Consensus formation in networks with neighbor-dependent synergy and observer effect

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Abstract

In this paper, we study the consensus formation over a directed hypergraph, which is an important generalization of standard graph structure by allowing possible neighbor-dependent synergy. The proposed model is situated in the social dynamics providing key features including social observer effect and bounded confidence. Under the minimal siphon condition of a directed hypergraph (Petri net), we show that global consensus can be reached with the final consensus value residing in the common comfortable range if it is non-empty. To achieved this, we establish an equivalent condition for the commensurate graph of a finite state machine to be strongly connected. Convergence analysis is performed based on the proposed nonlinear dynamic system model and Petri net method. The consensus result holds for any non-negative confidence bound, which distinguishes from traditional bounded confidence opinion models as we measure the difference among neighbors rather than the gap between neighbors and the ego. Numerical studies are conducted to unravel some insights in relation to the influence of observers, hypergraph architecture, and confidence bounds on opinion evolution. The results and methodologies presented here facilitate research of social consensus and also offer a way to make sense of synergy in networked complex systems.

MSC 2010: 91D30; 34H05; 05C65

Keywords: social dynamics; hypergraph; neighbor-dependent synergy; observer effect; confidence bound.

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1 Introduction and background

Consensus algorithms have become an essential constituent in the design of distributed coordination in multi-agent networks, which have found fundamental applications in aerospace engineering [1], sensor networks [2], evolutionary biology [3], and sociology [4], among others. In a standard consensus problem, agents update their states to achieve a common goal through local interactions using neighboring information flow over a communication graph. A broad spectrum of control-theoretic results on consensus problems have been presented in the literature [5–8].

Social dynamics, and more specifically, opinion evolutions in social networks, have attracted a great research attention in recent years partly due to the pervasive growth of social media and social networking sites [9]. As in the control-engineering studies, decision-making processes in the setting of social dynamics also entail the state agreement objective and distributed communications. Nevertheless, theoretical study of social consensus-building tends to be more challenging as social networks are often comprised of a large number of heterogeneous agents and complex socio-psychological processes need to be factored in [10–12]. The seminal DeGroot model [13] proposed a state update rule, where an individual is influenced by its neighbors as a weighted average. Opinion dynamics models under bounded confidence [14] capture homophily between individuals as opinion update occurs only when two individuals have opinions sufficiently close according to a given confidence bound. The discrepancy between private and expressed opinions arising from conformity pressure has been examined in [15, 16]. The unique roles of psychological aversion/contrarian [18], affinity [19], first impression [20], and biased assimilation [21] have also been investigated. We refer the reader to [11, 22] for updated surveys of opinion formation models.

All the above mentioned works model agent interaction using a network structure, which implicitly assumes that communication among agents happens in pairs independently [12]. However, this assumption is only an approximation in many real applications. For example, co-authors of a paper all have to approve the final version before it can be published; directors sitting on the board in a corporation need to reach a majority to sanction or veto a decision; a metabolic reaction requires systematic synergy of participatory metabolites in their respective roles. In social networks, it is also common for an individual to adjust their opinion only if a group of their neighbors becomes unanimous. In all these situations, the edge structure in networks is a simplification, which could arguably engender a bias of the underlying real decision-making process because the interactions among agents are essentially neighbor-dependent.

A handy generalization of the standard network structure is called hypergraph, and in the directed case, Petri net (a directed hypergraph) [23, 24]. Hypergraph structures allow hyperedges involving more than two nodes and ideally accommodate complex synergy among multiple nodes. Although these mathematical tools have found numerous remarkable applications in the structural characterization in biology and computer science [25, 26] with Petri nets mostly appeared in discrete event systems [27, 28], much less has been done regarding their implications in collective decision-making dynamics. In fact, consensus formation over hypergraphs is a challenging issue yet to be solved and presents an important and appealing direction in the generalization of standard network-based distributed coordination. Some sufficient criteria for synchronization problems over a 3-uniform undirected hypergraph have been proposed in [31] by using a joint degree notion. As the hypergraph considered there is 3-uniform, some form of matrix algebra is still applicable. For a general directed hypergraph, it is shown in [32] that a topological condition called siphon overlapping is sufficient to guarantee global consensus. The high-level method utilized there, however, does not extend to many interesting scenarios in e.g., sociology, as specific features of the dynamics cannot be easily fed into the convergence analysis. It is worth noting that in the physics literature, synchronization dynamics of hypernetworks have been extensively studied [29, 30], where "hypernetwork" is an alias of multilayer network [12] disparate to the hypergraph considered here.

The decision-making dynamics in social networks relates to not only the communication architectures but the complicated socio-psychological processes. Many of them have been dealt with by using agent-based modeling and simulations in consensus formation problems as mentioned above [11, 22]. The observer effect, or sometimes referred to as the Hawthorne effect, is well-known and has a long history in anthropological research [33]. This phenomenon refers to the behavioral change of individual or group being aware of the presence of observers – their behavior tends to become more moderate or neutral than otherwise. This potential bias has been widely recognized in observational data and field studies in the likes of police research [34], ethnography [35], health care [36] and sociology [37], etc. The social observer effect is certainly reminiscent of its physical counterpart in quantum mechanics [42], which nonetheless is manifested in the microscopic world of particles.

Theoretical research in consensus formation related to observer effect can be largely found in the category of state constrained consensus problems; see e.g. [38–41]. States of the agents are usually forced to be contained in some geometrically convex sets (comfortable ranges) typically by means of projection operators and barrier functions. The regulation protocols have been mainly designed in the control theory community and do not take into consideration of substantive social interactions or neighbor dependent synergy. Several recent work [15–17] has scoped the difference between expressed and private opinions in social networks, but they again do not directly speak to the observer effect in social dynamics.

Here, we aim to bridge the gap between cooperative consensus control and opinion formation dynamics featuring neighboring synergy and social observer effect, by developing a consensus formation framework over directed hypergraphs. To model the neighbor-dependent interactions, we adopt the language of Petri nets and establish an equivalent condition for the graph-theoretic strongly connectedness associated with a finite state machine. The neighborhood of an agent can be divided into multiple subgroups, each of which has a collective influence on the agent governed by a bounded confidence mechanism. The group decision-making process designed is informed by the bounded confidence theory [14], the social trust theory [47], as well as the social influence theory [43], where agents are bound to conform with neighbors' opinion to some extent according to different trust levels but their strategies can be fluid and state-dependent. This is essentially different from weighted-average type control protocols, where the weights in protocols are often specified initially and independent of states [6]. Moreover, the opinion value of each agent has their individual upper and lower comfortable bounds, which define their comfortable range of expressed opinion under external observation [34, 35]. We show that the consensus can be achieved with the ultimate agreement value lying in the intersection of all comfortable ranges if the communication topology satisfies a hypergraph connectivity condition and the opinions within each subgroup are asymptotically unanimous. The proposed framework is purely distributed. Note that a spiritually similar concept of comfortable range has also been examined recently in [46] recently, but the focus there is resilient consensus against external attacks and the methods adopted are totally different.

Numerical simulations are performed to further explore the consensus behavior in relation to the observer effect, the hypergraph structure, as well as the confidence bounds. Some key impacts of comfortable bounds and disparity between traditional bounded confidence models are revealed. The rest of the paper is organized as follows. In Section 2, some preliminary concepts are given and the model is formulated. We present our main convergence result and numerical studies in Section 3 with the proofs deferred to Section 4. Finally, conclusions are provided in Section 5.

2 Preliminaries and problem statement

2.1 Petri net formulation for neighbor-dependent synergy

The topology relationship between agents in a social network is traditionally modeled by using a directed graph $G = (P, E)$ with $P = \{p_1, p_2 \cdots, p_n\}$ representing the agents and $E \subseteq P \times P$ the arcs characterizing the information flows between agents. For example,

$(p_1, p_2) \in E$ means p_1 can influence p_2. For $p_i \in P$, let $\tilde{N}_i = \{p \in P : (p, p_i) \in E\}$ be the neighborhood of p_i. Its cardinality $|\tilde{N}_i|$, i.e. the number of agents that can send information directly to p_i, is called the degree of p_i. For $t \geq 0$, let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathbb{R}^n$ capture the states of agents at time t. We will often suppress t when the time dependence is clear for the context. Such a graph-theoretical characterization has shown to be very powerful and successful in analyzing multi-agent coordination [5] as matrix algebra methods can be readily applied. However, it ignores a critical information regarding joint neighbor interaction.

We hereby borrow the language of Petri net theory, which was first introduced by Carl Petri [44] as a process modeling technique having formal semantics and analysis methods. It has found a wide range of applications in control engineering (e.g. in the field of discrete event systems), computer science and biological signaling networks [25–28]. We formally consider a Petri net to be a directed bipartite graph (P, T, F), encompassing two finite node sets, places (i.e., agents) P and transitions $T = \{t_1, t_2, \cdots, t_m\}$, and $F = F_1 \cup F_2$ is a set of arcs, where $F_1 \subseteq T \times P$ and $F_2 \subseteq P \times T$, respectively. In this paper, with some ambiguity we use t both for time and a transition by convention since the exact meaning will always be clear from the context. For any set $S_P \subset P$, we denote the input transition set of S_P by $\mathrm{InP}(S_P) = \{t \in T : (t, p) \in F_1 \text{ for some } p \in S_P\}$ and the output transition set of S_P by $\mathrm{OutP}(S_P) = \{t \in T : (p, t) \in F_2 \text{ for some } p \in S_P\}$. Similarly, for any set $S_T \subset T$, we denote the input place set of S_T by $\mathrm{InT}(S_T) = \{p \in P : (p, t) \in F_2 \text{ for some } t \in S_T\}$ and the output place set of S_T by $\mathrm{OutT}(S_T) = \{p \in P : (t, p) \in F_1 \text{ for some } t \in S_T\}$. Here, each place $p \in P$ is an agent in the network G, and each transition $t \in T$ represents a joint interaction between agents (the precise meaning will be clear below). We assume each transition $t \in T$ has exactly one out-going arc in F_1, namely, $|\,\mathrm{OutT}(\{t\})| = 1$ for any $t \in T$. See Fig. 1(a) for an example of Petri net.

Given an agent $p_i \in P$, we partition the indices of the agents in $\tilde{N}_i$ as mutually exclusive sets $\{k : p_k \in \tilde{N}_i\} = \cup_{j=1}^{j_i} J_{ij}$, where $1 \leq j_i \leq |\tilde{N}_i|$ is an integer. Denote by $N_i = \{J_{ij} : j = 1, 2, \cdots, j_i\}$ the collection of these subgroups. We call each J_{ij} a minimal influence on the agent p_i in the sense that the state of p_i can be influenced only by all agents (with indices) in J_{ij} but not any proper subset of it. In Fig. 1(b), for example, J_{11} is a minimum influence on p_1 because p_2 or p_3 by its own cannot influence p_1 by definition but some synergy (i.e., simultaneous change of states) in them may influence p_1. Concrete examples are given in the system description in Section 2.2. Clearly, the traditional graph presentation $G = (P, E)$ can be regarded as a special case with $j_i = |\tilde{N}_i|$ for all $i \in [n] := \{1, 2, \cdots, n\}$, i.e. all subgroups are simply single agent. The other extreme situation is $j_i = 1$ for all i, which means the agent p_i views all neighbors as a whole group and any proper subset of it will not be able to influence its state.

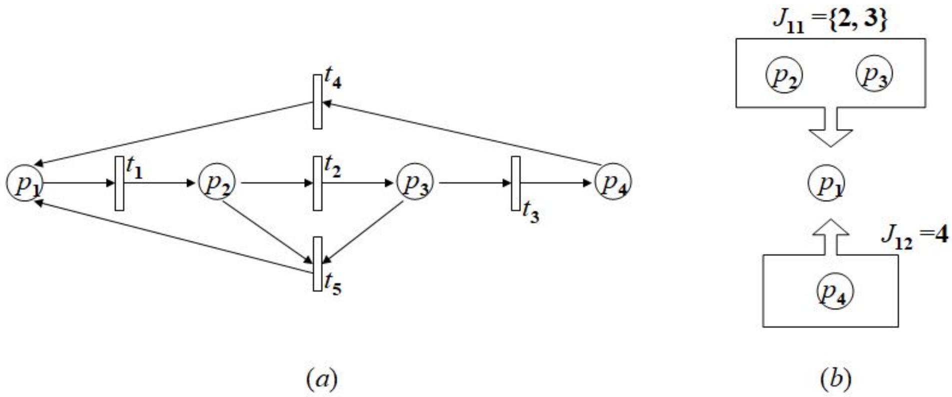

Figure 1: (a) A Petri net (P, T, F) with $P = \{p_1, p_2, p_3, p_4\}$, $T = \{t_1, \cdots, t_5\}$, $F_1 = \{(t_1, p_2), (t_2, p_3), (t_3, p_4), (t_4, p_1), (t_5, p_1)\}$, and $F_2 = \{(p_1, t_1), (p_2, t_2), (p_2, t_5), (p_3, t_3), (p_3, t_5), (p_4, t_4)\}$. (b) A schematic illustration of the minimal influences of J_{11} and J_{12} on p_1, where $\tilde{N}_1 = \{p_2, p_3, p_4\}$ and $N_1 = \{J_{11}, J_{12}\}$.

Remark 1. The above type of interactions between subgroups $\{J_{ij}\}_{j=1}^{j_i}$ and the agent p_i is referred to as "neighbor-dependent synergy" since in general each minimum influence J_{ij} can have more than one node. Only their consistent and simultaneous action could influence the behavior of p_i. As such, Petri net is a more appropriate tool than the classical graph theory.

A set $S_P \subseteq P$ is called a minimal siphon of the Petri net if $\mathrm{InP}(S_P) \subseteq \mathrm{OutP}(S_P)$ and no proper subset of S_P satisfies this condition [44,45]. Roughly, a minimal siphon is a minimal set of agents whose influence comes from themselves at least partly. For instance, P is a minimal siphon in the Petri net in Fig. 1(a). Therefore, it is also the only minimal siphon in this example. Intuitively, this Petri net is well-connected and we can make this concept precise. If for any $t \in T$ we have $|\,\mathrm{InT}(\{t\})| = |\,\mathrm{OutT}(\{t\})| = 1$, then the Petri net (P, T, F) is called a finite state machine. For a state machine, we associate the commensurate graph $G = (P, E)$ with it by linking p_{i_1} to p_{i_2} if $p_{i_1} \in \mathrm{InT}(\{t\})$ and $p_{i_2} \in \mathrm{OutT}(\{t\})$ for some t. The following result relates minimal siphon to graph connectivity.

Lemma 1. *Suppose (P, T, F) is a finite state machine and its commensurate graph is $G = (P, E)$. G is strongly connected if and only if the minimal siphon of the state machine is P.*

Proof. (Sufficiency). Assume that P is the only minimal siphon of the Petri net. For any $p \in P$, $\mathrm{InP}(\{p\}) \neq \emptyset$ since the atomic set $\{p\}$ is not a minimal siphon. There exists $t \in T$ such that $t \in \mathrm{InP}(\{p\})$, i.e., $(t, p) \in F_1$. As (P, T, F) is a state machine, $(p_{i_1}, t) \in F_2$ for some (unique) $p_{i_1} \in P$. Again we know $p_{i_1} \neq p$ since $\{p\}$ is not a minimal siphon. By definition of the commensurate graph, we have $(p_{i_1}, p) \in E$.

Next, we consider the set $\{p, p_{i_1}\}$. Repeating the above argument (by using the fact that $\{p, p_{i_1}\}$ is not a minimal siphon), we obtain that there exists an agent $p_{i_2} \notin \{p, p_{i_1}\}$ such that p_{i_2} is connected to $\{p, p_{i_1}\}$ in G. Since G is a finite graph, by recursively applying the argument, we see that every agent in G will be connected to p through a directed path. Hence, G is strongly connected as p is an arbitrary agent.

(Necessity). Suppose G is strongly connected. We first show that $\mathrm{InP}(P) \subseteq \mathrm{OutP}(P)$ holds. In fact, take any $p \in P$, we know $\mathrm{InP}(\{p\}) \neq \emptyset$ because G is strongly connected. Take any $t \in \mathrm{InP}(\{p\})$ and there is a unique $p_{i_1} \in \mathrm{InT}(\{t\})$. In other words, $t \in \mathrm{OutP}(P)$. This proves that $\mathrm{InP}(P) \subseteq \mathrm{OutP}(P)$.

What remains to show is that for any proper subset $S_P \subseteq P$, $\mathrm{InP}(S_P) \nsubseteq \mathrm{OutP}(S_P)$. Since S_P is a proper subset and G is strongly connected, we can choose $(p_{i_1}, p) \in E$ satisfying $p_{i_1} \notin S_P$ and $p \in S_P$. We have some $t \in \mathrm{InP}(\{p\})$ and $p_{i_1} \in \mathrm{InT}(\{t\})$ by the definition of commensurate graph. Since the Petri net is a state machine, $|\mathrm{InT}(\{t\})| = 1$. Hence, $t \notin \mathrm{OutP}(S_P)$. This leads to the desired result. Therefore, P is the minimal siphon. □

2.2 Modeling observer effect and neighbor-dependent synergy

Recall that $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathbb{R}^n$ describes the states of agents at time $t \geq 0$. Here we are interested in consensus formation for the group of agents in P.

Definition 1 (consensus formation). We say that the agents in P reach a consensus if there is $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all $p_i \in P$ and $x(0) \in \mathbb{R}^n$.

Under external observation, an agent in the social network tends to express their opinion within a moderate comfortable range [34, 35, 37]. Hence, we consider a range $R_i = [\underline{r}_i, \overline{r}_i]$ for each $p_i \in P$, and the expressed opinion of p_i is characterized by a mask function $\varphi_i(\cdot) : \mathbb{R} \longrightarrow R_i$ given by

$$\varphi_i(z) := \begin{cases} \overline{r}_i, & z > \overline{r}_i; \\ z, & \underline{r}_i \leq z \leq \overline{r}_i; \\ \underline{r}_i, & z < \underline{r}_i. \end{cases} \tag{1}$$

Given an index set $J \subseteq [n] := \{1, 2, \cdots, n\}$, denote by $x_J = \{x_j : j \in J\}$. Recall that J is a minimal influence on the agent $p_i \in P$ if x_i is influenced only by all agents with indices in J as a whole group but not any proper subset of J. Such a minimal influence can be delineated by a function $f_J(x_J)$. For example, given x_1, the minimal influence of p_2 and p_3 on p_1 in Fig. 1 can be encoded in the intrinsically nonlinear function $f_{\{2,3\}}(x_2, x_3) = 2^{-1}(\min\{\max\{x_2, x_3\}, x_1\} + \max\{\min\{x_2, x_3\}, x_1\})$. Clearly, only when both x_2 and x_3 are greater than (or less than) x_1, the result will be deviated from x_1. Another example could be $f_{\{2,3\}}(x_2, x_3) = \tan\sqrt{2^{-1}(|h(x_2)h(x_3)| + h(x_2)h(x_3))}$, where

$h(x) = \arctan(x - x_1)$. On the other hand, the minimal influence of a single agent p_4 on p_1 is typically represented by a linear function such as $f_{\{4\}}(x_4) = x_4$ [5].

For each $i \in [n]$, recall the partition of neighborhood $N_i = \{J_{ij} : j = 1, 2, \cdots, j_i\}$ with $1 \le j_i \le |\tilde{N}_i|$ in Section 2.1. We propose an agent p_{j_0} in each J_{ij} as the trusted agent of agent i and associate a weight $a_{ij_0} > 0$ with the subgroup J_{ij}. The trusted agent is used to reconcile disagreement in the case of excessive disagreement within a subgroup (cf. Remark 4). Combining the observer effect and neighbor-dependent synergy, the dynamics of agent $p_i \in P$ is captured by the following equation

$$\dot{x}_i(t) = \sum_{J \in N_i} a_{iJ}\big(f_J(\varphi_J(x_J(t))) - x_i(t)\big), \tag{2}$$

where $t \ge 0$, $J = J_{ij}$, and $a_{iJ} = a_{ij_0}$ when $J = J_{ij}$. Here in Equation (2) we use the notation J for ease of presentation. Recall that $x_J = \{x_k : k \in J\}$ is a collection of states in the subgroup J, and hence $\varphi_J(x_J) = \{\varphi_k(x_k) : k \in J\}$ captures the collection of expressed opinions modified by the mask functions φ_k defined in (1). The basic idea here is there is a trusted node p_{j_0} in each subgroup J_{ij}; it coordinates with other members in the subgroup to together influence (via (3) below) the node p_i. The system (2) specifies how the state of p_i responds to the influence of all its neighboring subgroups $N_i = \{J_{ij} : j = 1, 2, \cdots, j_i\}$.

Given a non-negative locally Lipschitz continuous function $\delta = \delta(t)$, we define the function $f_J(\cdot)$ in (2) as follows:

$$f_J(x_J) = \begin{cases} g_J(x_J), & \max\{|x_{k_1} - x_{k_2}| : k_1, k_2 \in J\} \le \delta; \\ x_{j_0}, & \text{otherwise}, \end{cases} \tag{3}$$

where $g_J(x_J)$ can be any locally Lipschitz continuous function satisfying $\min_{k \in J} x_k \le g_J(x_J) \le \max_{k \in J} x_k$.

Several remarks are in order.

Remark 2 (regarding $\delta(t)$). The function $\delta(t)$ controls the disagreement within each subgroup, which may depend on J. However, since there are only finite agents, without loss of generality we consider a global $\delta(t)$ instead of $\delta_J(t)$. The introduction of δ here is reminiscent of the classical bounded confidence models [14], where the confidence bound is measured between p_i and any of its neighbors. For example, the critical confidence bound is determined as 0.5 if the initial opinions of all agents are taken uniformly within the unit interval $[0, 1]$ and the underlying network is a connected graph [14]. In general, the critical bound is proved to be related to the expected value of the initial opinion distribution [20]. Our function $\delta(t)$ can be viewed as a "transverse" confidence bound in the hypergraph setting as the difference between neighbors are measured here. The classical bounded confidence models, on the other hand, can be regarded as using "longitudinal" confidence bounded in the standard network setting. Interestingly, a critical confidence bound in our case is non-existent (see Theorem 1 and Section 3.4) regardless of the initial opinion configuration.

Remark 3 (regarding g_J). Equation (3) means that $f_J(x_J)$ may take any value between the two extreme opinions if the discrepancy within each subgroup is controlled by δ, otherwise it adopts the opinion of the trusted agent. Typical examples of $g_J(\cdot)$ can be the arithmetic average function $g_J(x_J) = |J|^{-1} \sum_{k \in J} x_k$ and $g_J(x_J) = x_{j_0}$, which implies that p_i has absolute trust in p_{j_0}. The choice of g_J is fluid: (a) it can depend on the subgroup J as well as the states of neighbors, and (b) it does not coupled with the communication topology or affect convergence (see Theorem 1 below). This flexibility is desirable in the social network setting and different from rigid control protocols typically seen in the control theory literature on consensus problems either in a standard network environment [5–7, 15, 16] or a hypergraph environment [28–32].

Remark 4 (regarding p_{j_0}). For each $i \in [n]$, any agent in the subgroup J_{ij} can be chosen as a trusted agent p_{j_0}, who has a higher "social influence" [43]. In other words, p_{j_0} may depend on the agent p_i in question, and it is specified initially at time $t = 0$. This is in line with the social trust theory [47, 48], which predicts that individual tends to only trust a very limited number of acquaintances (or strong ties), who one has the most intimate knowledge, in cooperation involving more than a few individuals. The trusted agent introduced here also naturally extends the standard network situation, where each subgroup J_{ij} contains only one node (hence every neighbor is a trusted agent).

3 Main result and numerical studies

3.1 Consensus formation

Our main result is the following consensus formation which will be shown in Section 4 through a series of lemmas.

Theorem 1. *Consider the system (2), and assume that the Petri net (P, T, F) has the minimal siphon P and $\cap_{i=1}^n R_i \neq \emptyset$. Then for any $x(0) \in \mathbb{R}^n$, there exists $c \in \cap_{i=1}^n R_i$ such that $\lim_{t\to\infty} x_i(t) = c$ for all $i \in [n]$.*

Remark 5. We refer to the above assumption of the minimal siphon being P as the "minimal siphon condition". This condition is essentially necessary. Suppose (P, T, F) is a finite state machine. By Lemma 1, this condition is tantamount to strongly connectedness of its commensurate graph G. If G is not strongly connected, we can decompose it in several strongly connected components. Let C_1 be the component which has no in-coming arcs. Take $x_i(0) \notin \cap_{i=1}^n R_i$ for all $i \in C_1$. If consensus along the system (2) is achieved, then the final consensus value, say c, must satisfy $c \notin \cap_{i=1}^n R_i$ because c only depends on the agents in C_1. Therefore, for finite state machines, this condition is both sufficient and necessary for guaranteeing consensus value lying in the intersection of comfortable ranges.

In the following subsections we will demonstrate the result through some numerical simulations, which will further shed lights on the impact of observers, hypergraph structure, and confidence bound on the consensus formation.

3.2 Influence of observer effect

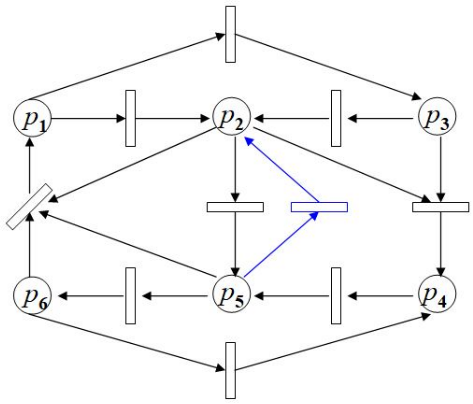

Figure 2: A Petri net (P, T, F) with $P = \{p_1, p_2, \cdots, p_6\}$.

We first consider the influence of the comfortable bounds $\underline{r}_i$ and $\overline{r}_i$ on the system behavior of (2). The directed hypergraph (P, T, F) here is illustrated in Fig. 2. It has 6 agents in $P = \{p_1, p_2, \cdots, p_6\}$ and its only siphon is P itself. The comfortable bounds for each agent are chosen as $R_1 = [1.5, 5], R_2 = [3, 6.5], R_3 = [3, 6], R_4 = [2.5, 5], R_5 = [2, 5.5], R_6 = [4, 7]$, respectively. The intersection becomes $\cap_{i=1}^{6} R_i = [4, 5]$. We choose $\delta(t) \equiv 1$, $a_{iJ} = 0.3$ and $g_J(x_J)$ being the arithmetic average function for all $i \in [6]$ and $J \in N_i$. Note that there are two groups of neighbor-dependent synergy in this Petri net: $J_{11} = \{2, 5, 6\}$ and $J_{41} = \{2, 3\}$, where agent p_2 is chosen as the trusted node in both cases.

State trajectories for two different sets of different initial states $x^{(1)}(0) = (1.5, 3.5, 1, 0, 2.5, 0.5)$ and $x^{(2)}(0) = (2, 6.5, 8, 4.5, 3.5, 1)$ are shown in Fig. 3. We observe from Fig. 3(a) and Fig. 3(b) that the system reaches consensus at the equilibrium value 4 at the boundary of $\cap_{i=1}^{6} R_i$. With a different initial condition in Fig. 3(d) and Fig. 3(e), we observe that the equilibrium is approximately at 4.19, which is inside the overlapping range $\cap_{i=1}^{6} R_i$. These results are in line with the prediction in Theorem 1. In Fig. 3(c) and Fig. 3(f), we plotted the extreme values of states, $\max_{i\in[6]} x_i(t)$ and $\min_{i\in[6]} x_i(t)$, and the gap between maximum and minimum. It is interesting to notice that the gap is not necessarily a monotonically non-increasing function. This is in contrast to many other consensus multi-agent systems without taking observer effect into consideration [5–7].

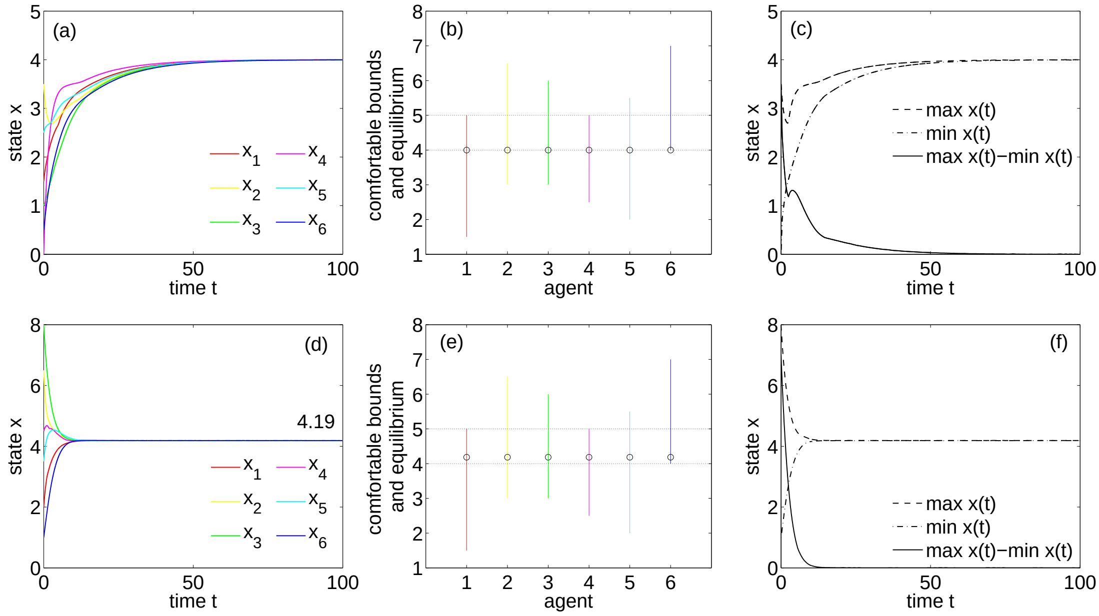

Figure 3: Top row: consensus formation of the system (2) under initial condition $x^{(1)}(0) = (1.5, 3.5, 1, 0, 2.5, 0.5)$ for (a) state trajectories; (b) comfortable ranges and final consensus equilibrium; (c) values of max, min and the gap of states. Bottom row: consensus formation of the system (2) under initial condition $x^{(2)}(0) = (2, 6.5, 8, 4.5, 3.5, 1)$ for (d) state trajectories; (e) comfortable ranges and final consensus equilibrium; (f) values of max, min and the gap of states.

In Fig. 4(a) and Fig. 4(b), we display the states of the agents in the system (2) with comfortable bounds $R_1 = [2, 5], R_2 = [3.5, 6.5], R_3 = [2, 4], R_4 = [1, 3], R_5 = [1.5, 3], R_6 = [3.5, 6]$ and the initial condition $x^{(2)}(0)$. All the other parameters are the same as those adopted in Fig. 3. Noting that $\cap_{i=1}^{6} R_i = \emptyset$, consensus is not achieved as one would expect.

In Fig. 4(c), we show the evolution of states for the system (2) with the same parameters as in Fig. 4(a) and Fig. 3(d) except that the comfortable ranges are removed. In other words, the observer effect is no longer at play. We observe that consensus is achieved at value 4.14 in agreement with Theorem 1. However, this final consensus value is slightly different from the value 4.19 shown in Fig. 3(d). This confirms that the observer effect essentially influences the opinion dynamics – not only impacts on the trajectories but the consensus value (cf. Fig. 3(d) and Fig. 4(c)).

3.3 Influence of hypergraph structure

If we remove the blue transition (and its associated two arcs) in Fig. 2, it is direct to check that the resulting Petri net (P, T', F') will not satisfy the condition of Theorem 1. In fact, $S_P = \{p_1, p_2, p_3\}$ will be a (minimal) siphon and hence P is no longer the minimal siphon

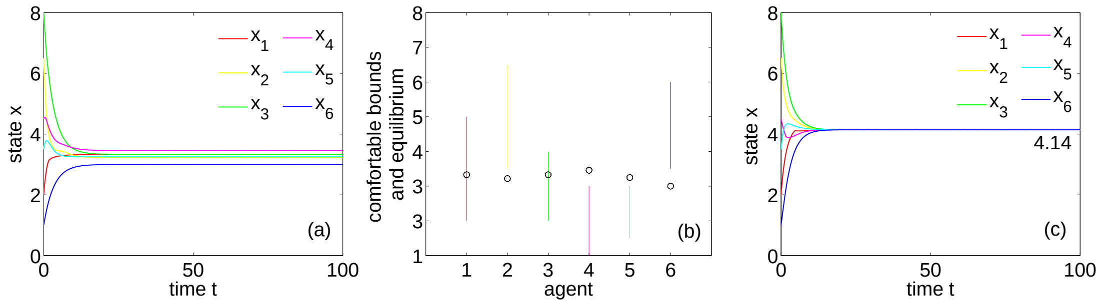

Figure 4: (a) State trajectories for the system (2) under initial condition $x^{(2)}(0) = (2, 6.5, 8, 4.5, 3.5, 1)$ with $\cap_{i=1}^{6} R_i = \emptyset$; (b) Comfortable ranges and final values; (c) State trajectories for the system (2) under initial condition $x^{(2)}(0) = (2, 6.5, 8, 4.5, 3.5, 1)$ with $\cap_{i=1}^{6} R_i = \mathbb{R}$.

of (P, T', F'). In Fig. 5 we show the state evolution of the system (2) over the modified hypergraph (P, T', F') with initial conditions $x^{(1)}(0)$ and $x^{(3)}(0) = (0, 1.5, 3.5, 4, 5.5, 3)$, respectively, both with $\delta = 0.1$ and all the other parameters kept the same as those in Fig. 3.

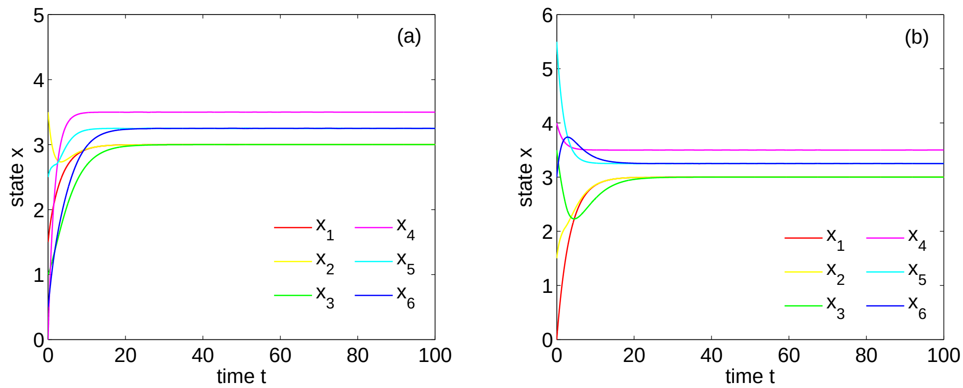

Figure 5: State trajectories for the system (2) with $\delta = 0.1$ under initial condition (a) $x^{(1)}(0) = (1.5, 3.5, 1, 0, 2.5, 0.5)$ and (b) $x^{(3)}(0) = (0, 1.5, 3.5, 4, 5.5, 3)$. All the other parameters are the same as those in Fig. 3.

We observe that the systems fail to reach global consensus and the final values are no longer in the range $[4, 5]$. The idea of choosing a smaller confidence bound δ here is inspired by the comment given in Remark 5. For a smaller δ, the group $\{p_1, p_2, p_3\}$ is more likely to have no in-coming arcs from outside as p_2 is set as the trusted agent. Therefore, these three agents are likely to form a separate cluster as observed in Fig. 5(a) and Fig. 5(b). However, the cause effect relationship is complicated by the observer effect and neighboring synergy in our scenario. In fact, if we adopt exactly the same parameters as in Fig. 3, global consensus will still be reached.

3.4 Influence of confidence bound $\delta(t)$

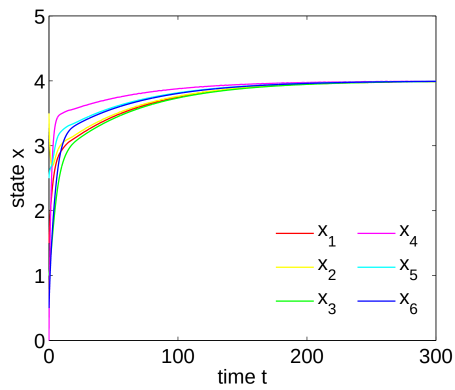

Figure 6: State trajectories for the system (2) with $\delta = 0$ and all the other parameters the same as those in Fig. 3(a).

In Fig. 6, we show the state convergence for the system (2) with $\delta = 0$ and all the other parameters are the same as those in Fig. 3(a). We observe that the global consensus is reached at around $t = 200$, which is much more slowly than the case in Fig. 3(a) with $\delta = 1$. The slow convergence in this example can be intuitively seen as the result of dilution of arcs in the hypergraph. When δ decreases, the confidence bounds become more stringent and the trusted nodes are more likely to be utilized. For example, for the minimal influence of J_{11} on p_1, the other two arcs will be forfeited when x_2 is taken as per the rule (3).

Define the consensus time as $t^*(\varepsilon) := \min\{t : |x_{i_1}(t) - x_{i_2}(t)| < \varepsilon, \text{ for any } i_1, i_2 \in [n]\}$ for $\varepsilon > 0$. In Table 1, we show the the values of $t^*(0.01)$. Clearly, the consensus time decreases with respect to δ. As shown in Theorem 1, consensus will be achieved for all $\delta(t) \geq 0$, which distinguishes it from the traditional bounded confidence models [14]. It also worth noting that when δ is sufficiently large, for example larger than $\max\{|x_{i_1}(0) - x_{i_2}(0)| :$ for any $i_1, i_2 \in [n]\}$, the consensus time will no longer decrease as one would expect from the system dynamics (2) and (3).

$\delta(t)$	t^{-1}	0	0.01	0.05	0.1	0.5	1	2	3
$t^*(0.01)$	255.51	259.42	257.28	223.35	189.56	105.52	68.06	66.58	66.58

Table 1: Consensus time $t^*(0.01)$ for the system (2) with different $\delta(t)$ and all the other parameters the same as those in Fig. 3(a).

4 Proof of Theorem 1

In this section, we will prove Theorem 1 in two steps: in Step I we additionally assume $\lim_{t\to\infty}\delta(t)=0$, and then in Step II we lift this condition.

4.1 Step I

Let $\underline{r} := \max_{i\in[n]}\underline{r}_i$ and $\overline{r} := \min_{i\in[n]}\overline{r}_i$. Define $\overline{\rho}(x(t)) = \max\{\max_{i\in[n]} x_i(t), \overline{r}\}$ and $\underline{\rho}(x(t)) = \min\{\min_{i\in[n]} x_i(t), \underline{r}\}$. They are locally Lipschitz continuous functions. Moreover, define the gap $\Delta(x(t)) := \overline{\rho}(x(t)) - \underline{\rho}(x(t)) \geq 0$. Recall that the Dini derivative of a continuous function $h(t)$ is defined as $d^+h(t) := \limsup_{\varepsilon\to 0+}\varepsilon^{-1}(h(t+\varepsilon)-h(t))$ [49]. The gap $\Delta(x(t))$ is non-increasing for any $t\geq 0$, as stated in the following lemma.

Lemma 2. *For $t \geq 0$, we have $d^+\Delta(x(t)) \leq 0$.*

Proof. Note that the intersection of comfortable ranges is $\cap_{i=1}^n R_i = [\underline{r}, \overline{r}]$. For any $t \geq 0$, if $\overline{\rho}(x(t)) > \overline{r}$, then $\max_{i\in[n]} x_i(t) > \overline{r}$ in the interval $[t, t+\varepsilon)$ for some $\varepsilon > 0$. Denote by $K(t) := \{k \in [n] : x_k(t) = \max_{i\in[n]} x_i(t)\}$. We have

$$\begin{aligned} d^+\overline{\rho}(x(t)) &= d^+\max_{i\in[n]} x_i(t) = \max_{k\in K(t)} \dot{x}_k(t) \\ &= \max_{k\in K(t)}\left[-\sum_{J\in N_k} a_{kJ}\big(x_k(t) - f_J(\varphi_J(x_J(t)))\big)\right], \end{aligned} \tag{4}$$

where we have taken the Dini derivative along the system (2) and applied the basic Dini derivative property [49,50]. For any $k_0 \in K(t)$, we have $x_i(t) \leq x_{k_0}(t)$ for each $i \in [n]$. We know for each $i\in[n]$, $\underline{r}_i \leq \underline{r} < \overline{r} \leq \overline{r}_i$. Consequently, when $x_i(t) > \overline{r}$, $\varphi_i(x_i(t)) \leq x_i(t)$; when $x_i(t) \leq \overline{r}$, $\varphi_i(x_i(t)) \leq \overline{r}$. Therefore, for each $p_i \in \tilde{N}_k$, $\varphi_i(x_i(t)) \leq \max\{x_i(t), \overline{r}\} \leq x_{k_0}(t)$. According to our model definition (2) and (3), $f_J(\varphi_J(x_J(t))) \leq x_{k_0}(t)$. It then follows from (4) that $d^+\overline{\rho}(x(t)) \leq 0$ under the assumption $\overline{\rho}(x(t)) > \overline{r}$.

Note that $\overline{\rho}(x(t)) \geq \overline{r}$ for any $t \geq 0$. If $\overline{\rho}(x(t')) = \overline{r}$ for some t', then $d^+\overline{\rho}(x(t')) = 0$, and hence for any $t \geq t'$ we have $\overline{\rho}(x(t)) = \overline{r}$. This indicates that $\overline{\rho}(x(t))$ is non-decreasing for all $t \geq 0$. Similar arguments can be applied to $\underline{\rho}(x(t))$ and we obtain $d^+\underline{\rho}(x(t)) \geq 0$ for all $t \geq 0$. This means that $\underline{\rho}(x(t))$ is non-increasing for $t \geq 0$. Therefore, we have $d^+\Delta(x(t)) \leq 0$ and the proof is complete. □

If we rewrite the system (2) using the vector field form $\dot{x}(t) = F(x(t))$, then the Dini derivative along the vector field is equivalent to the Dini derivative along the trajectory of the solution [51]. Namely, if $x(\tilde{t}) = \tilde{x}$ at time $\tilde{t} > 0$, then

$$d^+\Delta(x)|_{x=\tilde{x}} := \limsup_{\varepsilon\to 0+}\varepsilon^{-1}(\Delta(x+\varepsilon F(x)) - \Delta(x))|_{x=\tilde{x}} = d^+\Delta(x(t))|_{t=\tilde{t}}.$$

Define $S := \{x \in \mathbb{R}^n : d^+\Delta(x) = 0\}$.

Lemma 3. *If the Petri net (P,T,F) admits the minimal siphon P, $S \subseteq [\underline{r},\overline{r}]^n$.*

Proof. We prove this result by assuming the opposite and deriving contradiction. Suppose there is a vector $x' = (x'_1, x'_2, \cdots, x'_n) \in S$ and $x' \notin [\underline{r},\overline{r}]^n$. Without loss of generality, we take $x'_i := \max_{k\in[n]} x'_k$ and $x'_i > \overline{r}$. (The other case $x'_i := \min_{k\in[n]} x'_k$ and $x'_i < \underline{r}$ can be shown likewise.)

Consider the solution of system (2) with $x(0) = x'$. Denote by $K := \{k \in [n] : x'_k = x'_i\} \neq \emptyset$. Recall that (P,T,F) admits the minimal siphon P. For any $i \in [n]$ and any $j \in [j_i]$, if we remove all other nodes in J_{ij} (apart from p_{j_0}), $|J_{ij}| - 1$ arcs are deleted from the Petri net. The resulting directed hypergraph is a state machine. Clearly, P is still the minimal siphon of it. By Lemma 1, its commensurate graph G is strongly connected. The system (2) over G is a standard average system [5]. The value of any node in K will be pulled downwards by other nodes in $P\backslash K$ who have value less than x'_i, or by $\overline{r}$ even if $P\backslash K = \emptyset$. Accordingly, there is time $t' > 0$ such that $x_k(t') < x'_i$ for any $k \in [n]$. This implies $\overline{\rho}(x(t')) < \overline{\rho}(x(0))$ and hence $\Delta(x(t')) < \Delta(x(0))$. It contradicts with $x(0) = x' \in S$ (recall the definition of S). The proof is complete. □

Signify by $\Lambda^+(x(0))$ the positive limit set of the system (2). The LaSalle's invariance principle tells us that $\Lambda^+(x(0)) \subseteq S$ for any $x(0) \in \mathbb{R}^n$. If follows from Lemma 3 that $\Lambda^+(x(0)) \subseteq [\underline{r},\overline{r}]^n$. Hence, $x(t) \in [\underline{r},\overline{r}]^n$ for any sufficiently large t. The next two results show that the gap between any two agents is vanishing.

Lemma 4. *Consider the following system defined over a directed n-node graph $G = (V,E,A)$ containing a spanning tree:*

$$\dot{x}_i(t) = \sum_{j:(j,i)\in E} a_{ij}(x_j(t) - x_i(t)) + y_i(t), \quad i \in V,\ t \geq 0,$$

where $V = [n]$, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the adjacency matrix with $a_{ij} > 0$ if $(j,i) \in E$, and the function $y_i(t)$ is continuous except a set of measure zero. Denote by $y = (y_1, y_2, \cdots, y_n)$, and $\|y(t)\|_{[0,\infty)} := \max_{i\in V} \sup_{t\geq 0} |y_i(t)|$. Then for any $\alpha > 0$, there is $\varepsilon > 0$ satisfying the following: If $\|y(t)\|_{[0,\infty)} \leq \varepsilon$, then

$$\limsup_{t\to\infty} \max_{i_1,i_2\in V} |x_{i_1}(t) - x_{i_2}(t)| \leq \alpha$$

holds for any $x(0) \in \mathbb{R}^n$.

Proof. This is a special case of Theorem 4.1 in [52]. □

Lemma 5. $\lim_{t\to\infty} x_{i_1}(t) - x_{i_2}(t) = 0$ *for any $i_1, i_2 \in [n]$ and $x(0) \in \mathbb{R}^n$.*

Proof. It follows from (3) and Lemma 3 that there exists $t' > 0$ such that along the solution of the system (2), we have

$$|x_{j_0}(t) - f_{J_{ij}}\big(\varphi_{J_{ij}}(x_{J_{ij}}(t))\big)| \leq \delta(t) \tag{5}$$

for all $t \geq t'$, $i \in [n]$, and $j \in [j_i]$. The system (2) can be rewritten as

$$\dot{x}_i(t) = -\sum_{J_{ij} \in N_i} a_{iJ_{ij}} \big(x_i(t) - x_{j_0}(t) \big) + y_i(t), \tag{6}$$

where

$$y_i(t) = \sum_{J_{ij} \in N_i} a_{iJ_{ij}} \big(f_{J_{ij}}(\varphi_{J_{ij}}(x_{J_{ij}}(t))) - x_{j_0}(t) \big).$$

By (5) and $\lim_{t\to\infty} \delta(t) = 0$, we know that for any $\varepsilon > 0$ there exists $t' > 0$ such that $\|y(t)\|_{[t',\infty)} \leq \varepsilon$, where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$.

Arguing similarly as in the proof of Lemma 3, we know that the commensurate graph G of the resulting state machine is strongly connected. We take a sequence $\alpha_k = k^{-1}$ for integer $k \geq 1$. It follows from Lemma 4 that for each k, there exists $\varepsilon_k > 0$ such that when $\|y(t)\|_{[t',\infty)} \leq \varepsilon_k$ for some $t' > 0$, we have

$$\limsup_{t\to\infty} \max_{i_1, i_2 \in [n]} |x_{i_1}(t) - x_{i_2}(t)| \leq \alpha_k \tag{7}$$

for any $x(0) \in \mathbb{R}^n$. Utilizing the comment in the previous paragraph, we obtain $\lim_{t\to\infty} x_{i_1}(t) - x_{i_2}(t) = 0$ for any $i_1, i_2 \in [n]$ as desired. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1 (Step I). In the light of the comments below Lemma 3, any limit point of $x_i(t)$ must lie in the range $[\underline{r}, \overline{r}]$ for all $i \in [n]$. Fix an arbitrary i and take a limit point, say c. Then $c \in [\underline{r}, \overline{r}]$. If $\underline{r} = \overline{r}$, the theorem holds immediately. We only need to consider the case of $\underline{r} < \overline{r}$. It follows from (7) and the proof of Lemma 5, for any $\varepsilon > 0$, there exists $t' > 0$ such that $|x_i(t') - c| \leq \varepsilon$ holds for all $i \in [n]$. We consider the location of c in three situations.

If $\underline{r} < c < \overline{r}$, then we choose a small $\varepsilon > 0$ satisfying $\underline{r} < c - \varepsilon \leq x_i(t') \leq c + \varepsilon < \overline{r}$ for all $i \in [n]$. By definition $x_i(t) = \varphi_i(x_i(t))$ for each i. Therefore, the system (2) can be simplified as

$$\dot{x}_i(t) = \sum_{J \in N_i} a_{iJ} \big(f_J(x_J(t)) - x_i(t) \big),$$

whose solution converges to the solution of

$$\dot{x}_i(t) = \sum_{\substack{j_0 \in J_{ij} \\ j \in [j_i]}} a_{ij_0} (x_{j_0}(t) - x_i(t))$$

by invoking the Gronwall's inequality; cf. [53, Thm. 2.1]. The latter is a standard consensus system over strongly connected graph. Hence, we have $\lim_{t\to\infty} x_i(t) = c$ for all $i \in [n]$.

If $c = \overline{r}$, we choose a small $\varepsilon > 0$ satisfying $\underline{r} < c - \varepsilon \leq x_i(t') \leq c + \varepsilon$, which implies $\underline{r} < x_i(t') \leq c + \varepsilon$ for all $i \in [n]$. Arguing similarly as in Lemma 2 by setting

$\overline{\rho}'(x(t)) = \max\{\max_{i\in[n]} x_i(t), \overline{r}\}$, we can show $\overline{\rho}'(x(t))$ is non-increasing and bounded from below. Hence, $\lim_{t\to\infty} \overline{\rho}'(x(t)) = b$ for some b. Letting $\varepsilon \to 0$, we obtain $b = c$ and $\lim_{t\to\infty} \max_{i\in[n]} x_i(t) = c$. By (7) and the proof of Lemma 5, we have $\lim_{t\to\infty} \min_{i\in[n]} x_i(t) = c$. Combining these estimates, we arrive at $\lim_{t\to\infty} x_i(t) = c$ for any $i \in [n]$.

The case of $c = \underline{r}$ can be shown similarly. The proof of Stage I is complete. □

4.2 Step II

For a general confidence bound $\delta(t)$, it is critical to observe that the change of $\delta(t)$ essentially gives rise to the change of the underlying communication hypergraph structure in (P, T, F) in view of (2) and (3). As shown in the Step I, the extreme case of $\delta = 0$ corresponds to an underlying strongly connected directed graph reducing from $|\,\mathrm{InT}(\{t\})| \geq 1$ to $|\,\mathrm{InT}(\{t\})| = 1$ for all $t \in T$ (by removing all but one arcs in F_2 for each transition $t \in T$). For a larger δ, the communication structure alters between the strongly connected directed graph and the full structure of (P, T, F).

Starting from a sufficiently large δ, where all arcs in (P, T, F) are present, some arcs in $\mathrm{InT}(\{t\})$ will be removed for some transitions t as δ decreases. Since P is the minimal siphon of (P, T, F), any such removal yields a Petri net (P, T, F') with P as its minimal siphon. Noting that (i) the proof of Step I does not relies on the specific choice of j_0 and (ii) $\min_{k\in J} x_k \leq g_J(x_J) \leq \max_{k\in J} x_k$, we know that the system (2) is equivalent to the following system

$$\dot{x}_i(t) = \sum_{j\in N_i} a_{ij}\big(\varphi_j(x_j(t)) - x_i(t)\big),$$

over a strongly connected directed graph G corresponding to the above adjacency matrix structure specified by $(a_{ij}) \in \mathbb{R}^{n\times n}$. Following the same line of Step I, we can similarly establish Lemmas 2-5 by replacing j_0 with j. Then Theorem 1 can be shown likewise following the proof of Theorem 1 (Step I).

5 Conclusion

Interaction in many real networks is better characterized as a directed hypergraph, which is a natural generalization of graphs by allowing possible neighbor-dependent synergy. In this work we showed that consensus can be reached in directed hypergraphs featuring social observer effect as well as confidence bounds among neighbors. To do this we introduced a Petri net triad (P, T, F) and established a sufficient and necessary condition for the commensurate graph of a state machine to be strongly connected. We showed that when the network has the minimal siphon P, global consensus is guaranteed as long as there is a common non-empty comfortable range for all agents. The final consensus will lie in this common range and the result holds for all non-negative confidence bounds. This deviates from traditional bounded confidence opinion models as we measure the difference among neighbors rather than the gap between neighbors and the ego. Numerical simulations unraveled some interesting insights in relation to the influence of model parameters on opinion evolution. Although our methods are applied here to investigate the social opinion dynamics, the theory can be similarly extended to study other networked dynamical processes, to yield better understanding of their system behaviors.

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