodic travelling waves exist. This
67 notation comes from singularity theory and is elaborated further in §2, as the geometry
68 of Σ^1 appears in the phase modulation theory. A typical Σ^1 curve is shown in Figure
69 1.

70 As far as we are aware, a modulation theory near an elliptic-hyperbolic transition
71 curve, generalizing Whitham modulation theory, has not been attempted heretofore.
72 One strategy for deriving a new modulation equation near a Σ^1 curve is to take the

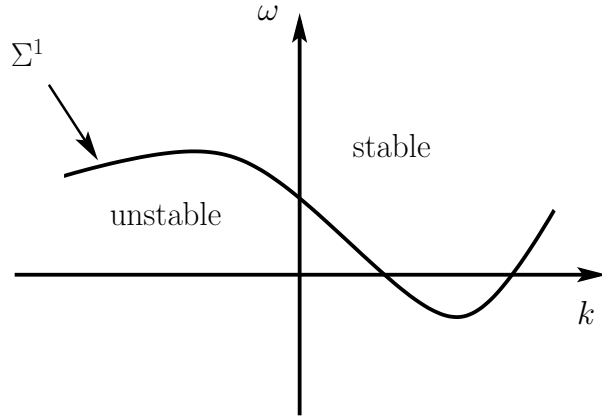


FIG. 1. A typical curve defined by $\Delta_L = 0$ in the (ω, k) plane.

73 Whitham theory to higher order. Luke [25] has given a theory and algorithm for
 74 deducing higher-order Whitham equations. However, the theory is quite complicated
 75 after the first order, and a clear closed system does not immediately emerge.

76 Another strategy is to change the time scale. The breakdown of the WMEs can
 77 be interpreted as a signal that a change in time scale, from $T = \varepsilon t$ to $T = \varepsilon^2 t$, is
 78 appropriate. Another feature of points on Σ^1 curves with $\mathcal{A}_k \neq 0$ is that the linearized
 79 WMEs have a double characteristic with nonzero speed, suggesting a moving frame
 80 is appropriate. Since $\mathcal{A}_k = \mathcal{B}_\omega$, the speed at the double characteristic is

81 (9)
$$c_g = \frac{\mathcal{A}_k}{\mathcal{A}_\omega}.$$

82 The symbol c_g is used as this velocity is a form of nonlinear group velocity. It is inter-
 83 pretable as (minus) the derivative of the frequency with respect to the wavenumber
 84 with wave action fixed. There are various generalizations of group velocity to the non-
 85 linear regime in Whitham theory (e.g. [17, 30]). The definition (9) is preferred here
 86 as it is the velocity at the double characteristic, and arises naturally in the nonlinear
 87 modulation theory.

88 Our strategy for developing a nonlinear modulation theory near Σ^1 curves is
 89 to slow down the time scale, go into a c_g -boosted moving frame, and slow down
 90 the phase, wavenumber and frequency modulation. The modulation mapping (1) is
 91 replaced by

92 (10)
$$k \mapsto k + \varepsilon^2 q(X, T, \varepsilon),$$

93 and

94 (11)
$$\omega \mapsto \omega - c_g \varepsilon^2 q(X, T, \varepsilon) + \varepsilon^3 \Omega(X, T, \varepsilon),$$

95 with

96
$$X = \varepsilon(x - c_g t) \quad \text{and} \quad T = \varepsilon^2 t.$$

97 Substitution into the governing equations, which are the Euler-Lagrange equations
 98 based on a general abstract Lagrangian, then leads at fifth order in ε via a solvability

99 condition to the new modulation equations replacing (2) and (3),

$$100 \quad (12) \quad q_T = \Omega_X \quad \text{and} \quad \mathcal{A}_\omega \Omega_T + \kappa q q_X + \mathcal{K} q_{XX} = 0.$$

101 Differentiating the second equation with respect to X and using the first equation
 102 shows that it is a variant of the two-way Boussinesq equation, but with coefficients that
 103 are universal in the same sense that the Whitham equations are universal (that they
 104 follow from the abstract properties of the Lagrangian). The importance of $\mathcal{A}_\omega \neq 0$
 105 shows up in the first coefficient. The second coefficient, κ is the second derivative of the
 106 mapping $(\omega, k) \mapsto (\mathcal{A}(\omega, k), \mathcal{B}(\omega, k))$ evaluated on the kernel of the first derivative,
 107 and the coefficient of dispersion \mathcal{K} is determined by a Jordan chain argument. The
 108 details of the derivation leading to (12) are given in §4.

109 A two-way Boussinesq equation is derived via phase modulation in [34], but in that
 110 case the Whitham theory has a double *zero* characteristic, and the phase modulation
 111 is *relative to a stationary frame of reference*. Moreover, that theory requires two
 112 parameters and is not associated directly with a stability-instability transition. The
 113 theory in this paper addresses the stability-instability transition directly, and will be
 114 more prevalent in applications as it only requires the variation of a single parameter.

115 There are several interesting consequences due to the modulation equation (12)
 116 near Σ^1 curves: dispersion is generated, thereby admitting coherent structures (e.g.
 117 solitary waves), and a wide range of complex solutions are generated (multi-pulse
 118 solitary waves [18], breathers [13], blowup [7, 36], integrable structures [6]) and it has
 119 its own elliptic-hyperbolic dichotomy. The two-way Boussinesq equation is said to be
 120 elliptic (“bad”) if it is linearly ill-posed (corresponding in this case to $\mathcal{A}_\omega \mathcal{K} < 0$) and
 121 hyperbolic (“good”) for the reverse sign. The good Boussinesq equation moderates
 122 the modulational instability, whereas the bad Boussinesq equation enhances the in-
 123 stability. In either case, dispersion identifies a cut-off wavenumber for the modulation
 124 instability which is absent in the dispersionless WMEs.

125 There are two familiar examples in the literature where Σ^1 curves arise. The first
 126 is stabilization of the Benjamin-Feir instability, for water waves on infinite depth,
 127 at large amplitude [24, 28, 39]. This case is interpreted in terms of the theory here
 128 in §6. The second is stabilization of the Benjamin-Feir instability when the depth
 129 parameter is below a critical threshold, $kh_0 \approx 1.363$ [3, 16]. This latter case occurs in
 130 the weakly nonlinear regime, and a theory for this case is developed by Johnson [19]
 131 near the threshold by extending the nonlinear Schrödinger equation to higher order. A
 132 new example has recently been discovered by Maiden & Hofer [26] where an elliptic-
 133 hyperbolic transition has been discovered in modulation of viscous fluid conduit waves.
 134 However, in the latter two examples the modulation is *multiphase* and so the theory
 135 of this paper does not directly apply (see comments in §8). Here an example, based
 136 on modulation of a one-phase periodic travelling wave solution of a nonlinear complex
 137 Klein-Gordon equation, is presented where all the details can be worked out explicitly
 138 and it illustrates the key features induced by the elliptic-hyperbolic transition.

139 There is an interesting geometry associated with the mapping

$$140 \quad (\omega, k) \mapsto (\mathcal{A}(\omega, k), \mathcal{B}(\omega, k)),$$

141 and it is developed in §2. The condition $\Delta_L = 0$ defines a curve in the (ω, k) plane
 142 which locally separates stable and unstable regions. The image defines a curve in
 143 $(\mathcal{A}, \mathcal{B})$ space. The geometry of these curves appears in the modulation theory. The

144 modulation theory is developed for general conservative PDEs generated by a La-
 145 grangian, and the background for this is developed in §3. The details of the modu-
 146 lation theory are presented in §4. In §5 features of the emergent two-way Boussinesq
 147 equation are discussed. Two examples of the application of the theory are presented:
 148 §6 applies the theory to the instability-stability transition of the Benjamin-Feir in-
 149 stability of Stokes waves in deep water, and §7 computes Σ^1 curves, and the reduced
 150 Boussinesq equation for periodic travelling waves of a nonlinear complex Klein-Gordon
 151 equation.

152 **2. The frequency-wavenumber mapping.** The geometry of the *frequency-*
 153 *wavenumber map*

154 (13)
$$(\omega, k) \mapsto \begin{pmatrix} \mathcal{A}(\omega, k) \\ \mathcal{B}(\omega, k) \end{pmatrix} := \mathbf{F}(\omega, k),$$

155 appears centrally within the modulation theory. The Jacobian of this mapping,

156
$$\mathbf{DF}(\omega, k) := \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix},$$

157 is degenerate on the Σ^1 curves (7). With the assumptions (7), the trace of \mathbf{DF} is
 158 nonzero and so the zero eigenvalue of \mathbf{DF} is simple with geometric eigenvector

159
$$\mathbf{DF}(\omega, k)\mathbf{n} = 0.$$

160 Since \mathbf{DF} is symmetric, \mathbf{n} is both a left and right eigenvector. In terms of c_g

161 (14)
$$\mathbf{n} = \begin{pmatrix} -c_g \\ 1 \end{pmatrix},$$

162 modulo a nonzero multiplicative constant. Although this eigenvector is not unique
 163 the choice (14) is canonical in that it will be shown to be relevant in the modulation
 164 theory.

165 The symbol \mathbf{n} is used for the eigenvector in (14) because it is a normal vector.
 166 However, it is not the normal vector to the curve Σ^1 , it is the normal vector to the
 167 image of this curve in the $(\mathcal{A}, \mathcal{B})$ plane. To see this first look at the geometry of the
 168 curve defined by Σ^1 . To lighten the notation define

169
$$f(\omega, k) := \Delta_L(\omega, k).$$

170 Then the normal vector to the curve $\Delta_L = 0$ is proportional to ∇f . A schematic is
 171 shown on the left in Figure 2. Now parameterize the curve $\Delta_L = 0$ by $(\omega(s), k(s))$.
 172 Then a tangent vector on the image of the mapping $\mathbf{F}(\omega(s), k(s))$ is

173
$$\begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} \begin{pmatrix} \dot{\omega} \\ \dot{k} \end{pmatrix}.$$

174 The left eigenvector \mathbf{n} of \mathbf{DF} is orthogonal to this direction, giving a normal vector
 175 on the image curve in $(\mathcal{A}, \mathcal{B})$ -space. A schematic is shown on the right in Figure 2.

176 The geometry of mappings from a plane to a plane is a fundamental problem
 177 in singularity theory and the basic results can be found in the first few chapters of

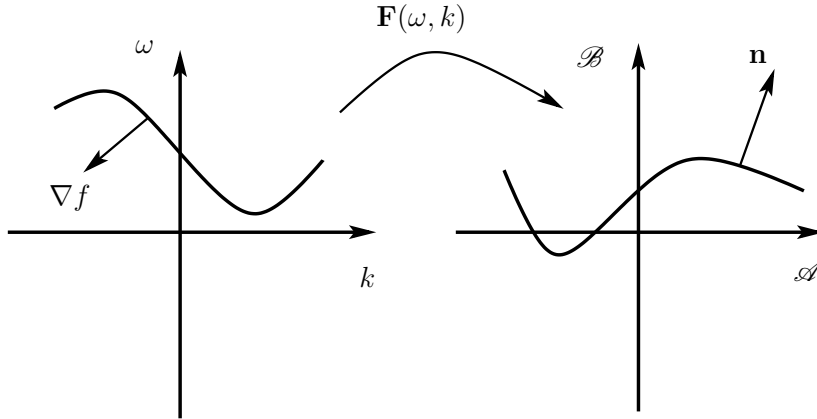


FIG. 2. The singular curve defined by $\Delta_L = 0$ on the left, and its image curve under the mapping \mathbf{F} is on the right.

178 Arnold et al. [1]. For a mapping from the plane to the plane with a Σ^1 singularity,
 179 there are generically two types of curves: either

$$180 \quad \begin{aligned} T_p \Sigma^1 \oplus \text{Ker}(\mathbf{DF}) &= \mathbb{R}^2 && \text{(fold)} \\ T_p \Sigma^1 &= \text{Ker}(\mathbf{DF}) && \text{(cusp)}, \end{aligned}$$

181 where $p = (\omega, k) \in \Sigma^1$. Since $T_p \Sigma^1 = \text{Ker}(\nabla f)$, the fold condition is

$$182 \quad (15) \quad \langle \nabla f, \mathbf{n} \rangle \neq 0,$$

183 where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^2 . The cusp condition is simply

$$184 \quad (16) \quad \langle \nabla f, \mathbf{n} \rangle = 0.$$

185 All other potential singularities of mappings from the plane to the plane are not
 186 stable under perturbation, a result known as Whitney's theorem [1], although one
 187 can potentially have many cusps [21]. This geometry plays a central role in the
 188 modulation theory, as it turns out that $\kappa \neq 0$ in (12) is precisely related to (15).

189 Define

$$190 \quad (17) \quad \kappa = \langle \mathbf{n}, \mathbf{D}^2 \mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle, \quad (\omega, k) \in \Sigma^1,$$

191 with \mathbf{n} in the canonical form (14). The expression on the right is the *intrinsic second*
 192 *derivative* [31, 1]. It is the ordinary second derivative of the mapping \mathbf{F} but evaluated
 193 on the kernel of the first derivative. It is widely used in singularity theory (cf. Chapter
 194 3 of [1]).

195 The connection between κ in (17) and the fold condition (15) is the following

$$196 \quad (18) \quad \langle \nabla f, \mathbf{n} \rangle = \left(\frac{\mathcal{A}_\omega + \mathcal{B}_k}{\|\mathbf{n}\|^2} \right) \langle \mathbf{n}, \mathbf{D}^2 \mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle.$$

197 The coefficient on the right is nonzero since the zero eigenvalue of \mathbf{DF} is simple. The
 198 formula (18) is proved as follows. The function f can be characterized as

$$199 \quad f(\omega, k) = \det[\mathbf{DF}(\omega, k)],$$

200 and so, using the formula for the derivative of a determinant,

201
$$f_\omega = \text{Trace}(\mathbf{DF}^\# \mathbf{DF}_\omega) \quad \text{and} \quad f_k = \text{Trace}(\mathbf{DF}^\# \mathbf{DF}_k),$$

202 where $\mathbf{DF}^\#$ is the adjugate of \mathbf{DF} . Combining

203
$$\langle \nabla f, \mathbf{n} \rangle = n_1 f_\omega + n_2 f_k = \text{Trace}(\mathbf{DF}^\# (n_1 \mathbf{DF}_\omega + n_2 \mathbf{DF}_k)).$$

204 Now note that the adjugate of a 2×2 matrix of rank 1 is proportional to \mathbf{nn}^T , and
 205 in this case it is exactly

206
$$\mathbf{DF}^\# = \frac{\text{Tr}(\mathbf{DF})}{\|\mathbf{n}\|^2} \mathbf{nn}^T,$$

207 a formula which can be confirmed by direct calculation. Since $\text{Tr}(\mathbf{nn}^T \mathbf{A}) = \langle \mathbf{n}, \mathbf{An} \rangle$
 208 for any 2×2 matrix \mathbf{A} , the formula (18) follows.

209 Writing out (17) using the canonical form for \mathbf{n} in (14),

210 (19)
$$\kappa = (\mathcal{B}_{kk} - c_g \mathcal{A}_{kk}) - 2c_g (\mathcal{B}_{\omega k} - c_g \mathcal{A}_{\omega k}) + c_g^2 (\mathcal{B}_{\omega\omega} - c_g \mathcal{A}_{\omega\omega}).$$

211 It is this form of the intrinsic second derivative κ that shows up in the modulation
 212 theory as the coefficient of nonlinearity in the modulation equation (12).

213 It is important to note that the “intrinsic” nature of the second derivative does
 214 not mean that the value of κ is independent of the choice of \mathbf{n} . As an eigenvector \mathbf{n} is
 215 not unique and multiplication of \mathbf{n} by a nonzero constant multiplies κ by that constant
 216 cubed, and so it can even change the sign of κ . The intrinsic label signifies that the
 217 affine part of the second derivative is removed, and the trilinear form of the second
 218 derivative remains the same. See [31, 1] for further detail on intrinsic derivatives.

219 **3. Lagrangian setup and basic state.** The starting point for the modulation
 220 theory is a general class of PDEs generated by an abstract Lagrangian,

221 (20)
$$\mathcal{L}(U) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L}(U, U_x, U_t) \, dx dt,$$

222 where $U(x, t)$ is a vector-valued field on the rectangle $[x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2$. It is
 223 advantageous to first transform the Lagrangian density to multisymplectic form,

224 (21)
$$\mathcal{L}(Z) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left[\frac{1}{2} \langle Z, \mathbf{M}Z_t \rangle + \frac{1}{2} \langle Z, \mathbf{J}Z_x \rangle - S(Z) \right] \, dx dt,$$

225 where now $Z \in \mathbb{R}^n$ for each (x, t) and n is assumed to be even. The Lagrangian density
 226 is the same in going from (20) to (21) but the representation (21) has more structure.
 227 The operators \mathbf{M} and \mathbf{J} are constant skew-symmetric $n \times n$ matrices and $S : \mathbb{R}^n \rightarrow \mathbb{R}$
 228 is a given smooth function. The transformation from (20) to (21), effectively a double
 229 Legendre transform, is discussed in previous papers [10, 8, 33, 34]. The Euler-Lagrange
 230 equation deduced from the Lagrangian (21) takes the concise form

231 (22)
$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z), \quad Z \in \mathbb{R}^n.$$

232 The theory could be developed directly on the primitive abstract Lagrangian (20)
 233 but partitioning the Lagrangian density as in (21) gives added structure that greatly
 234 simplifies the theory.

235 The basic state is a periodic travelling wave solution of wavelength $2\pi/k$ and
236 period $2\pi/\omega$ of the form

$$237 \quad (23) \quad Z(x, t) = \widehat{Z}(\theta, \omega, k), \quad \widehat{Z}(\theta + 2\pi, \cdot) = \widehat{Z}(\theta, \cdot), \quad \theta = kx + \omega t + \theta_0,$$

238 with arbitrary phase shift θ_0 . There is the usual assumption on existence and smooth-
239 ness of this solution so that the necessary differentiation in θ , k , and ω is meaningful.
240 The basic state satisfies

$$241 \quad (24) \quad \omega \mathbf{M} \widehat{Z}_\theta + k \mathbf{J} \widehat{Z}_\theta = \nabla S(\widehat{Z}).$$

242 An important property of the structure is multisymplectic Noether theory [10]
243 associated with conservation of wave action, that is,

$$244 \quad (25) \quad \nabla A(\widehat{Z}) = \mathbf{M} \widehat{Z}_\theta \quad \text{and} \quad \nabla B(\widehat{Z}) = \mathbf{J} \widehat{Z}_\theta,$$

245 where A, B are the components of the action conservation law, $\widehat{Z}(\theta, \omega, k)$ is the basic
246 state, and the gradient is defined with respect to the inner product including averaging
247 over θ ,

$$248 \quad (26) \quad \langle\langle U, V \rangle\rangle := \frac{1}{2\pi} \int_0^{2\pi} \langle U, V \rangle d\theta,$$

249 where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n .

250 To get the components of the conservation law for wave action, average the La-
251 grangian, evaluated on the family of travelling waves, over θ ,

$$252 \quad \mathcal{L}(\omega, k) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{\omega}{2} \langle \mathbf{M} \widehat{Z}_\theta, \widehat{Z} \rangle + \frac{k}{2} \langle \mathbf{J} \widehat{Z}_\theta, \widehat{Z} \rangle - S(\widehat{Z}) \right] d\theta,$$

253 and differentiate with respect to ω and k , giving

$$254 \quad (27) \quad \begin{aligned} \mathcal{L}_\omega &:= \mathcal{A}(\omega, k) = \frac{1}{2} \langle\langle \mathbf{M} \widehat{Z}_\theta, \widehat{Z} \rangle\rangle \\ \mathcal{L}_k &:= \mathcal{B}(\omega, k) = \frac{1}{2} \langle\langle \mathbf{J} \widehat{Z}_\theta, \widehat{Z} \rangle\rangle. \end{aligned}$$

255 The key feature here is that the wave action and wave action flux, evaluated on the
256 family of periodic travelling waves, are related to the tangent vectors of the waves via
257 the structure matrices \mathbf{M} and \mathbf{J} . This is multisymplectic Noether theory in action.

258 The first derivatives needed for \mathbf{DF} and Δ_L are

$$259 \quad (28) \quad \begin{aligned} \mathcal{A}_\omega &= \langle\langle \mathbf{M} \widehat{Z}_\theta, \widehat{Z}_\omega \rangle\rangle, \quad \mathcal{A}_k = \langle\langle \mathbf{M} \widehat{Z}_\theta, \widehat{Z}_k \rangle\rangle, \\ \mathcal{B}_\omega &= \langle\langle \mathbf{J} \widehat{Z}_\theta, \widehat{Z}_\omega \rangle\rangle, \quad \mathcal{B}_k = \langle\langle \mathbf{J} \widehat{Z}_\theta, \widehat{Z}_k \rangle\rangle. \end{aligned}$$

260 The second derivatives needed in the construction of κ can be simplified by using a
261 boosted symplectic structure. Define

$$262 \quad (29) \quad \mathbf{K} := \mathbf{J} - c_g \mathbf{M}.$$

263 Then differentiating (28) and combining gives

$$264 \quad (30) \quad \begin{aligned} \mathcal{B}_{\omega\omega} - c_g \mathcal{A}_{\omega\omega} &= \langle\langle \mathbf{K} \widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K} \widehat{Z}_\theta, \widehat{Z}_{\omega\omega} \rangle\rangle \\ \mathcal{B}_{\omega k} - c_g \mathcal{A}_{\omega k} &= \langle\langle \mathbf{K} \widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K} \widehat{Z}_\theta, \widehat{Z}_{\omega k} \rangle\rangle \\ \mathcal{B}_{kk} - c_g \mathcal{A}_{kk} &= \langle\langle \mathbf{K} \widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\rangle + \langle\langle \mathbf{K} \widehat{Z}_\theta, \widehat{Z}_{kk} \rangle\rangle. \end{aligned}$$

265 **3.1. Linearization about the periodic basic state.** Define the linear oper-
 266 ator

$$267 \quad (31) \quad \mathbf{L}W = \left[D^2S(\widehat{Z}) - k\mathbf{J}\frac{d}{d\theta} - \omega\mathbf{M}\frac{d}{d\theta} \right] W,$$

268 obtained by linearizing (24). Then differentiating (24) with respect to θ , k and ω
 269 gives,

$$\begin{aligned} 269 \quad D^2S(\widehat{Z})\widehat{Z}_\theta &= k\mathbf{J}\widehat{Z}_{\theta\theta} + \omega\mathbf{M}\widehat{Z}_{\theta\theta}, \\ 270 \quad D^2S(\widehat{Z})\widehat{Z}_k &= k\mathbf{J}\widehat{Z}_{\theta k} + \omega\mathbf{M}\widehat{Z}_{\theta k} + \mathbf{J}\widehat{Z}_\theta \\ 270 \quad D^2S(\widehat{Z})\widehat{Z}_\omega &= k\mathbf{J}\widehat{Z}_{\theta\omega} + \omega\mathbf{M}\widehat{Z}_{\theta\omega} + \mathbf{M}\widehat{Z}_\theta, \end{aligned}$$

271 or

$$272 \quad (32) \quad \mathbf{L}\widehat{Z}_\theta = 0, \quad \mathbf{L}\widehat{Z}_k = \mathbf{J}\widehat{Z}_\theta, \quad \text{and} \quad \mathbf{L}\widehat{Z}_\omega = \mathbf{M}\widehat{Z}_\theta,$$

273 with other derivatives following a similar pattern. The first equation of (32) shows
 274 that \widehat{Z}_θ is in the kernel of \mathbf{L} , and it is natural to assume that the kernel is no larger.
 275 Hence assume

$$276 \quad (33) \quad \text{Kernel}(\mathbf{L}) = \text{span}\{\widehat{Z}_\theta\}.$$

277 For inhomogeneous equations that arise in the modulation theory and the Jordan
 278 chain theory, a solvability condition will be needed. With the assumption (33) and
 279 the symmetry of \mathbf{L} , the solvability condition for the inhomogeneous equation $\mathbf{L}W = F$
 280 is

$$281 \quad (34) \quad \mathbf{L}W = F \quad \text{is solvable if and only if} \quad \langle\langle \widehat{Z}_\theta, F \rangle\rangle = 0.$$

282 **3.2. A twisted symplectic Jordan chain.** The second and third equation of
 283 (32) show that there are potentially two non-trivial Jordan chains associated with
 284 the zero eigenvalue of \mathbf{L} with geometric eigenvector \widehat{Z}_θ . In previous work [8, 33, 34],
 285 the phase modulation theory required a longer Jordan chain formed from *either* a
 286 \mathbf{J} -chain or an \mathbf{M} -chain. Here the intertwining of these two chains will be required
 287 in the phase modulation theory. Then, using (32),

$$288 \quad (35) \quad \mathbf{L}(\widehat{Z}_k - c_g\widehat{Z}_\omega) = (\mathbf{J} - c_g\mathbf{M})\widehat{Z}_\theta = \mathbf{K}\widehat{Z}_\theta,$$

289 using the boosted symplectic structure \mathbf{K} (29) in the last equality. Therefore, define

$$290 \quad (36) \quad \xi_1 = \widehat{Z}_\theta \quad \text{and} \quad \xi_2 = \widehat{Z}_k - c_g\widehat{Z}_\omega.$$

291 Then a mixed \mathbf{K} -Jordan chain of length two is formed

$$292 \quad (37) \quad \mathbf{L}\xi_1 = 0 \quad \text{and} \quad \mathbf{L}\xi_2 = \mathbf{K}\xi_1.$$

293 It is the extension of this chain and its connection with the singularity (7) that will
 294 appear in the modulation theory. Since the symplectic structure assures that the
 295 chain length is even, a proposed longer chain is

$$\begin{aligned} 296 \quad (38) \quad \mathbf{L}\xi_1 &= 0 \\ \mathbf{L}\xi_2 &= \mathbf{K}\xi_1 \\ \mathbf{L}\xi_3 &= \mathbf{K}\xi_2 \\ \mathbf{L}\xi_4 &= \mathbf{K}\xi_3. \end{aligned}$$

297 In this chain it is either assumed that \mathbf{K} is invertible or $\mathbf{K}\xi_j \neq 0$ for $j = 1, 2, 3$.

298 The second equation in (38) is solvable due to (32), and the third equation is
299 solvable since

$$\begin{aligned}
 \langle\langle \widehat{Z}_\theta, \mathbf{K}\xi_2 \rangle\rangle &= \langle\langle \widehat{Z}_\theta, \mathbf{K}(\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\rangle \\
 &= -\langle\langle \mathbf{K}\widehat{Z}_\theta, (\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\rangle \\
 300 \quad (39) \quad &= -\langle\langle \mathbf{J}\widehat{Z}_\theta - c_g \mathbf{M}\widehat{Z}_\theta, (\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\rangle \\
 &= -\mathcal{B}_k + c_g \mathcal{B}_\omega + c_g \mathcal{A}_k - c_g^2 \mathcal{A}_\omega \\
 &= -\frac{1}{\mathcal{A}_\omega} \Delta_L,
 \end{aligned}$$

301 using (28), and $\Delta_L = 0$ on Σ^1 curves. The fourth equation in (38) is solvable due to
302 even-ness of the Jordan chain, but it can be confirmed explicitly,

$$\begin{aligned}
 \langle\langle \widehat{Z}_\theta, \mathbf{K}\xi_3 \rangle\rangle &= -\langle\langle \mathbf{K}\xi_1, \xi_3 \rangle\rangle \\
 &= -\langle\langle \mathbf{L}\xi_2, \xi_3 \rangle\rangle \\
 303 \quad &= -\langle\langle \xi_2, \mathbf{L}\xi_3 \rangle\rangle \\
 &= -\langle\langle \xi_2, \mathbf{K}\xi_2 \rangle\rangle \\
 &= 0,
 \end{aligned}$$

304 with the last line following from skew-symmetry of \mathbf{K} . This Jordan chain terminates
305 at four if the next equation

$$306 \quad \mathbf{L}\xi_5 = \mathbf{K}\xi_4,$$

307 is not solvable; that is, when

$$308 \quad (40) \quad \langle\langle \widehat{Z}_\theta, \mathbf{K}\xi_4 \rangle\rangle := -\mathcal{K} \neq 0.$$

309 It is this coefficient \mathcal{K} that shows up as the coefficient of dispersion in the modulation
310 equation (12).

311 To summarize: for $(\omega, k) \in \Sigma^1$, with the assumption (33), the algebraic multiplic-
312 ity of the zero eigenvalue of \mathbf{L} is at least four and is exactly four when $\mathcal{K} \neq 0$.

313 **4. Modulation ansatz.** Given the family of basic states, $\widehat{Z}(\theta, \omega, k)$, the classical
314 Whitham modulation equations (2) are obtained using the modulation ansatz

$$315 \quad (41) \quad Z(x, t) = \widehat{Z}\left(\theta + \frac{1}{\varepsilon}\phi, \omega + \Omega, k + q\right) + \varepsilon W\left(\theta + \frac{1}{\varepsilon}\phi, X, T, \varepsilon\right),$$

316 with ϕ dependent on (X, T, ε) ,

$$317 \quad q = \phi_X, \quad \Omega = \phi_T, \quad X = \varepsilon x, \quad T = \varepsilon t.$$

318 Substitution of the ansatz (41) into the Euler-Lagrange equation (22) leads, via a
319 solvability condition at order ε^1 , to the dispersionless conservation of wave action in
320 (2). This modulation ansatz is valid away from a Σ^1 curve.

321 For $(\omega, k) \in \Sigma^1$ the ansatz needs to be modified. *A posteriori* it is confirmed that
322 the appropriate modification of (41) is

$$323 \quad (42) \quad Z(x, t) = \widehat{Z}(\theta + \varepsilon\phi, \omega - c_g \varepsilon^2 q + \varepsilon^3 \Omega, k + \varepsilon^2 q) + \varepsilon^3 W(\theta, X, T, \varepsilon).$$

324 The conservation of waves is still operational

$$325 \quad (43) \quad q = \phi_X, \quad \Omega = \phi_T, \quad \text{and} \quad q_T = \Omega_X,$$

326 but the scaling of the independent variables is changed to

$$327 \quad (44) \quad X = \varepsilon(x - c_g t) \quad \text{and} \quad T = \varepsilon^2 t, \quad \text{with} \quad c_g := \frac{\mathcal{L}_{\omega k}}{\mathcal{L}_{\omega\omega}} = \frac{\mathcal{A}_k}{\mathcal{A}_\omega}.$$

328 The strategy is then to substitute the ansatz (42) into the Euler-Lagrange equation
 329 (22), expand everything in powers of ε , and solve order by order in ε . While the ansatz
 330 (42) is new, particularly in how the speed c_g affects the modulation, the machinations
 331 of the expansions is similar to previous work [8, 33, 34], and so only a summary is
 332 given. The zeroth, first, and second order equations in ε reproduce the equation for
 333 the basic state, the linearization, and conservation of waves (43). At third order the
 334 resulting equation is

$$335 \quad (45) \quad \begin{aligned} \mathbf{L}W_3 &= q_X \left[\mathbf{J}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_k + c_g^2 \mathbf{M}\widehat{Z}_\omega - c_g \mathbf{J}\widehat{Z}_\omega \right] \\ &= q_X \mathbf{K}(\widehat{Z}_k - c_g \widehat{Z}_\omega) = \mathbf{K}\xi_2, \end{aligned}$$

336 using (29) and (36). Here, W_3 is obtained from the Taylor expansion of W ,

$$337 \quad \varepsilon^3 W(\theta, X, T, \varepsilon) = \varepsilon^3 W_3(\theta, X, T) + \varepsilon^4 W_4(\theta, X, T) + \varepsilon^5 W_5(\theta, X, T) + \dots$$

338 The equation (45) is solvable for $(\omega, k) \in \Sigma^1$ due to (39). Hence

$$339 \quad (46) \quad W_3 = q_X \xi_3 + \alpha \xi_1,$$

340 where at this point $\alpha(X, T)$ is an arbitrary function.

341 **4.1. Fourth order equation.** The fourth order equation simplifies to

$$342 \quad \begin{aligned} \mathbf{L}(W_4 - q_{XX}\xi_4 - \alpha_X \xi_2 - \phi q_X(\xi_3)_\theta - \alpha \phi \widehat{Z}_{\theta\theta}) \\ = q_T \left(\mathbf{M}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_\omega \right) + \Omega_X \left(\mathbf{J}\widehat{Z}_\omega - c_g \mathbf{M}\widehat{Z}_\omega \right). \end{aligned}$$

343 A curiosity in the theory is that the q_T and Ω_X terms are exactly solvable for $(\omega, k) \in$
 344 Σ^1 since

$$345 \quad q_T \langle\langle \widehat{Z}_\theta, (\mathbf{M}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_\omega) \rangle\rangle = q_T (-\mathcal{A}_k + c_g \mathcal{A}_\omega) = 0,$$

346 and

$$347 \quad \Omega_X \langle\langle \widehat{Z}_\theta (\mathbf{J}\widehat{Z}_\omega - c_g \mathbf{M}\widehat{Z}_\omega) \rangle\rangle = \Omega_X (\mathcal{B}_\omega - c_g \mathcal{A}_\omega) = 0,$$

348 using the definition of c_g and the cross-derivatives $\mathcal{A}_k = \mathcal{B}_\omega$.

349 The complete solution for W_4 is therefore

$$350 \quad (47) \quad W_4 = q_T \eta + q_{XX} \xi_4 + \alpha_X \xi_2 + \phi q_X (\xi_3)_\theta + \alpha \phi \widehat{Z}_{\theta\theta} + \beta \xi_1,$$

351 where $\beta(X, T)$ is arbitrary at this point, and η is a particular solution of

$$352 \quad (48) \quad \mathbf{L}\eta = \mathbf{M}\widehat{Z}_k - 2c_g \mathbf{M}\widehat{Z}_\omega + \mathbf{J}\widehat{Z}_\omega.$$

353 The solution η of this equation will not be needed explicitly in the theory, only its
 354 abstract definition in (48).

355 **4.2. Solvability at fifth order.** After some simplification, the fifth order terms
 356 reduce to

$$357 \quad (49) \quad \mathbf{L}\widetilde{W}_5 = \Omega_T \mathbf{M}\widehat{Z}_\omega + qq_X \left[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - D^3 S(\widehat{Z})(\xi_2, \xi_3) \right] \\ + q_{XXX} \mathbf{K}\xi_4 + \Omega_{XX} (\mathbf{M}\xi_3 + \mathbf{K}\eta),$$

358 where \widetilde{W}_5 incorporates all terms that are exactly solvable and,

$$359 \quad \Upsilon := \widehat{Z}_{kk} - 2c_g \widehat{Z}_{\omega k} + c_g^2 \widehat{Z}_{\omega\omega}.$$

360 An explicit expression for \widetilde{W}_5 can be constructed but is not needed as solvability
 361 delivers the modulation equation (12).

362 The awkward term in (49) is the Ω_{XX} term which would make the resulting
 363 modulation equation non-conservative. However, it too is in the range of \mathbf{L} , and it is
 364 the abstract definition of the function η in (48) that is used to show that this term is
 365 removable,

$$\begin{aligned} \langle\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 + \mathbf{K}\eta \rangle\rangle &= \langle\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 \rangle\rangle - \langle\langle \mathbf{K}\widehat{Z}_\theta, \eta \rangle\rangle \\ &= \langle\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 \rangle\rangle - \langle\langle \mathbf{L}\xi_2, \eta \rangle\rangle \\ &= -\langle\langle \mathbf{M}\widehat{Z}_\theta, \xi_3 \rangle\rangle - \langle\langle \xi_2, \mathbf{L}\eta \rangle\rangle \\ &= -\langle\langle \mathbf{L}\widehat{Z}_\omega, \xi_3 \rangle\rangle - \langle\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\rangle \\ 366 \quad &= -\langle\langle \widehat{Z}_\omega, \mathbf{L}\xi_3 \rangle\rangle - \langle\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\rangle \\ &= -\langle\langle \widehat{Z}_\omega, \mathbf{K}\xi_2 \rangle\rangle - \langle\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g \mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\rangle \\ &= -\langle\langle \xi_2, \mathbf{M}(\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\rangle \\ &= -\langle\langle \xi_2, \mathbf{M}\xi_2 \rangle\rangle \\ &= 0, \end{aligned}$$

367 using skew-symmetry of \mathbf{M} and \mathbf{K} , symmetry of \mathbf{L} , the Jordan chain, and the function
 368 η (48). Therefore there exists a function δ such that

$$369 \quad \mathbf{L}\delta = \mathbf{M}\xi_3 + \mathbf{K}\eta.$$

370 This simplifies the fifth order equation to

$$371 \quad \mathbf{L}(\widetilde{W}_5 - \Omega_{XX}\delta) = \Omega_T \mathbf{M}\widehat{Z}_\omega + q_{XXX} \mathbf{K}\xi_4 \\ + qq_X \left[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - D^3 S(\widehat{Z}^o)(\xi_2, \xi_3) \right].$$

372 This equation is solvable if and only if the right hand side is orthogonal to \widehat{Z}_θ , giving

$$373 \quad (50) \quad a_1 \Omega_T + a_2 qq_X + a_3 q_{XXX} = 0,$$

374 with

$$375 \quad (51) \quad a_1 = \langle\langle \widehat{Z}_\theta, \mathbf{M}\widehat{Z}_\omega \rangle\rangle \\ a_2 = \langle\langle \widehat{Z}_\theta, \left[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - D^3 S(\widehat{Z})(\xi_2, \xi_3) \right] \rangle\rangle \\ a_3 = \langle\langle \widehat{Z}_\theta, \mathbf{K}\xi_4 \rangle\rangle.$$

376 Now

$$377 \quad a_1 = \langle\langle \widehat{Z}_\theta, \mathbf{M}\widehat{Z}_\omega \rangle\rangle = -\langle\langle \mathbf{M}\widehat{Z}_\theta, \widehat{Z}_\omega \rangle\rangle = -\mathcal{A}_\omega,$$

378 using (28), and appeal to (40) shows that

$$379 \quad a_3 = -\langle\langle \mathbf{K}\xi_1, \xi_4 \rangle\rangle = -\mathcal{K}.$$

380 Using the geometry of the frequency-wavenumber map it is shown below that $a_2 = -\kappa$,
 381 where κ is defined in (17), giving the final form of the modulation equation as

$$382 \quad (52) \quad \mathcal{A}_\omega \Omega_T + \kappa q q_X + \mathcal{K} q_{XX} = 0 \quad \text{and} \quad q_T = \Omega_X,$$

383 and for non-degeneracy of this equation it is assumed that

$$384 \quad (53) \quad \mathcal{A}_\omega \neq 0, \quad \kappa \neq 0, \quad \text{and} \quad \mathcal{K} \neq 0.$$

385 **4.3. The geometry of the frequency-wavenumber map and a_2 .** It is not
 386 at all obvious that the intrinsic second derivative that arises from the geometry of
 387 the frequency-wavenumber map (17) should be related to the above coefficient a_2 that
 388 appears from the modulation analysis as the coefficient of nonlinearity. However, with
 389 the canonical choice of normal vector \mathbf{n} , they are exactly equal and this is proved as
 390 follows. The expression that arises in the modulation analysis and solvability condition
 391 is

$$392 \quad (54) \quad a_2 = \left\langle\left\langle \widehat{Z}_\theta, \left[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - D^3 S(\widehat{Z})(\xi_2, \xi_3) \right] \right\rangle\right\rangle.$$

393 Differentiating $\mathbf{L}\xi_3 = \mathbf{K}\xi_2$ with respect to θ ,

$$394 \quad \mathbf{L}(\xi_3)_\theta + D^3 S(\widehat{Z})(\xi_1, \xi_3) = \mathbf{K}(\xi_2)_\theta,$$

395 Hence, with $\widehat{Z}_\theta = \xi_1$, the second term in (54) is

$$\begin{aligned} \langle\langle \widehat{Z}_\theta, \mathbf{K}(\xi_3)_\theta \rangle\rangle &= -\langle\langle \mathbf{K}\xi_1, (\xi_3)_\theta \rangle\rangle \\ &= -\langle\langle \mathbf{L}\xi_2, (\xi_3)_\theta \rangle\rangle \\ &= -\langle\langle \xi_2, \mathbf{L}(\xi_3)_\theta \rangle\rangle \\ 396 \quad &= -\langle\langle \xi_2, \mathbf{K}(\xi_2)_\theta - D^3 S(\widehat{Z})(\xi_1, \xi_3) \rangle\rangle \\ &= -\langle\langle \xi_2, \mathbf{K}(\xi_2)_\theta \rangle\rangle + \langle\langle \xi_2, D^3 S(\widehat{Z})(\xi_1, \xi_3) \rangle\rangle \\ &= -\langle\langle \xi_2, \mathbf{K}(\xi_2)_\theta \rangle\rangle + \langle\langle \xi_1, D^3 S(\widehat{Z})(\xi_2, \xi_3) \rangle\rangle, \end{aligned}$$

397 using $\mathbf{L}\xi_2 = \mathbf{K}\xi_1$, skew-symmetry of \mathbf{K} , symmetry of \mathbf{L} , and permutation of the
 398 trilinear form in the last line. Substitution of the expression for $\langle\langle \widehat{Z}_\theta, \mathbf{K}(\xi_3)_\theta \rangle\rangle$ into a_2
 399 reduces it to

$$400 \quad (55) \quad a_2 = \langle\langle \mathbf{K}\xi_2, (\xi_2)_\theta \rangle\rangle + \langle\langle \widehat{Z}_\theta, \mathbf{K}\Upsilon \rangle\rangle.$$

401 Start with this expression for a_2 , and substitute for Υ and ξ_2 ,

$$\begin{aligned}
-a_2 &= -\langle\langle \mathbf{K}\xi_2, (\xi_2)_\theta \rangle\rangle - \langle\langle \widehat{Z}_\theta, \mathbf{K}\Upsilon \rangle\rangle \\
&= \langle\langle \mathbf{K}(\widehat{Z}_{k\theta} - c_g \widehat{Z}_{\omega\theta}), \widehat{Z}_k - c_g \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K}\widehat{Z}_\theta, \Upsilon \rangle\rangle \\
&= \langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\rangle - c_g \langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\rangle - c_g \langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\rangle \\
&\quad + c_g^2 \langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K}\widehat{Z}_\theta, \Upsilon \rangle\rangle \\
&= \langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\rangle - c_g \langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\rangle - c_g \langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\rangle + c_g^2 \langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\rangle \\
402 \quad &\quad + \langle\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{kk} - 2c_g \widehat{Z}_{\omega k} + c_g^2 \widehat{Z}_{\omega\omega} \rangle\rangle \\
&= (\langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\rangle + \langle\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{kk} \rangle\rangle) + c_g^2 (\langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{\omega\omega} \rangle\rangle) \\
&\quad - c_g (\langle\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\rangle + \langle\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\rangle + 2\langle\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{\omega k} \rangle\rangle) \\
&= (\mathcal{B}_{kk} - c_g \mathcal{A}_{kk}) - 2c_g (\mathcal{B}_{\omega k} - c_g \mathcal{A}_{\omega k}) + c_g^2 (\mathcal{B}_{\omega\omega} - c_g \mathcal{A}_{\omega\omega}) \\
&= \langle \mathbf{n}, \mathbf{D}^2 \mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle \\
&= \kappa,
\end{aligned}$$

403 when \mathbf{n} is in canonical form (14). The third to last step follows from the substitution
404 of the identities (30). This completes the derivation of the phase modulation equations
405 (52) on Σ^1 curves.

406 **4.4. Invariance under coordinate change.** Since the modulation equation
407 (52) relies on two eigenvector choices there is a potential non-uniqueness in the final
408 form. The first potential non-uniqueness is the choice of geometric eigenvector ξ_1 of
409 the zero eigenvalue of \mathbf{L} ,

$$410 \quad \mathbf{L}\xi_1 = 0 \quad \Rightarrow \quad \xi_1 = b\widehat{Z}_\theta,$$

411 where b is an arbitrary multiplicative constant. This constant is then multiplied by
412 each element in the Jordan chains. Hence a_1 and a_3 in (50) would be multiplied by b^2 .
413 However, the signs of a_1 and a_3 would not change and the factor b^2 can be removed
414 by scaling. The other eigenvector choice is \mathbf{n} and

$$415 \quad \mathbf{D}\mathbf{F}(\omega, k)\mathbf{n} = 0 \quad \Rightarrow \quad \mathbf{n} = b \begin{pmatrix} -c_g \\ 1 \end{pmatrix},$$

416 for some nonzero constant b . In this case the only change would be a scale factor
417 on κ , $\kappa \mapsto b^3 \kappa$. Since κ multiplies a nonlinearity, scaling q (or ϕ) using b^3 would
418 eliminate this scale factor in κ . A change in sign of κ is eliminated by a change in sign
419 of q . Hence, with the canonical choices $\xi_1 = \widehat{Z}_\theta$ and \mathbf{n} as in (14), and the modulation
420 ansatz (42), the modulation equation (52) is uniquely defined.

421 **4.5. Unfolding from Σ^1 curves.** Instead of taking Δ_L to be identically zero, it
422 can be taken to be of order ε^2 giving an unfolding of the two-way Boussinesq equation

$$423 \quad (56) \quad \mathcal{A}_\omega \Omega_T + \mu q_X + \kappa q q_X + \mathcal{H} q_{XX} = 0 \quad \text{and} \quad q_T = \Omega_X,$$

424 where $\text{sign}(\mu) = \text{sign}(\mathcal{A}_\omega \Delta_L)$. In this case, the combined equation is the classical
425 two-way Boussinesq equation with a second derivative in X term

$$426 \quad (57) \quad \mathcal{A}_\omega q_{TT} + \mu q_{XX} + \left(\frac{1}{2}\kappa q^2\right)_{XX} + \mathcal{H} q_{XXX} = 0.$$

427 This unfolded version allows one to extend the discussion from solely along the Σ^1
428 curves to the neighbourhood around them, characterised by the small parameter ε .

429 **5. The two-way Boussinesq equation.** Once the modulation equation (52)
 430 is derived in a specific context, analysis of the solutions follows the standard strategy.
 431 Assuming all the coefficients are non-zero, the dependent and independent variables
 432 can be scaled so that the coefficients are ± 1 , simplifying the form of the equation.

433 Starting with (57), scale X , T , and q and let

434
$$s_1 = \text{sign}(\Delta_L) \quad \text{and} \quad s_2 = \text{sign}(\mathcal{A}_\omega \mathcal{H}).$$

435 Denote the scaled space and time variables by ξ and τ , and the scaled q by $u(\xi, \tau)$.
 436 Then the two-way Boussinesq equation is reduced to the standard form

437 (58)
$$u_{\tau\tau} + s_1 u_{\xi\xi} + \left(\frac{1}{2}u^2\right)_{\xi\xi} + s_2 u_{\xi\xi\xi\xi} = 0, \quad s_1, s_2 = \pm 1.$$

438 The set Σ^1 locally separates the subset of the (ω, k) for which travelling waves exist
 439 into two regions: elliptic ($s_1 = +1$) and hyperbolic ($s_1 = -1$). The sign s_2 indicates
 440 whether the resulting two-way Boussinesq equation is good ($s_2 = +1$) or bad ($s_2 =$
 441 -1). In the latter case, the initial value problem for the linearized system is ill posed.

442 Consider the linearization of (58) about the trivial solution and consider a normal
 443 mode solution of the form $e^{i(\hat{k}\xi + \hat{\omega}\tau)}$, then the dispersion relation is of the form

444
$$\hat{\omega}^2 = -s_1 \hat{k}^2 + s_2 \hat{k}^4.$$

445 There are four cases depending on the signs s_1 and s_2 , and they are shown in Figure
 446 3. The figure plots $\hat{\omega}^2$ against \hat{k}^2 and so $\hat{\omega}^2 < 0$ indicates linear instability of the
 trivial solution which in turn reflects linear instability of the basic travelling wave.

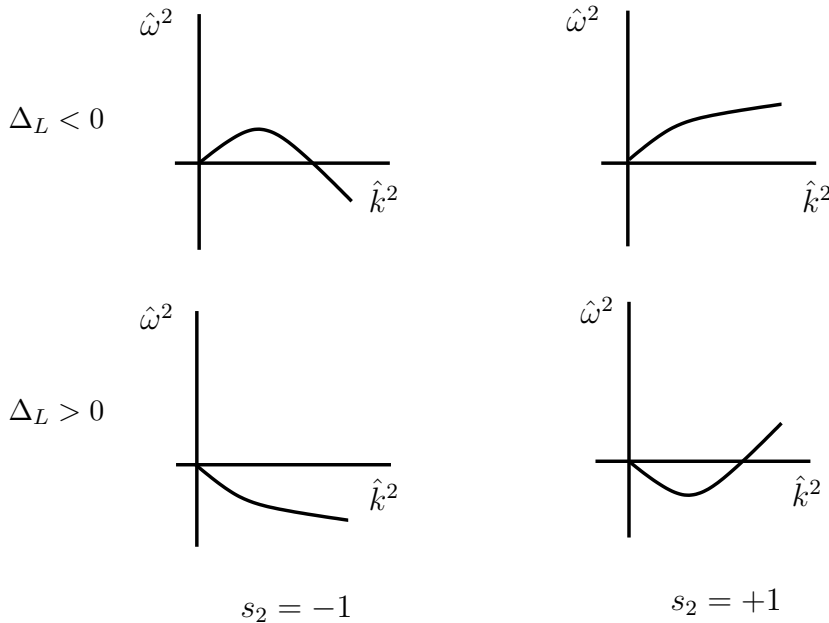


FIG. 3. The four cases determined by the signs $s_1 = \text{sign}(\Delta_L)$ and $s_2 = \text{sign}(\mathcal{A}_\omega \mathcal{H})$ in the two-way Boussinesq equation near a Σ^1 curve.

447

448 When $s_1 < 0$ (the upper two cases in Figure 3) then either an unstable band
 449 emerges at finite \hat{k} when $s_2 = -1$ or the Boussinesq equation is also hyperbolic
 450 ($s_2 = +1$). When $s_1 > 0$ (lower two cases in Figure 3) then either a cutoff wave
 451 number emerges with re-stabilization at finite \hat{k} (as in the lower right diagram with
 452 $s_2 = +1$), or instability is further enhanced ($s_1 = +1$ and $s_2 = -1$).

453 The simplest class of nonlinear solutions of (58) are travelling solitary wave solu-
 454 tions, for example,

$$455 \quad u(\xi, \tau) = \hat{u}(\xi + \gamma\tau),$$

456 which satisfies the ODE

$$457 \quad (\gamma^2 \hat{u} + s_1 \hat{u} + \frac{1}{2} \hat{u}^2 + s_2 \hat{u}'')'' = 0.$$

458 Integrating and taking the function of integration to be constant

$$459 \quad s_2 \hat{u}'' + (s_1 + \gamma^2) \hat{u} + \frac{1}{2} \hat{u}^2 = h.$$

460 The constant of integration h is fixed by initial data or the value of \hat{u} at infinity. For
 461 appropriate parameter values, this planar ODE has a family of periodic solutions and
 462 a homoclinic orbit which represent periodic travelling waves and a solitary travelling
 463 wave solution of (58). The implication of these solutions is that the transition from
 464 elliptic to hyperbolic of a periodic travelling wave of the original system generates a
 465 coherent structure in the transition, which is represented by the above solitary wave.
 466 However, there is much more complexity generated at the transition. Hirota [18]
 467 shows that there is a large family of N -soliton solutions to (58) as well. Further
 468 details especially in the case $N = 2$ are given in [18]. Numerical simulations of the
 469 case $N = 2$ are presented in [27].

470 The two-way Boussinesq equation is also generated by a Lagrangian, and has
 471 both a Hamiltonian and multisymplectic structure (e.g. [6], §10 of [9], and [12]).

472 **6. Example: finite-amplitude stabilization of Stokes waves.** The four
 473 scenarios in Figure 3 can be used to identify the type of stability-instability transition
 474 in the water wave problem at finite-amplitude, linearized about Stokes waves on deep
 475 water. It was first shown by Longuet-Higgins [24] that the Benjamin-Feir instability of
 476 Stokes travelling waves in deep water stabilizes at finite amplitude. This stabilization
 477 can be seen most clearly in the numerics of McLean [28]. Linear stability exponents
 478 for finite-amplitude Stokes waves in deep water are computed, and in Figure 2 of [28]
 479 stability regions are plotted as a function of the modulation wavenumbers, for a se-
 480 quence of amplitudes. Three-dimensional instabilities (two modulation wavenumbers)
 481 are plotted but only the two-dimensional (one modulation wavenumber) instabilities
 482 are of interest here. At low amplitude the Benjamin-Feir instability is operational
 483 and it persists as the amplitude increases, until a wave steepness of $h/\lambda \approx 0.108$ is
 484 reached, where h is crest to trough distance and λ the wavelength. At this value, the
 485 region of modulation instability in wavenumber space detaches from the origin (see
 486 the transition in going from Figure 2(c) to 2(d) in [28]).

487 Independently, in the same year, Whitham [39] showed that the stabilization point
 488 was precisely a transition point associated with $\Delta_L = 0$. Whitham first transforms
 489 the averaged Lagrangian into a functional \mathcal{H} based on the energy,

$$490 \quad \mathcal{L}(\omega, k, I) = \omega I - \mathcal{H}(k, I),$$

491 where I is the value of the wave action. The amplitude of the wave is parameterized in
 492 terms of wave action (see [17] for discussion of Whitham modulation theory in terms
 493 of $\mathcal{H}(k, I)$). In terms of $\mathcal{H}(k, I)$ the Lighthill determinant is

494
$$\det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} = \frac{\mathcal{H}_{kk}}{\mathcal{H}_{II}}.$$

495 An explicit transformation from \mathcal{L} to \mathcal{H} is given in the introduction and Appendix
 496 A of [17]. The sign here differs from [17] and [39] as they define wave-action flux with
 497 the opposite sign.

498 Whitham [39] then argues (see §10 in [39]) that the energy takes a self similar
 499 form

500
$$\mathcal{H}(k, I) = \frac{g}{k^2} W(\zeta) \quad \text{with} \quad \zeta := \frac{k^3 I}{\sqrt{gk}}.$$

501 He then appeals to the tabulated values of the energy in Longuet-Higgins [23] to show
 502 that $\mathcal{H}_{II} > 0$ and does not change sign along a branch of Stokes waves, but shows
 503 that $\mathcal{H}_{kk} = 0$ precisely at $h/\lambda = 0.109$ which agrees, to numerical accuracy, with the
 504 change of stability found in [24] and [28].

505 With this association between the stability-instability transition point and van-
 506 ishing of the Lighthill determinant, the theory of this paper can be used to deduce
 507 that the two-way Boussinesq equation is generated at the transition.

508 Going by the transition in Figure 2 of [28], the appropriate Boussinesq model
 509 is the bad Boussinesq with $s_2 = -1$, and Δ_L goes from positive to negative as the
 510 amplitude increases, corresponding to the two left graphs in Figure 3. Since the
 511 sign of the coefficient of the nonlinearity in (52) is not important, and generically it
 512 is nonzero, the appropriate Boussinesq model for water waves near the instability-
 513 stability transition of Stokes waves is

514 (59)
$$u_{\tau\tau} + s_1 u_{\xi\xi} \pm \left(\frac{1}{2}u^2\right)_{\xi\xi} - u_{\xi\xi\xi\xi} = 0, \quad s_1 = \pm 1,$$

515 with $s_1 = +1$ below the amplitude threshold and $s_1 = -1$ above.

516 This example is not of much interest physically since the numerics of [28] show
 517 that the above threshold point is surrounded by unstable Stokes waves. Below the
 518 threshold the waves are modulationally unstable, and above the threshold other finite-
 519 wavenumber instabilities and multidimensional (two modulation wavenumbers) take
 520 over. However, it is of theoretical interest in that it shows how limited qualitative
 521 information, obtained numerically, is sufficient to predict the nature of the modulation
 522 equation near the transition point.

523 **7. Example: Σ^1 curves and explicit reduction for a nonlinear wave**
 524 **equation.** Consider the nonlinear wave equation, a complex Klein-Gordon (CKG)
 525 equation,

526 (60)
$$\Psi_{tt} = \Psi_{xx} - \Psi + |\Psi|^2 \Psi,$$

527 for the complex-valued function $\Psi(x, t)$, which is a model for the nonlinear dynamics
 528 near the Kelvin-Helmholtz instability [2]. The CKG equation is generated by the
 529 Lagrangian

530 (61)
$$\mathcal{L}(\Psi, \bar{\Psi}) = \frac{1}{2} \int_{t_1}^{t_2} \int_{x_1}^{x_2} [-|\Psi_t|^2 + |\Psi_x|^2 + |\Psi|^2 - \frac{1}{2}|\Psi|^4] \, dxdt,$$

531 on the set $[x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2$. The variation $\delta\mathcal{L}/\delta\bar{\Psi} = 0$, with fixed endpoints,
 532 generates (60), and $\delta\mathcal{L}/\delta\Psi = 0$ generates the conjugate of (60). Multisymplectification
 533 of CKG will be introduced below when required for the calculation of the dispersion
 534 coefficient \mathcal{K} .

535 **7.1. Periodic travelling waves.** The CKG equation (60) has a family of exact
 536 periodic travelling wave solutions

$$537 \quad (62) \quad \Psi(x, t) = \Psi_0 e^{i\theta}, \quad \theta = kx + \omega t + \theta_0,$$

538 and substitution into (60) gives the nonlinear dispersion relation, relating amplitude
 539 to the frequency and wavenumber

$$540 \quad (63) \quad |\Psi_0|^2 = 1 - \omega^2 + k^2.$$

541 This solution set consists of a hyperboloid of one sheet in the three dimensional space
 542 (ω, k, r) with $r = |\Psi_0| > 0$. The projection of this hyperboloid onto the (ω, k) plane
 543 is shown in Figure 4. The unshaded region is the set where solutions of (63) exist and
 544 it consists of

$$545 \quad (64) \quad U = \{(\omega, k) \in \mathbb{R}^2 : \omega^2 < 1 + k^2, k \neq 0\}.$$

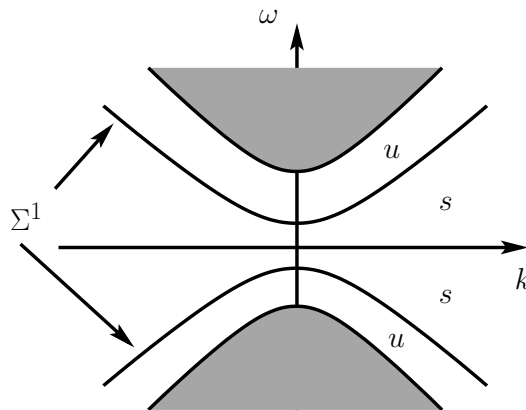


FIG. 4. Regions of existence and Σ^1 curve for the family of periodic travelling wave solutions of CKG. The symbols s (u) denote regions where the periodic travelling wave is stable (unstable).

546

547 **7.2. Conservation law and Σ^1 curves.** The conservation law which represents
 548 conservation of wave action is due to an S^1 -symmetry: $e^{is}\Psi$ is a solution of CKG
 549 whenever Ψ is a solution for any $s \in \mathbb{R}$. The conservation law is

$$550 \quad A_t + B_x = 0, \quad \text{with } A = -\text{Im}(\bar{\Psi}\Psi_t), \quad B = \text{Im}(\bar{\Psi}\Psi_x).$$

551 Evaluate the components of the conservation law on the family of periodic travelling
 552 waves

$$553 \quad (65) \quad \begin{aligned} \mathcal{A}(\omega, k) &= -\omega|\Psi_0|^2 = -\omega(1 + k^2 - \omega^2) \\ \mathcal{B}(\omega, k) &= k|\Psi_0|^2 = k(1 + k^2 - \omega^2). \end{aligned}$$

554 They can also be obtained by substituting (62) into (61), averaging, and differentiating
 555 with respect to ω and k . The matrix in the Lighthill determinant is

$$556 \quad \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} = \begin{bmatrix} -1 - k^2 + 3\omega^2 & -2\omega k \\ -2\omega k & 1 + 3k^2 - \omega^2 \end{bmatrix}.$$

557 Setting the determinant to zero gives

$$\begin{aligned} \Delta_L &= \mathcal{A}_\omega \mathcal{B}_k - \mathcal{A}_k \mathcal{B}_\omega \\ &= (-1 - k^2 + 3\omega^2)(1 + 3k^2 - \omega^2) - 4\omega^2 k^2 \\ 558 \quad &= -(1 - \omega^2 + k^2)(1 - 3\omega^2 + 3k^2) \\ &= -|\Psi_0|^2(1 - 3\omega^2 + 3k^2). \end{aligned}$$

559 Hence the only non-trivial points in U where $\Delta_L = 0$ are when the second factor
 560 vanishes

$$561 \quad (66) \quad \Sigma^1 = \{(\omega, k) \in U : \omega^2 - k^2 = \frac{1}{3}\},$$

562 with U defined in (64). The singular set Σ^1 consists of two curves and they are labelled
 563 in Figure 4, and the stable (unstable) regions in the (ω, k) -plane are labelled with s
 564 (u). The image of Σ^1 in the $(\mathcal{A}, \mathcal{B})$ plane consists of the two curves

$$565 \quad \mathcal{A}^2 - \mathcal{B}^2 = \frac{4}{27}.$$

566 All the points in Σ^1 are fold points. There are no cusp points in this example, and so
 567 $\kappa \neq 0$. Explicitly,

$$568 \quad \kappa = (\mathcal{B}_{kk} - c_g \mathcal{A}_{kk}) - 2c_g(\mathcal{B}_{\omega k} - c_g \mathcal{A}_{\omega k}) + c_g^2(\mathcal{B}_{\omega\omega} - c_g \mathcal{A}_{\omega\omega}).$$

569 Computing

$$570 \quad c_g = \left. \frac{\mathcal{A}_k}{\mathcal{A}_\omega} \right|_{\Sigma^1} = -\frac{\omega}{k},$$

571 and

$$\begin{aligned} 572 \quad [\mathcal{B}_{\omega\omega} - c_g \mathcal{A}_{\omega\omega}] \Big|_{\Sigma^1} &= 4k + \frac{2}{k} \\ [\mathcal{B}_{\omega k} - c_g \mathcal{A}_{\omega k}] \Big|_{\Sigma^1} &= -4\omega \\ [\mathcal{B}_{kk} - c_g \mathcal{A}_{kk}] \Big|_{\Sigma^1} &= 4k - \frac{3}{2k}. \end{aligned}$$

573 Combining gives

$$574 \quad \kappa = \frac{2}{3k^3}.$$

575 Since $\mathcal{A}_\omega \Big|_{\Sigma^1} = 2k^2$, the emergent two-way Boussinesq equation is

$$576 \quad 2k^2 q_{TT} + \frac{2}{3k^3} (qq_X)_X + \mathcal{K} q_{XXXX} = 0.$$

577 It remains to compute the coefficient of dispersion. It can be computed in this case
 578 by deriving the dispersion relation for the linearization of (60) about the periodic
 579 travelling wave, but the Jordan chain strategy is used instead to illustrate it in an
 580 example, and because it is the most general strategy for more complex problems.

581 **7.3. Multisymplectification, linearization and \mathcal{K} .** A Legendre transform
 582 can be used to develop the multisymplectic formulation of CKG, but it is simple enough
 583 to write down directly. Let

$$584 \quad \mathbf{a} = \begin{pmatrix} \operatorname{Re}(\Psi) \\ \operatorname{Im}(\Psi) \end{pmatrix}, \quad \mathbf{b} = \mathbf{a}_t, \quad \text{and} \quad \mathbf{c} = \mathbf{a}_x.$$

585 Then CKG has the multisymplectic formulation

$$586 \quad \begin{bmatrix} 0 & -\mathbf{I}_2 & 0 \\ \mathbf{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}_t + \begin{bmatrix} 0 & 0 & \mathbf{I}_2 \\ 0 & 0 & 0 \\ -\mathbf{I}_2 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}_x = \begin{pmatrix} \mathbf{a} - \|\mathbf{a}\|^2 \mathbf{a} \\ \mathbf{b} \\ -\mathbf{c} \end{pmatrix},$$

587 where \mathbf{I}_2 is the 2×2 identity matrix, or

$$588 \quad \mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z),$$

589 with

$$590 \quad \mathbf{K} = \mathbf{J} - c_g \mathbf{M} = \begin{bmatrix} 0 & c_g \mathbf{I}_2 & \mathbf{I}_2 \\ -c_g \mathbf{I}_2 & 0 & 0 \\ -\mathbf{I}_2 & 0 & 0 \end{bmatrix}, \quad Z = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \in \mathbb{R}^6,$$

591 and

$$592 \quad S(Z) = \frac{1}{2} \|\mathbf{b}\|^2 - \frac{1}{2} \|\mathbf{c}\|^2 + \frac{1}{2} \|\mathbf{a}\|^2 - \frac{1}{4} \|\mathbf{a}\|^4.$$

593 In these coordinates the basic state is

$$594 \quad \widehat{Z}(\theta, \omega, k) = G_\theta \begin{pmatrix} \widehat{\mathbf{a}} \\ \widehat{\mathbf{b}} \\ \widehat{\mathbf{c}} \end{pmatrix}, \quad \mathbf{b} = \omega \mathbf{J}_2 \widehat{\mathbf{a}}, \quad \mathbf{c} = k \mathbf{J}_2 \widehat{\mathbf{a}},$$

595 with $\|\widehat{\mathbf{a}}\|^2 = 1 - \omega^2 + k^2$,

$$596 \quad G_\theta = R_\theta \oplus R_\theta \oplus R_\theta, \quad R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \text{and} \quad \mathbf{J}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

597 The linearized operator \mathbf{L} is

$$598 \quad (67) \quad \mathbf{L} = \begin{bmatrix} (1 - \|\widehat{\mathbf{a}}\|^2) \mathbf{I} - 2\widehat{\mathbf{a}}\widehat{\mathbf{a}}^T & \omega \mathbf{J}_2 & -k \mathbf{J}_2 \\ -\omega \mathbf{J}_2 & \mathbf{I}_2 & 0 \\ k \mathbf{J}_2 & 0 & -\mathbf{I}_2 \end{bmatrix},$$

599 and the Jordan chain satisfies $\mathbf{L}\xi_j = \mathbf{K}\xi_{j-1}$, $j = 1, 2, 3, 4$ with $\xi_0 = 0$. Computing

$$600 \quad \xi_1 = \widehat{Z}_\theta = G_\theta \begin{pmatrix} \mathbf{J}_2 \widehat{\mathbf{a}} \\ -\omega \widehat{\mathbf{a}} \\ -k \widehat{\mathbf{a}} \end{pmatrix},$$

601 and, with $\gamma = (k + \omega c_g) \|\widehat{\mathbf{a}}\|^{-2}$,

$$602 \quad \xi_2 = G_\theta \begin{pmatrix} \gamma \widehat{\mathbf{a}} \\ -(c_g - \omega \gamma) \mathbf{J}_2 \widehat{\mathbf{a}} \\ (1 + k \gamma) \mathbf{J}_2 \widehat{\mathbf{a}} \end{pmatrix} + \mathbb{R} \xi_1, \quad \xi_3 = G_\theta \begin{pmatrix} 0 \\ -\gamma c_g \widehat{\mathbf{a}} \\ \gamma \widehat{\mathbf{a}} \end{pmatrix} + \mathbb{R} \xi_1,$$

603 where $\mathbb{R}\xi_1$ represents the arbitrary amount of homogeneous solution. The first three
 604 terms in the Jordan chain will be sufficient for computing \mathcal{K} since

605
$$\mathcal{K} := \langle\langle \mathbf{K}\xi_1, \xi_4 \rangle\rangle = \langle\langle \mathbf{L}\xi_2, \xi_4 \rangle\rangle = \langle\langle \xi_2, \mathbf{L}\xi_4 \rangle\rangle = \langle\langle \xi_2, \mathbf{K}\xi_3 \rangle\rangle.$$

606 Hence

607
$$\mathcal{K} = \langle\langle \xi_2, \mathbf{K}\xi_3 \rangle\rangle = \gamma^2(1 - c_g^2)\|\widehat{\mathbf{a}}\|^2.$$

608 Now using the restrictions

609
$$c_g = -\frac{\omega}{k}, \quad \|\widehat{\mathbf{a}}\|^2 = \frac{2}{3} \quad \text{and} \quad \gamma = -\frac{1}{2k} \quad \text{when} \quad (\omega, k) \in \Sigma^1,$$

610 it follows that

611
$$\mathcal{K} = -\frac{1}{18k^4}.$$

612 **7.4. CKG to Boussinesq reduction.** The Boussinesq model for $(\omega, k) \in \Sigma^1$
 613 is therefore

614 (68)
$$2k^2 q_{TT} + \frac{2}{3k^3}(qq_X)_X - \frac{1}{18k^4}q_{XXX} = 0.$$

615 The importance of the assumption $k \neq 0$ in U (64) is evident here. The resulting
 616 Boussinesq equation is the linearly ill-posed version since $\mathcal{A}_\omega \mathcal{K} < 0$. Unfolding and
 617 scaling leads to the following canonical form

618
$$u_{\tau\tau} + s_1 u_{\xi\xi} + \left(\frac{1}{2}u^2\right)_{\xi\xi} - u_{\xi\xi\xi\xi} = 0, \quad s_1 = \pm 1,$$

619 where $s_1 = -1$ ($s_1 = +1$) on the stable (unstable) side of the Σ^1 curve (66).

620 To summarize, the CKG equation (60) has a family of exact periodic travelling
 621 waves. Modulation of these travelling waves in the neighbourhood of the Σ^1 curves
 622 (66) leads to a reduction to the two-way Boussinesq equation (68). The reduced
 623 equation contains a range of bounded periodic, quasiperiodic and localized solutions,
 624 but it also portends more dramatic behaviour in the original CKG equation in that
 625 it is linearly ill-posed and so general initial data may be dramatically unstable.

626 **8. Coalescing characteristics and multiphase wavetrains.** The theory in
 627 this paper is for basic states with one phase. However there are many examples in
 628 the literature where at least two phases are present. Examples are modulation of the
 629 cnoidal wave solutions of the KdV equation (§16.14 of [38]), modulation of Stokes
 630 waves in finite depth coupled to mean flow (§16.6-16.11 in [38]), and modulation of
 631 viscous fluid conduit periodic waves (MAIDEN & HOEFER [26]). In the latter two
 632 examples there is an elliptic-hyperbolic transition. However the theory of this paper
 633 does not apply directly and needs to be generalized to multiphase wavetrains. A
 634 theory for bifurcation of multiphase wavetrains near a zero characteristic has recently
 635 been developed by RATLIFF & BRIDGES [32]. Hence there is some optimism that
 636 the theory of this paper can be generalized to the elliptic-hyperbolic transition in
 637 multiphase wavetrains, but is outside the scope of this paper.

638 **9. Concluding remarks.** The modulation equations derived here

639 (69)
$$q_T = \Omega_X \quad \text{and} \quad \mathcal{A}_\omega \Omega_T + \kappa qq_X + \mathcal{K} q_{XXX} = 0,$$

640 are *asymptotically* valid in that the modulation ansatz (42) satisfies the governing
 641 equation (22) exactly with an error of order ε^6 . However, this theory gives no indica-
 642 tion of convergence to all orders in ε .

643 *Rigorous* validity of the theory presented here is an open question, and outside the
 644 scope of this paper. Rigorous validity is generally done in three steps: show that the
 645 original equation has a well-defined existence theory, show that the reduced equation
 646 has a well-defined existence theory, and then show that the difference between the
 647 exact and approximate solution stays close for a time interval of order ε^{-p} , for some
 648 $p > 0$.

649 Even considering validity of the CKG reduction to Boussinesq as an example,
 650 rather than reduction from an abstract Lagrangian, there is still a difficulty with the
 651 fact that the reduced equation (69) may not be well posed in general, particularly in
 652 the case where $\mathcal{A}_\omega \mathcal{K} < 0$, which arises in the CKG example. Hence methodology
 653 based on Cauchy-Kowalevskaya in a space of functions which are complex analytic in a
 654 strip would be required. This approach was successfully used by Düll & Schneider [14]
 655 in their proof of the validity of *elliptic* Whitham modulation equations in a reduction
 656 from the nonlinear Schrödinger equation.

657

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