Resilient consensus in multi-agent systems with state constraints ⋆

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Abstract

This paper investigates resilient consensus problems over directed networks with state constraints. Cooperative agents in the network can potentially be influenced by uncooperative neighbors, who are knowledgable, anonymous and able to spread misinformation. We formulate the resilient constrained consensus problem for high-dimensional multi-agent systems. A projection based resilient constrained consensus protocol is presented so that the agent's state will be pushed back to the constraint set when it approaches the boundary. We show that resilient constrained consensus can be reached for robust networks when the constraint sets are convex and share a non-empty overlap. The proposed algorithm is of low complexity, purely distributed, and can be performed in tandem with a max-consensus process to estimate the allowed number of uncooperative neighbors.

Key words: resilient consensus; state constraint; network; distributed algorithm; multi-agent system.

1 Introduction

Complex networked multi-agent systems abound in the world of nature and technology, where non-local information of each agent is usually not assumed. The problem of how a group of autonomous agents can achieve an agreement or consensus under distributed interactions is extensively investigated (Ge, Yang, & Han, 2017; Olfati-Saber, Fax, & Murray, 2007). The problem becomes the resilient consensus problem when some agents are infected due to malicious attacks or system level faults. Instead of losing just those compromised agents, the performance of the whole network could be corrupted. Recently, the graph concept of r-robustness is introduced in LeBlanc *et al.* (2013); Zhang, Fata, & Sundaram (2015) to facilitate the resilient consensus on networks. Using the Mean-Subsequence-Reduced (MSR) algorithm, agents of r-robust networks are enabled to reach consensus even when there are r malicious agents in the neighborhood of each cooperative agent (LeBlanc *et al.*, 2013). The results are generalized to double-integrator dynamics with time delays in Dibaji & Ishii (2017). Resilient consensus frameworks unifying both continuous- and discrete-time agents have been developed for switched (Shang, 2018) and hybrid (Shang, 2020) multi-agent systems. A modified MSR algorithm is introduced in Fiore & Russo (2019) to handle both fault tolerance and differential privacy requirements on the initial conditions. Resilient control strategies have been effectively applied in asynchronous systems (Senejohnny *et al.*, 2019) and distributed optimization (Zhao, He, & Wang, 2019). We refer the readers to Yang *et al.* (2020) for an updated overview on the trends and methodologies of resilient distributed coordination.

All the aforementioned works concern about unconstrained consensus meaning that the states of agents are not constrained. In many real world applications, agents need to reach consensus and at the same time their state trajectories and equilibrium have to be in some constrained sets due to physical limitation (Lin & Ren, 2014; Zhou & Wang, 2018). Such restrictions negatively impact the system's ability to reach consensus. In distributed target formation of unmanned aerial vehicles, for example, consensus may fail to form if the speed of some vehicles is constrained. State constrained consensus problem is introduced in Nedic, Ozdaglar, & Parrilo (2010) using nonlinear projection over balanced networks with doubly stochastic adjacency structures. A discarded consensus algorithm is proposed in Liu & Chen (2012) to achieve constrained consensus for strongly connected communication networks over both discrete-time and continuous-time dynamics. An adaptive neural network based consensus protocol is designed in Meng *et al.* (2017) to achieve constrained consensus in fixed undirected networks. Different state

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constrained protocols have also been studied in the context of, e.g., optimal consensus (Qiu, Liu, & Xie, 2016) and robust consensus against state uncertainties (Nguyen, Narikiyo, & Kawanishi, 2018).

In this paper, we study resilient consensus in continuous-time multi-agent systems with state constraints. The malicious agents present in the network are assumed to be able to change their states arbitrarily and unconstrainedly, know complete information of the entire network, and keep their identity anonymous. To our knowledge, this is the first work considering state constrained consensus problems against malicious agents. The contributions are as follows. First, we present a novel projection based resilient consensus protocol for agents with high-dimensional continuous-time dynamics, which is purely distributed (namely, each agent only knows its own and neighbors' states and its own constraint set) and can withstand locally bounded malicious agents. In the existing literature of resilient consensus, the state of an agent is typically assumed to be scalar; see e.g. Dibaji & Ishii (2017); Fiore & Russo (2019); LeBlanc *et al.* (2013); Shang (2018, 2020); Zhang, Fata, & Sundaram (2015). With high-dimensional state space, our strategy will push the agents' states back to the constraint sets whilst guarantee the smoothness of their trajectories. Second, we present sufficient conditions for resilient consensus on directed robust networks. Due to the existence of malicious agents, the previous constrained consensus algorithms do not work and many of them even require restrictive conditions. For example in Liu & Chen (2012), all constraint sets are assumed to be identical for continuous-time dynamics, while we allow different constraint sets provided their intersection is nonempty. Moreover, unlike the work Meng *et al.* (2017); Nedic, Ozdaglar, & Parrilo (2010); Nguyen, Narikiyo, & Kawanishi (2018); Zhou & Wang (2018), no balanced topology condition is assumed here.

The rest of the paper is organized as follows. Section 2 formulates the problem. Section 3 describes the proposed distributed strategy. Section 4 is devoted to convergence analysis. Section 5 presents a simulation example and Section 6 concludes the paper.

2 Problem statement

2.1 Graph theory

The set of integers is denoted by $\mathbb{N}$ and the cardinality of a set is given by $|\cdot|$. For $n \in \mathbb{N}$, the n-dimensional real space is denoted by $\mathbb{R}^n$ and the Euclidean norm of a vector in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. The topology of a multi-agent system can be modeled by a directed graph $G = (V, E)$, in which the node set is $V = \{1, 2, \cdots, N\}$ representing the agents and the edge set is $E \subseteq V \times V$ describing the interaction among agents. To cope with compromised nodes, the node set V is partitioned in to two subsets $V = C \cup U$, where C consists of cooperative agents with $|C| = N_C$ and U consists of uncooperative agents with $|U| = N_U$. The uncooperative agents are defined in Definition 2 below. Clearly, $N = N_C + N_U$. A directed edge $(i, j) \in E$ means that agent i can convey information to agent j. The (in-degree) neighborhood of agent i is given by $\mathcal{N}_i = \{j \in V : (j, i) \in E\}$. A directed path of length l from $i_0 \in V$ to $i_l \in V$ is an order sequence of nodes $i_0, i_1, i_2, \cdots, i_l$, in which each consecutive pair forms an edge of G. The graph G contains a directed spanning tree with root $i \in V$ when i can be connected to any other nodes in G via a directed path starting from i. Let $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ be the adjacency matrix of G, in which $a_{ij} > 0$ when $(j, i) \in E$ and $a_{ij} = 0$ otherwise.

We say that a set $S \subseteq V$ is r-reachable if there exists a node $i \in S$ with $|\mathcal{N}_i \backslash S| \geq r$, where $r \in \mathbb{N}$ (LeBlanc *et al.*, 2013; Zhang, Fata, & Sundaram, 2015). On the basis of reachability, G is said to be r-robust if for any two mutually exclusive node sets S_1 and S_2 in G at least one of them is r-reachable. Robustness is a refined characterization of structural connectivity as shown in the following lemma.

Lemma 1.(LeBlanc *et al.*, 2013) *Assume that a directed graph G is r-robust. H is obtained by deleting at most s incoming edges of each node in G. If $s < r$, then H is $(r - s)$-robust. Furthermore, 1-robustness is equivalent to containing a directed spanning tree.*

2.2 Problem formulation

Consider a multi-agent system characterized by the directed graph $G = (V, E)$ with $V = C \cup U$. The system state of each agent $i \in V$ at time $t \geq 0$ is given by $x_i(t) = (x_{i1}(t), \cdots, x_{in}(t))^{\mathsf{T}} \in \mathbb{R}^n$, where $n \in \mathbb{N}$ is the dimension of the state vector. The following assumption on the state constraints for cooperative agents is made.

Assumption 1. For each $i \in C$, the constraint set $\Omega_i = \{x_i \in \mathbb{R}^n : g_i(x_i) \leq 1\}$ and $\Omega = \cap_{i=1}^{N_C} \Omega_i \neq \emptyset$, where $g_i : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable convex function.

Remark 1. The set $\Omega_i \subseteq \mathbb{R}^n$ is called level set of the convex function g_i (Bertsekas, 2009). The above assumption is tantamount to requiring the convexity of the constraint set Ω_i, which is commonly assumed in state constraint consensus problems; see e.g. Lin & Ren (2014); Liu & Chen (2012); Meng *et al.* (2017); Nedic, Ozdaglar, & Parrilo (2010); Zhou & Wang (2018). For example, if $n = 1$, then Ω_i is a closed interval.

We define the following resilient state constrained consensus, which requires all cooperative agents achieve consensus with their trajectories within constraints.

Definition 1 (Resilient State Constrained Consensus). The cooperative agents in G are said to reach resilient state constrained consensus if for any initial conditions $\{x_i(0)\}_{i\in V}$ such that $x_i(0) \in \Omega_i$ for all $i \in C$, we have (i) $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j \in C$ and (ii) $x_i(t) \in \Omega_i$ for all $i \in C$ and $t \geq 0$.

By Definition 1, we readily reproduce the ordinary consensus definition when $V = C$ and $\Omega_i = \mathbb{R}^n$ (e.g., $g_i(x) \equiv 1$) for all $i \in C$. In general, the dynamics of cooperative agent $i \in C$ are described as

$$\dot{x}_i(t) = u_i(t) := \varphi_i\left(\{x^i_j(t) : j \in \mathcal{N}_i \cup \{i\}\}\right), \quad (1)$$

where $x^i_j(t) = (x^i_{j1}(t), \cdots, x^i_{jn}(t))^\top \in \mathbb{R}^n$ delineates the values conveyed from agent j to agent i at time t, and we assume cooperative agents send their own states, namely, $x^i_j(t) = x_j(t)$ for all $j \in C$. The control input u_i or the function φ_i in (1) is the rule to be followed by cooperative agents aiming to reach agreement on their states. We will design u_i in Section 3 for cooperative agents. The uncooperative agents on the other hand are able to apply different strategies trying to sabotage the system performance.

Definition 2 (Uncooperative Agents). Any agent $i \in U$ is said to be uncooperative. It uses a different update rule $\tilde{\varphi}_i$ from the cooperative agents in (1), or at some time $t > 0$ it conveys different values to different neighbors.

The uncooperative agents are often difficult to cope with as they may collude with other uncooperative agents and spread wrong/malicious information to its neighbors via broadcasting or peer-to-peer communication. They are often referred to as Byzantine nodes (LeBlanc *et al.*, 2013; Yang *et al.*, 2020; Zhang, Fata, & Sundaram, 2015), for instance, in wireless communications and sensor networks. We allow the uncooperative agents to change their states arbitrarily and unconstrainedly, know complete information of the entire network, and keep their identity anonymous. To deal with such adversaries, we need an upper bound of the number of uncooperative agents in $\mathcal{N}_i$ for every $i \in C$. We denote this parameter by $r \in \mathbb{N}$, which satisfies $|\mathcal{N}_i \cap U| \leq r$ for all $i \in C$ bounding the adversaries in the neighborhoods of cooperative agents.

3 Projection based resilient consensus algorithm

To realize resilient state constrained consensus (c.f. Definition 1), we propose the following projection based resilient consensus strategy for cooperative agents on the basis of MSR-like algorithms, which typically deal with scalar state and unconstrained convergence (Dibaji & Ishii, 2017; Fiore & Russo, 2019; LeBlanc *et al.*, 2013; Shang, 2018, 2020; Zhang, Fata, & Sundaram, 2015).

Given $r \in \mathbb{N}$, the common upper bound of the number of uncooperative neighbors of the cooperative agents, each agent $i \in C$ receives its neighbors' state vectors $\{x^i_j(t)\}_{j\in\mathcal{N}_i}$ at time t and arranges them in the descending order for each coordinate independently. Namely, for each $1 \leq k \leq n$ the real sequence $\{x^i_{jk}(t)\}_{j\in\mathcal{N}_i}$ is sorted in the descending order. For each $1 \leq k \leq n$, the indices of the highest r values that are greater than $x_{ik}(t)$ and the indices of the lowest r values that are smaller than $x_{ik}(t)$ are put in a set $\mathcal{R}_{ik}(t)$. If there are less than r such values respectively, all of these indices are put in the set $\mathcal{R}_{ik}(t)$. See Fig. 1 for a data flow model for $i \in C$.

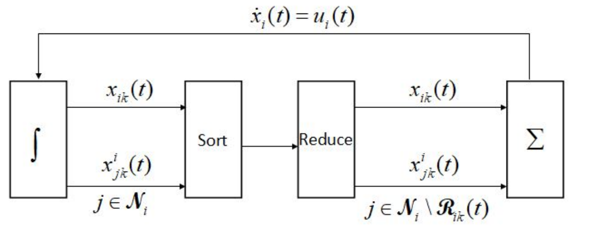

Fig. 1. Data flow model of MSR algorithm for agent i.

Next, we define

$$y_i(t) = \sum_{k=1}^{n} 1_{kk} \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_{ik}(t)} a_{ij}\varphi_{ij}(x^i_j(t), x_i(t)), \quad (2)$$

where $y_i(t) = (y_{i1}(t), \cdots, y_{in}(t))^\top \in \mathbb{R}^n$, $1_{kk} = 1_k 1_k^\top$, $1_k \in \mathbb{R}^n$ is the k-th unit vector, and $\varphi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (i) φ_{ij} is locally Lipschitz continuous, (ii) $\varphi_{ij}(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, and (iii) $(x_{1k} - x_{2k})1_k^\top\varphi_{ij}(x_1, x_2) > 0$ for all $1 \leq k \leq n$, where $x_1 = (x_{11}, \cdots, x_{1n})^\top$ and $x_2 = (x_{21}, \cdots, x_{2n})^\top$. Finally, we adopt the dynamics (1) with the control input $u_i(t) = (u_{i1}(t), \cdots, u_{in}(t))^\top$ designed as follows:

- If $f_i(x_i(t)) \leq 0$, then $u_i(t) = y_i(t)$,
- If $f_i(x_i(t)) > 0$ and $\frac{\partial f_i(x_i(t))}{\partial x_{ik}} y_{ik}(t) \leq 0$, then $u_{ik}(t) = 1_k^\top y_i = y_{ik}(t)$,
- If $f_i(x_i(t)) > 0$ and $\frac{\partial f_i(x_i(t))}{\partial x_{ik}} y_{ik}(t) > 0$, then

$$u_{ik}(t) = 1_k^\top \cdot [I_n - f_i(x_i(t))H_{ik}(x_i(t))] \cdot 1_{kk} \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_{ik}(t)} a_{ij}\varphi_{ij}(x^i_j(t), x_i(t)),$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, $f_i : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f_i(x_i) = \frac{g_i(x_i) - \delta_i}{1 - \delta_i}$$

with δ_i being freely chosen by each agent i such that $\inf_{x_i\in\Omega_i} g_i(x_i) < \delta_i < 1$, and $H_{ik} : \mathbb{D}_{ik} \to \mathbb{R}^{n\times n}$ is given by

$$H_{ik}(x_i) = \frac{\nabla f_i(x_i) \nabla f_i(x_i)^\mathsf{T}}{[\nabla f_i(x_i)^\mathsf{T} 1_k]^2}$$

with $\mathbb{D}_{ik} = \{x_i \in \mathbb{R}^n : \nabla f_i(x_i)^\mathsf{T} 1_k \neq 0\}$ and $\nabla f_i(x_i) \in \mathbb{R}^n$ being the gradient of f_i.

Remark 2. The conditions on the function φ_{ij} in (2) are fairly mild. For example, we can take $\varphi_{ij}(x_1, x_2) = (x_1 - x_2)$. This is in line with classical consensus protocols (Olfati-Saber, Fax, & Murray, 2007). Our strategy is purely distributed in the sense that each cooperative agent only knows the state of its neighbors, its own state, and its own constraint set. Moreover, in contrast to discrete-time MSR algorithms, the sorting and reducing mechanism here can be implemented by a Lipschitz continuous filter ϕ_k, which is a composition of concatenation, sorting, reducing, and sum functions (LeBlanc & Koutsoukos, 2011, Def. 1). In other words, the equation (2) can be formally written as

$$y_i(t) = \sum_{k=1}^{n} 1_{kk} \phi_k \bigg(\sum_{j \in \mathcal{N}_i \cup \{i\}} a_{ij} \varphi_{ij}(x_j^i(t), x_i(t)) \bigg).$$

Remark 3. If all constraints sets $\Omega_i \equiv \mathbb{R}^n$, namely, no state constraint is imposed, we obtain that both g_i and f_i are constants (e.g. $g_i = f_i \equiv 1$) for all $i \in C$. Hence, $\frac{\partial f_i(x_i)}{\partial x_{ik}} = 0$ and the control input $u_i(t) = y_i(t)$ for each $i \in C$. This can be viewed as a high-dimensional MSR algorithm (extending the previous scalar versions e.g. Dibaji & Ishii (2017); Fiore & Russo (2019); LeBlanc *et al.* (2013); Shang (2018, 2020); Zhang, Fata, & Sundaram (2015)), in which the state filtering procedure is performed for each coordinate independently. In particular, if additionally $n = 1$, we readily reproduce an ordinary scalar MSR algorithm.

Remark 4. By Assumption 1, we have $f_i(x_i) \leq 1$ for any $i \in C$. Each constraint set Ω_i can be represented as $\Omega_i = \overline{\Omega}_i \cup \underline{\Omega}_i$, where $\overline{\Omega}_i = \{x_i \in \mathbb{R}^n : 0 < f_i(x_i) \leq 1\}$ and $\underline{\Omega}_i = \{x_i \in \mathbb{R}^n : f_i(x_i) \leq 0\}$ (see Fig. 2). When the state $x_i \in \underline{\Omega}_i$ ('central area'), the control input is $u_i = y_i$. When $x_i \in \overline{\Omega}_i$ ('peripheral area'), the time derivative $\dot{f}_i = \nabla f_i(x_i) \dot{x}_i = \nabla f_i(x_i) y_i$ is checked. If the k-the coordinate is negative or zero, meaning that f_i is decreasing, we still use $u_{ik} = y_{ik}$. When this value is positive, i.e., f_i is increasing, that means x_{ik} has the tendency to move outside of Ω_i. In this situation, we modify the control input u_{ik} by using the projection. The choice of δ_i can change the relative size of $\overline{\Omega}_i$ and $\underline{\Omega}_i$: $\underline{\Omega}_i$ is larger when δ_i become closer to 1. This also tunes when the projection will be in use.

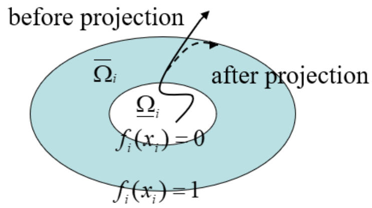

Fig. 2. A schematic illustration of state modification.

4 Consensus analysis

In this section, we present the resilient consensus with state constraints under our proposed distributed algorithm in Section 3. We first show that the state trajectories for cooperative agents remain invariably in their constraint sets.

Theorem 1. *Consider the multi-agent system (1) over G, in which each cooperative agent follows the projection based resilient consensus algorithm. If the initial condition $x_i(0) \in \Omega_i$ for all $i \in C$, then $x_i(t) \in \Omega_i$ holds for all $i \in C$ and $t \geq 0$.*

Proof. Fix $i \in C$. Since $x_i(0) \in \Omega_i$, we only need to consider the situation where x_i is in $\overline{\Omega}_i$ and has the tendency to move outside of Ω_i. Hence, for any $1 \leq k \leq n$, the k-th coordinate of the time derivative of f_i can be expressed as

$$\begin{aligned}
&\frac{\partial f_i(x_i(t))}{\partial x_{ik}} u_{ik}(t) \\
=&\frac{\partial f_i(x_i(t))}{\partial x_{ik}} \cdot 1_k^\mathsf{T} \cdot [I_n - f_i(x_i(t)) H_{ik}(x_i(t))] \\
&\cdot 1_{kk} \sum_{j \in (\mathcal{N}_i \cup \{i\}) \setminus \mathcal{R}_{ik}(t)} a_{ij} \varphi_{ij}(x_j^i(t), x_i(t)) \\
=&\frac{\partial f_i(x_i(t))}{\partial x_{ik}} 1_k^\mathsf{T} \sum_{j \in (\mathcal{N}_i \cup \{i\}) \setminus \mathcal{R}_{ik}(t)} a_{ij} \varphi_{ij}(x_j^i(t), x_i(t)) \\
&- f_i(x_i(t)) \frac{\partial f_i(x_i(t))}{\partial x_{ik}} \\
&\cdot 1_k^\mathsf{T} \sum_{j \in (\mathcal{N}_i \cup \{i\}) \setminus \mathcal{R}_{ik}(t)} a_{ij} \varphi_{ij}(x_j^i(t), x_i(t)) \\
=&(1 - f_i(x_i(t))) \frac{\partial f_i(x_i(t))}{\partial x_{ik}} \\
&\cdot 1_k^\mathsf{T} \sum_{j \in (\mathcal{N}_i \cup \{i\}) \setminus \mathcal{R}_{ik}(t)} a_{ij} \varphi_{ij}(x_j^i(t), x_i(t)) \\
:=&(1 - f_i(x_i(t))) \cdot \rho_{ik}(t). \qquad (3)
\end{aligned}$$

Note that $\rho_{ik}(t) = \frac{\partial f_i(x_i(t))}{\partial x_{ik}} y_{ik}(t) > 0$ by using (2) when x_i is in $\overline{\Omega}_i$ but has the tendency to move outside. When $x_i \in \overline{\Omega}_i$, we have $f_i(x_i) \in [0, 1]$. Thanks to (3), we see that as time t passes by, $f_i(x_i)$ is increasing (meaning $x_i(t)$ moves towards the boundary of Ω_i) but when it approaches the boundary (where $f_i(x_i) = 1$) it attains a plateau. In other words, as illustrated in Fig. 2, $x_i(t)$ will eventually stay on the boundary of the constraint set Ω_i as t grows. □

The next result shows that the states of cooperative agents will always be within the range of their initial configurations. More precisely, for $1 \le k \le n$, we define $\theta_{Mk}(t) = \max_{i\in C} x_{ik}(t)$ and $\theta_{mk}(t) = \min_{i\in C} x_{ik}(t)$ be the largest and smallest state values for cooperative agents in G, respectively.

Theorem 2. *Consider the multi-agent system (1) over G, in which each cooperative agent follows the projection based resilient consensus algorithm. For each $i \in C$, $x_{ik}(t) \in [\theta_{mk}(0), \theta_{Mk}(0)]$ for all $1 \le k \le n$ and $t \ge 0$.*

Proof. Fix $1 \le k \le n$. We only show $x_{ik}(t) \le \theta_{Mk}(0)$ and the proof for $x_{ik}(t) \ge \theta_{mk}(0)$ is likewise.

If the inequality is not true, there must be some time $\hat{t} > 0$ such that there is some agent $i_0 \in C$ with $x_{i_0k}(\hat{t}) = \theta_{Mk}(0)$ and $\dot{x}_{i_0k}(\hat{t}) > 0$ but for any $t \le \hat{t}$ and $i \in C$, $x_{ik}(t) \le \theta_{Mk}(0)$. On the basis of our resilient consensus algorithm, we consider three situations.

(1) When $f_{i_0}(x_{i_0}(\hat{t})) \le 0$, then $u_{i_0k}(\hat{t}) = y_{i_0k}(\hat{t})$. By (2), we have

$$\begin{aligned} 0 <&\dot{x}_{i_0k}(\hat{t}) \\ =& \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})} a_{i_0j} 1_k^\top \varphi_{i_0j}\left(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t})\right). \end{aligned} \quad (4)$$

Since there are no more than r uncooperative neighbors in $\mathcal{N}_{i_0}$, the filtering procedure guarantees $x_{jk}^{i_0}(\hat{t}) \le \theta_{Mk}(0) = x_{i_0k}(\hat{t})$ for any $j \in (\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})$. Note that j may be an uncooperative agent. Using our condition on φ_{i_0j}, we have $1_k^\top\varphi_{i_0j}(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t})) \le 0$. Since $a_{i_0j} > 0$, the right-hand side of (4) must be non-positive. We arrive at a contradiction.

(2) When $f_{i_0}(x_{i_0}(\hat{t})) > 0$ and $\frac{\partial f_{i_0}(x_{i_0}(\hat{t}))}{\partial x_{i_0k}} y_{i_0k}(\hat{t}) \le 0$, then $u_{i_0k}(\hat{t}) = y_{i_0k}(\hat{t})$. This case can be proved exactly in the same way as above.

(3) When $f_{i_0}(x_{i_0}(\hat{t})) > 0$ and $\frac{\partial f_{i_0}(x_{i_0}(\hat{t}))}{\partial x_{i_0k}} y_{i_0k}(\hat{t}) > 0$, then $u_{i_0k}(\hat{t}) = 1_k^\top \cdot [I_n - f_{i_0}(x_{i_0}(\hat{t}))\ H_{i_0k}(x_{i_0}(\hat{t}))] \cdot 1_{kk} \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})} a_{i_0j}\varphi_{i_0j}(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t}))$. We have

$$\begin{aligned} 0 <&\dot{x}_{i_0k}(\hat{t}) \\ =&1_k^\top \cdot [I_n - f_{i_0}(x_{i_0}(\hat{t}))H_{i_0k}(x_{i_0}(\hat{t}))] \cdot 1_k \\ &\cdot 1_k^\top \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})} a_{i_0j}\varphi_{i_0j}(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t})) \\ =&\left(1 - f_{i_0}(x_{i_0}(\hat{t}))\right) \\ &\cdot \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})} a_{i_0j} 1_k^\top\varphi_{i_0j}(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t})). \end{aligned} \quad (5)$$

As shown in case (1) we derive $\sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0k}(\hat{t})} a_{i_0j}$ $1_k^\top\varphi_{i_0j}(x_j^{i_0}(\hat{t}), x_{i_0}(\hat{t})) \le 0$. Moreover, $1 - f_{i_0}(x_{i_0}(\hat{t})) \ge 0$. Therefore, the right-hand side of (5) must be non-positive. We arrive at a contradiction. □

It is worth noting that the underlying communication topology is essentially time-varying in our resilient consensus protocol since the set $\mathcal{R}_{ik}(t)$ depends on time. It is therefore a good idea (conventional in distributed coordination (Olfati-Saber, Fax, & Murray, 2007)) to bound the dwell time in order to keep the switching rate in check. We assume the following.

Assumption 2. Denote by $\{b_\ell\}_{\ell\in\mathbb{N}}$ a sequence of time steps at which the set $\mathcal{R}_{ik}(t)$ changes for some $i \in C$ and $1 \le k \le n$. We assume there exists a positive number b satisfying $|b_{\ell+1} - b_\ell| \ge b > 0$ for all $\ell \in \mathbb{N}$.

Remark 5. Note that continuity of the state trajectories is neither sufficient nor necessary for Assumption 2. What required, say in the case of $r = n = 1$ and considering only the higher-end of our MSR algorithm for example, is whenever $\overline{j} = \arg\max_{j\in\mathcal{N}_i\cup\{i\}} x_{j1}^i(t_0)$ for some $i \in C$ and $t_0 > 0$, we have $x_{\overline{j}1}^i(t) = \max_{j\in\mathcal{N}_i\cup\{i\}} x_{j1}^i(t)$ for all $t \in [t_0, t_0 + b]$. This condition generally holds provided the state trajectories (of both cooperative and uncooperative agents) do not change arbitrarily frequently. For instance, if $\tilde{\varphi}_i$ in Definition 2 is continuous bounded, then by (1) and the mean value theorem $x_i(t)$ is Lipschitz continuous for $i \in U$, while (2) guarantees Lipschitz continuity for $i \in C$. Therefore, Assumption 2 holds.

Theorem 3. *Consider the multi-agent system (1) over G, in which each cooperative agent follows the projection based resilient consensus algorithm. If G is $(2r+1)$-robust, then resilient state constrained consensus is achieved.*

Proof. In view of Definition 1 and Theorem 2, what remains to show is the convergence of state. Fix $1 \le k \le n$. For $t \ge 0$, we define $\Theta_k(t) := \theta_{Mk}(t) - \theta_{mk}(t) \ge 0$. Recall the Dini derivative of a function $\phi(t)$ is $D\phi(t) = \limsup_{h\to 0+}(\phi(t+h) - \phi(t))/h$. We define two indices i_M and i_m satisfying, respectively, $\dot{x}_{i_Mk}(t) = \max_{i\in I_{Mk}(t)} \dot{x}_{ik}(t)$ with $I_{Mk}(t) := \{i \in C :$

$x_{ik}(t) = \theta_{Mk}(t)\}$ and $\dot{x}_{i_m k}(t) = \max_{i \in I_{mk}(t)} \dot{x}_{ik}(t)$ with $I_{mk}(t) := \{i \in C : x_{ik}(t) = \theta_{mk}(t)\}$. Thanks to the property of Dini derivative (Danskin, 1966), we have $D\theta_{Mk}(t) = \dot{x}_{i_M k}(t)$ and $D\theta_{mk}(t) = \dot{x}_{i_m k}(t)$.

We will show that $D\Theta_k(t) \le 0$. Depending on the choice of two agents i_M and i_m, we have $3 \times 3 = 9$ combinations in total under our resilient consensus algorithm (c.f. the proof of Theorem 2 where three cases are considered regarding i_0). In the following we only consider two typical cases. All the rest cases follow with essentially the same arguments.

(1) $f_{i_M}(x_{i_M}(t)) \le 0$ and $f_{i_m}(x_{i_m}(t)) \le 0$. In this case, we have $u_{i_M k}(t) = y_{i_M k}(t)$ and $u_{i_m k}(t) = y_{i_m k}(t)$. By (2) and the above analysis, we obtain

$$D\theta_{Mk}(t) = 1_k^\top \sum_{j \in (\mathcal{N}_{i_M} \cup \{i_M\}) \setminus \mathcal{R}_{i_M k}(t)} a_{i_M j} \varphi_{i_M j}(x_j^{i_M}(t), x_{i_M}(t)) \quad (6)$$

and

$$D\theta_{mk}(t) = 1_k^\top \sum_{j \in (\mathcal{N}_{i_m} \cup \{i_m\}) \setminus \mathcal{R}_{i_m k}(t)} a_{i_m j} \varphi_{i_m j}(x_j^{i_m}(t), x_{i_m}(t)). \quad (7)$$

By the choice of i_M and the fact that there are no more than r uncooperative agents in $\mathcal{N}_{i_M}$, we have $x_{i_M k}(t) \ge x_{jk}^{i_M}(t)$ for $j \in (\mathcal{N}_{i_M} \cup \{i_M\}) \setminus \mathcal{R}_{i_M k}(t)$ under our filtering procedure. Note that j may be uncooperative. Invoking the conditions on the function $\varphi_{i_M j}$, we see that $1_k^\top \varphi_{i_M j}(x_j^{i_M}(t), x_{i_M}(t)) \le 0$ in (6), leading to $D\theta_{Mk}(t) \le 0$. Likewise, using (7) we have $D\theta_{mk}(t) \ge 0$ and hence $D\Theta_k(t) = D\theta_{Mk}(t) - D\theta_{mk}(t) \le 0$.

(2) $f_{i_M}(x_{i_M}(t)) > 0$ with $\frac{\partial f_{i_M}(x_{i_M}(t))}{\partial x_{i_M k}} y_{i_M k}(t) > 0$ and $f_{i_m}(x_{i_m}(t)) > 0$ with $\frac{\partial f_{i_m}(x_{i_m}(t))}{\partial x_{i_m k}} y_{i_m k}(t) > 0$. In this case, we obtain $u_{i_M k}(t) = 1_k^\top [I_n - f_{i_M}(x_{i_M}(t)) H_{i_M k}(x_{i_M}(t))] 1_{kk} \sum_{j \in (\mathcal{N}_{i_M} \cup \{i_M\}) \setminus \mathcal{R}_{i_M k}(t)} a_{i_M j} \varphi_{i_M j} (x_j^{i_M}(t), x_{i_M}(t))$ and $u_{i_m k}(t) = 1_k^\top [I_n - f_{i_m}(x_{i_m}(t)) H_{i_m k}(x_{i_m}(t))] 1_{kk} \sum_{j \in (\mathcal{N}_{i_m} \cup \{i_m\}) \setminus \mathcal{R}_{i_m k}(t)} a_{i_m j} \varphi_{i_m j} (x_j^{i_m}(t), x_{i_m}(t))$. Therefore,

$$\begin{aligned} &D\theta_{Mk}(t) \\ =& 1_k^\top \cdot [I_n - f_{i_M}(x_{i_M}(t)) H_{i_M k}(x_{i_M}(t))] \cdot 1_k \\ &\cdot 1_k^\top \sum_{j \in (\mathcal{N}_{i_M} \cup \{i_M\}) \setminus \mathcal{R}_{i_M k}(t)} a_{i_M j} \varphi_{i_M j}(x_j^{i_M}(t), x_{i_M}(t)) \\ =& (1 - f_{i_M}(x_{i_M}(t))) \cdot \sum_{j \in (\mathcal{N}_{i_M} \cup \{i_M\}) \setminus \mathcal{R}_{i_M k}(t)} a_{i_M j} 1_k^\top \\ &\cdot \varphi_{i_M j}(x_j^{i_M}(t), x_{i_M}(t)) \end{aligned} \quad (8)$$

and in an analogous manner

$$D\theta_{mk}(t) = (1 - f_{i_m}(x_{i_m}(t))) \cdot \sum_{j \in (\mathcal{N}_{i_m} \cup \{i_m\}) \setminus \mathcal{R}_{i_m k}(t)} a_{i_m j} 1_k^\top \varphi_{i_m j}(x_j^{i_m}(t), x_{i_m}(t)). \quad (9)$$

As in case (1), we similarly obtain $1_k^\top \varphi_{i_M j}(x_j^{i_M}(t), x_{i_M}(t)) \le 0$ in (8) and $1_k^\top \varphi_{i_m j}(x_j^{i_m}(t), x_{i_m}(t)) \ge 0$ in (9). Combining this with non-negativity of adjacency element in A, $1 - f_{i_M}(x_{i_M}(t)) \ge 0$, and $1 - f_{i_m}(x_{i_m}(t)) \ge 0$, we arrive at $D\Theta_k(t) = D\theta_{Mk}(t) - D\theta_{mk}(t) \le 0$.

Combining all cases as commented above, $D\Theta_k(t) \le 0$ holds for all $t \ge 0$.

Next, we will show that $\lim_{t \to \infty} D\Theta_k(t) = 0$ by contradiction. If this limit is not true, then there exist $\varepsilon, \sigma_0 > 0$ and a list of time steps $\{s_\ell\}_{\ell \in \mathbb{N}}$ with $\lim_{\ell \to \infty} s_\ell = \infty$ such that $D\Theta_k(s_\ell) \le -\varepsilon$ and $|s_{\ell+1} - s_\ell| > \sigma_0$ for all $\ell \in \mathbb{N}$. Consider a time interval J with $J \cap \{b_\ell\}_{\ell \in \mathbb{N}} = \emptyset$. We know that $D\Theta_k(t)$ is uniformly continuous in J because $D\Theta_k(t)$ is continuous in J and $\dot{x}_{ik}(t)$ is bounded for all $i \in C$. As a result, there exists $\sigma_1 > 0$ such that for any $t_1, t_2 \in J$ with $|t_1 - t_2| < \sigma_1$, $|D\Theta_k(t_1) - D\Theta_k(t_2)| < \varepsilon/2$ holds. Next, we select $0 < \sigma_2 < \sigma_1$ such that for any $\ell \in \mathbb{N}$, the small interval $[s_\ell - \sigma_2, s_\ell + \sigma_2]$ is a subset of an interval J delineated as above. For $t \in [s_\ell - \sigma_2, s_\ell + \sigma_2]$, a grain of algebra gives $D\Theta_k(t) = -|D\Theta_k(s_\ell) - (D\Theta_k(s_\ell) - D\Theta_k(t))| \le -\varepsilon + \varepsilon/2 = -\varepsilon/2$. We choose $0 < \sigma < \sigma_2$ such that the family $\{[s_\ell - \sigma, s_\ell + \sigma]\}_{\ell \in \mathbb{N}}$ contains mutually exclusive sets. Based on the these discussions, we examine the integral of $D\Theta_k(t)$ as

$$\begin{aligned} \int_0^\infty D\Theta_k(t) dt &\le \lim_{L \to \infty} \sum_{\ell=1}^{L} \int_{s_\ell - \sigma}^{s_\ell + \sigma} D\Theta_k(t) dt \\ &\le -\lim_{L \to \infty} \sum_{\ell=1}^{L} \int_{s_\ell - \sigma}^{s_\ell + \sigma} \frac{\varepsilon}{2} dt = -\infty. \end{aligned} \quad (10)$$

This clearly contradicts with $\Theta_k(t) \ge 0$ for all $t \ge 0$. Therefore, we conclude $\lim_{t \to \infty} D\Theta_k(t) = 0$.

An immediate result of the vanishing of $D\Theta_k(t)$ is that both $D\theta_{Mk}(t)$ and $D\theta_{mk}(t)$ are vanishing as t goes to infinity (since they are bounded by zero). Furthermore, there exist two constants $\rho_{Mk} \ge \rho_{mk}$ satisfying $\lim_{t \to \infty} \theta_{Mk}(t) = \lim_{t \to \infty} x_{i_M k}(t) := \rho_{Mk}$ and $\lim_{t \to \infty} \theta_{mk}(t) = \lim_{t \to \infty} x_{i_m k}(t) := \rho_{mk}$. According to our assumption, the graph G is $(2r+1)$-robust. By Lemma 1, the underlying communication topology (corresponding to the k-th coordinate of the system) contains a directed spanning tree under our resilient consensus protocol. Assume that $\rho_{Mk} > \rho_{mk}$. In this case, there is time τ_1 and $\varepsilon > 0$ such that $x_{i_M k}(t) > \rho_{Mk} - \varepsilon > \rho_{mk} + \varepsilon > x_{i_m k}(t)$ for all $t \ge \tau_1$. In view of our resilient consensus protocol and the conditions on φ_{ij}, the limit $\lim_{t\to\infty} \dot{x}_{i_M k}(t) = 0$ implies $\lim_{t\to\infty} x^{i_M}_{jk}(t) - x_{i_M k}(t) = 0$ for all $j \in (\mathcal{N}_{i_M} \cup \{i_M\})\backslash\mathcal{R}_{i_M k}(t)$. Likewise, $\lim_{t\to\infty} x^{i_m}_{jk}(t) - x_{i_m k}(t) = 0$ for all $j \in (\mathcal{N}_{i_m} \cup \{i_m\})\backslash\mathcal{R}_{i_m k}(t)$. Due to finiteness of the node set V, there exists $\tau_2 > \tau_1$ such that there are two directed paths in the communication topology (corresponding to the k-th coordinate) at time τ_2, one beginning from the root node, say l, ending at i_M and the other from l to i_m, and the following holds: $x_{lk}(\tau_2) > \rho_{Mk} - \varepsilon$ and $x_{lk}(\tau_2) < \rho_{mk} + \varepsilon$. This clearly cannot happen. Hence, it must be $\rho_{Mk} = \rho_{mk}$. Since the argument applies to all $1 \le k \le n$, the proof is complete. □

Remark 6. The $(2r + 1)$-robustness condition in Theorem 3 is typical in MSR algorithms for resilient consensus problems of systems with 1-dimensional and unconstrained state space both in discrete time (Dibaji & Ishii, 2017; Fiore & Russo, 2019; LeBlanc *et al.*, 2013) and in continuous time (LeBlanc & Koutsoukos, 2011; Shang, 2018, 2020).

Remark 7. In real-world applications, cooperative agents may agree on a value r in a distributed manner. For instance, if each cooperative agent i has a conservative estimate r_i, the max-consensus process (Nakamura, Ishii, & Dibaji, 2018) can be followed (see Section 5 for an example). It is also worth noting that resilient consensus is not guaranteed if cooperative agents apply different values of r. This is because a smaller value might leave more malicious agents in the neighborhood while a larger value could lead to an isolated node.

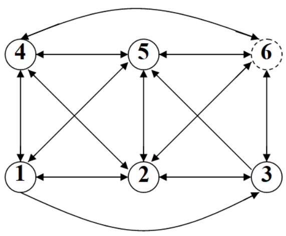

Fig. 3. A directed graph G with $N = 6$ agents.

5 Numerical simulations

Here we consider a multi-agent system with $N = 6$ agents, i.e., $V = \{1, 2, \cdots, 6\}$, having cooperative agents $C = \{1, \cdots, 5\}$ and an uncooperative agent $U = \{6\}$ over a directed graph G; see Fig. 3. The adjacency matrix is taken as a binary one and the agent state is of dimension $n = 2$. It can be verified directly that G is 3-robust. The initial states are set as $x_1(0) = (3.5, 4)^\top$, $x_2(0) = (-2, -2)^\top$, $x_3(0) = (0.5, -3)^\top$, $x_4(0) = (-1, 5.5)^\top$, $x_5(0) = (4, -2)^\top$, and $x_6(0) = (3, -4)^\top$.

For all agent $i \in C$, the state constraint sets are defined by the functions $g_i(x_i)$ as $g_1(x_1) = 2(x_{11} - 2)^2 + (x_{12} - 3)^2 - 6$, $g_2(x_2) = 3(x_{21} + 1)^2 + (x_{22} - 1)^2 - 2(x_{21} + 1)(x_{22} - 1) - 8$, $g_3(x_3) = 10(x_{31} - 1)^2 + (x_{32} + 1)^2 - 19$, and $g_4(x_4) = g_5(x_5) \equiv 1$. The sets are shown in Fig. 5 with $\Omega_4 = \Omega_5 = \mathbb{R}^2$, and their intersection $\Omega \neq \emptyset$. We take $\delta_1 = \delta_2 = -2$, $\delta_3 = \delta_4 = \delta_5 = -3$, and the function $\varphi_{ij}(v, w) = 0.5 \cdot (v - w)$ for all $i, j \in C$.

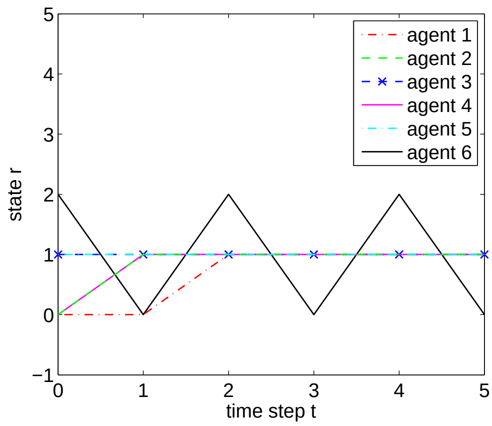

Fig. 4. Reaching consensus in r over the network G.

Prior to applying the projection based strategy, we need to agree on the parameter r in a distributed manner. To this end, we consider the max-consensus process following Nakamura, Ishii, & Dibaji (2018). Initially, we choose individual estimates $r_1(0) = r_2(0) = r_4(0) = 0$, $r_3(0) = r_5(0) = 1$, and the uncooperative agent takes value 2 at even time steps and 0 at odd time steps. It is shown in Fig. 4 that all cooperative agents are able to achieve an agreement at $r = 1$.

With cooperative agents following our projection based resilient consensus algorithm with $r = 1$ and the uncooperative agent 6 following its dynamics $\dot{x}_6(t) = \begin{pmatrix} -0.25 & 0 \\ 0 & -0.5 \end{pmatrix} x_6(t)$, the state trajectories are shown in Fig. 5. As one would expect from Theorem 3, resilient consensus has been reached for cooperative agents in C. The final consensus state, i.e., equilibrium, is located at the boundary of Ω_2. In practical applications, if the previously agreed r value does not lead to a consensus, we can gradually increase the estimate r_i. If the network G is sufficiently robust, the desired r will be ultimately reached and global consensus is guaranteed.

6 Conclusion

In this paper, we have investigated the resilient consensus problem for multi-agent systems over directed networks with state constraints. We formulated the resilient state constrained consensus in the presence of uncooperative agents with some extremely harmful features. On the basis of the projection based resilient consensus algorithm, we have shown that state constrained resilient consensus can be reached for robust networks when the

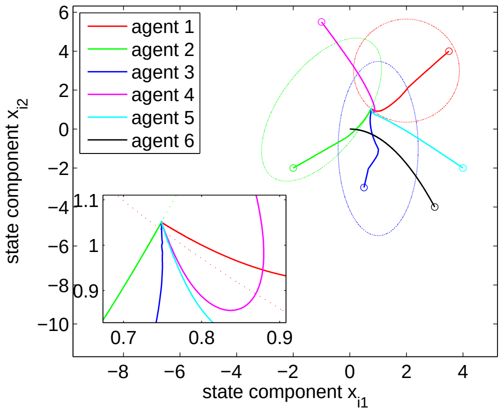

Fig. 5. Agents trajectories in the state space with initial states indicated by circles. The inset show a magnified view of the states around the equilibrium.

constraint sets are convex and have a non-empty overlap. The proposed consensus strategy is purely local and can be carried out in a distributed manner.

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