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Destabilization of rotating flows with positive shear by azimuthal magnetic fields

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According to Rayleigh's criterion, rotating flows are linearly stable when their specific angular momentum increases radially outward. The celebrated magnetorotational instability opens a way to destabilize those flows, as long as the angular velocity is decreasing outward. Using a local approximation we demonstrate that even flows with very steep positive shear can be destabilized by azimuthal magnetic fields which are current free within the fluid. We illustrate the transition of this instability to a rotationally enhanced kink-type instability in the case of a homogeneous current in the fluid, and discuss the prospects for observing it in a magnetized Taylor-Couette flow.

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From a purely hydrodynamic point of view, rotating flows are stable as long as their angular momentum is increasing radially outward [1]. Since this criterion applies to the Keplerian rotation profiles which are typical for low-mass accretion disks, the growth mechanism of central objects, such as protostars and black holes, had been a conundrum for many decades. Nowadays, magnetorotational instability (MRI) [2] is considered the main candidate to explain the turbulence and enhanced angular momentum in accretion disks. The standard version of MRI (SMRI), with a vertical magnetic field B_z applied to the rotating flow, requires both the rotation period and the Alfvén crossing time to be shorter than the time scale for magnetic diffusion [3]. This implies, for a disk of height H, that both the magnetic Reynolds number $\mathrm{Rm} = \mu_0\sigma H^2\Omega$ and the Lundquist number $S = \mu_0\sigma H v_A$ must be larger than one (Ω is the angular velocity, μ_0 is the magnetic permeability constant, σ the conductivity, and $v_A := B_z/\sqrt{\mu_0\rho}$ is the Alfvén velocity, with ρ denoting the density). While these conditions are safely fulfilled in well-conducting parts of accretion disks, the situation is less clear in the "dead zones" of protoplanetary disks, in stellar interiors and liquid cores of planets, because of the small value of the magnetic Prandtl number $\mathrm{Pm} = \nu/\eta$ [4], i.e., the ratio of viscosity ν to magnetic diffusivity $\eta := (\mu_0\sigma)^{-1}$.

This low Pm case is also the subject of intense theoretical and experimental research initiated by Hollerbach and Rüdiger [5]. Adding an azimuthal magnetic field B_ϕ to B_z, the authors found a new version of MRI, now called helical MRI (HMRI). It was proved to work also in the inductionless limit [6], $\mathrm{Pm} = 0$, and to be governed by the Reynolds number $\mathrm{Re} = \mathrm{Rm}\,\mathrm{Pm}^{-1}$ and the Hartmann number $\mathrm{Ha} = S\,\mathrm{Pm}^{-1/2}$, quite in contrast to standard SMRI that is governed by Rm and S.

A somewhat sobering limitation of HMRI was identified by Liu *et al.* [7] who used a local approximation [also called the short-wavelength, Wentzel-Kramers-Brillouin (WKB), or geometric optics approximation—see Ref. [8]] to find a minimum steepness of the rotation profile $\Omega(r)$, expressed by the Rossby number $\mathrm{Ro} := r(2\Omega)^{-1}\partial\Omega/\partial r$, of $\mathrm{Ro}_{\mathrm{LLL}} = 2(1-\sqrt{2}) \approx -0.828$. This *lower Liu limit* (LLL) implies that, at least for $B_\phi(r) \propto 1/r$, HMRI does not extend to the most relevant Keplerian case, characterized by $\mathrm{Ro}_{\mathrm{Kep}} = -3/4$. Surprisingly, in addition to the LLL, the authors also found a second threshold of Ro, which we call the *upper Liu limit* (ULL), at $\mathrm{Ro}_{\mathrm{ULL}} = 2(1+\sqrt{2}) \approx +4.828$. For $\mathrm{Ro} > \mathrm{Ro}_{\mathrm{ULL}}$ one expects a magnetic destabilization of those flows with strongly increasing angular velocity *that would be stable even with respect to SMRI*.

By relaxing the demand that the azimuthal field is current free in the liquid, i.e., $B_\phi(r) \propto 1/r$, and allowing fields with arbitrary radial dependence, we have recently shown [8,9] that the LLL and the ULL are just the endpoints of one common instability curve in a plane that is spanned by Ro and a corresponding steepness of the azimuthal magnetic field, called the magnetic Rossby number, $\mathrm{Rb} := r(2B_\phi/r)^{-1}\partial(B_\phi/r)/\partial r$. In the limit of large Re and Ha, this curve acquires the closed and simple form

$$\mathrm{Rb} = -\frac{1}{8}\frac{(\mathrm{Ro}+2)^2}{\mathrm{Ro}+1}. \tag{1}$$

A nonaxisymmetric "relative" of HMRI, the azimuthal MRI (AMRI) [10], which appears for purely or dominantly B_ϕ, has been shown to be governed by basically the same scaling behavior, and the same Liu limits [11]. Actually, the key parameter dependencies of HMRI and AMRI were confirmed in various liquid metal experiments at the PROMISE facility [12,13].

In the present Rapid Communication, we focus exclusively on the case of positive Ro, i.e., on flows whose *angular velocity* (not only the angular frequency) is increasing outward. From a purely hydrodynamic point of view, such flows are linearly stable (while nonlinear instabilities were actually observed in experiments [14]). Flows with positive Ro are indeed relevant for the equator-near strip (approximately between ±30°) of the solar tachocline [15], which is, interestingly, also the region of sunspot activity [16]. Up to now, the ULL at $\mathrm{Ro}_{\mathrm{ULL}} = +4.828$ has only been predicted in the framework of various local approximations [7–9], while attempts to confirm it in a one-dimensional (1D) modal stability code on the basis of Taylor-Couette (TC) flows have failed so far [17]. Hence, the questions arise: Is the magnetically triggered flow

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instability for Ro > $\mathrm{Ro_{ULL}}$ a real phenomenon (which would fundamentally modify the stability criteria for rotating flows in general), or just an artifact of the local approximation, and is there any chance to observe it in a TC experiment?

In order to tackle these problems we restrict our attention here to nonaxisymmetric instabilities, which are the relevant ones for pure B_ϕ, and further assume Pm = 0. Under these assumptions, we had recently [8] derived the closed equation

$$\mathrm{Re}^2 = \frac{1}{4}\frac{[(1+\mathrm{Ha}^2 n^2)^2 - 4\,\mathrm{Ha}^2\,\mathrm{Rb}(1+\mathrm{Ha}^2 n^2) - 4\,\mathrm{Ha}^4 n^2][1+\mathrm{Ha}^2(n^2-2\,\mathrm{Rb})]^2}{\mathrm{Ha}^4\,\mathrm{Ro}^2 n^2 - \{[1+\mathrm{Ha}^2(n^2-2\,\mathrm{Rb})]^2 - 4\,\mathrm{Ha}^4 n^2\}[\mathrm{Ro}+1]} \tag{2}$$

for the marginal curves of the instability, where the following definitions for Re, Ha, and the modified azimuthal wave number n are used:

$$\mathrm{Re} = \frac{\alpha}{|\mathbf{k}|^2}\frac{\Omega(r)}{\nu}, \tag{3}$$

$$\mathrm{Ha} = \frac{\alpha}{|\mathbf{k}|^2}\frac{B_\phi(r)}{r(\mu_0\rho\eta\nu)^{1/2}}, \tag{4}$$

$$n = m/\alpha, \tag{5}$$

with $\alpha = k_z/|\mathbf{k}|$ and $|\mathbf{k}|^2 = k_r^2 + k_z^2$ defined as functions of the axial and radial wave numbers k_r and k_z.

Because of its comparably simple form, and the absence of the ratio β of azimuthal to axial magnetic field (which would play a decisive role for HMRI), Eq. (2) allows one to easily visualize the transition from a shear-driven instability of the AMRI type to the current-driven, kink-type Tayler instability (TI) [18], when going over from Rb = −1 to Rb = 0.

Let us start with the current-free case, Rb = −1. Figure 1(a) shows, for varying values of Ro and the particular case $n = 1.4$, the marginal curves in the Ha-Re plane. We see that the critical Re increases steeply for Ro below 6, which reflects the fact that we approach $\mathrm{Ro_{ULL}} = 4.828$ from above. We ask now for the dominant wave numbers, as illustrated in Fig. 1(b) for the particular value Ro = 5.5. Evidently, the minimal values of Re and Ha (the "knee" of the curve) appear for $n \sim 1.4$, which represents a rather "benign" combination of wave numbers with $k_r \sim k_z$, so that neither the axial nor the radial wavelength of the perturbations diverge. From this point of view, there seems to be no contradiction with the underlying short-wavelength approximation.

While for Rb = −1 the only energy source of the instability is the shear of the rotating flow, we move now in the direction of Rb = 0 which corresponds to a constant current density in the fluid, for which the kink-type TI [18] is expected to occur. For the particular choice $n = 1.2$, this transition is illustrated in Fig. 2, where we have intensionally chosen, for all Rb, the same scales for Re and Ha. For Rb = −0.6 we observe the appearance of a crossing with the abscissa, i.e., a point where the instability draws all its energy from the electrical current instead of the shear. Actually, the lowest value where this can occur is $\mathrm{Rb} = n^2/4 - 1 = -0.64$ [8].

For Rb = 0 the instability is characterized in more detail in Fig. 3. Very similar to the results of Ref. [19], we observe in Fig. 3(a) that for Ro > 0 the curves move to the left with increasing Re (i.e., the flow *supports* the kink-type instability) and converge to well-defined values of Ha when Re goes to infinity. The dependence on the wave-number ratio α is quite interesting. Figure 3(b) shows that the mode with $n = 1$ (i.e., with $k_r = 0$), which is still dominant at Re = 0, is replaced by modes with higher values of n for increasing Re. The limits of the critical Ha for Re = 0 and Re → ∞ can be determined by setting to zero, in Eq. (2), the nominator or denominator, respectively, which leads (for Rb = 0) to

$$\mathrm{Ha}_{\mathrm{Re}=0} = 1/\sqrt{n(2-n)}, \tag{6}$$

$$\mathrm{Ha}_{\mathrm{Re}\to\infty} = \sqrt{\frac{(\mathrm{Ro}+1)+\sqrt{(\mathrm{Ro}+1)(\mathrm{Ro}+2)/n}}{\mathrm{Ro}^2+(\mathrm{Ro}+1)(4-n^2)}}. \tag{7}$$

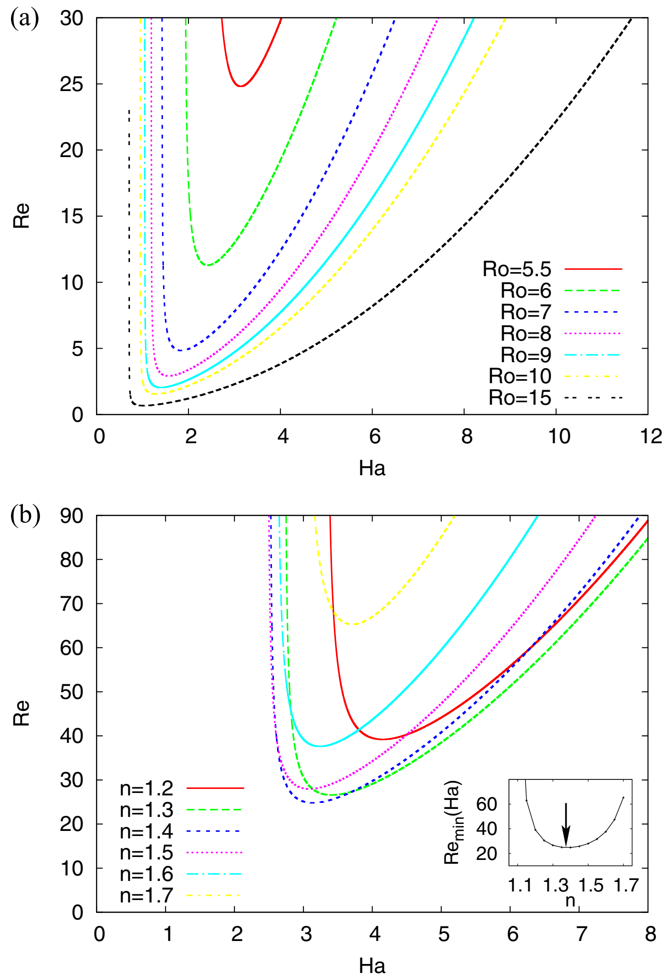

FIG. 1. (Color online) Marginal curves for Rb = −1. (a) Dependence on Ro for $n = 1.4$. (b) Dependence on n for Ro = 5.5. The inset shows the dependence of the minimum value (with respect to Ha) of the critical Re on n. The arrow points to the optimum $n \approx 1.35$ that leads to the lowest critical Re.

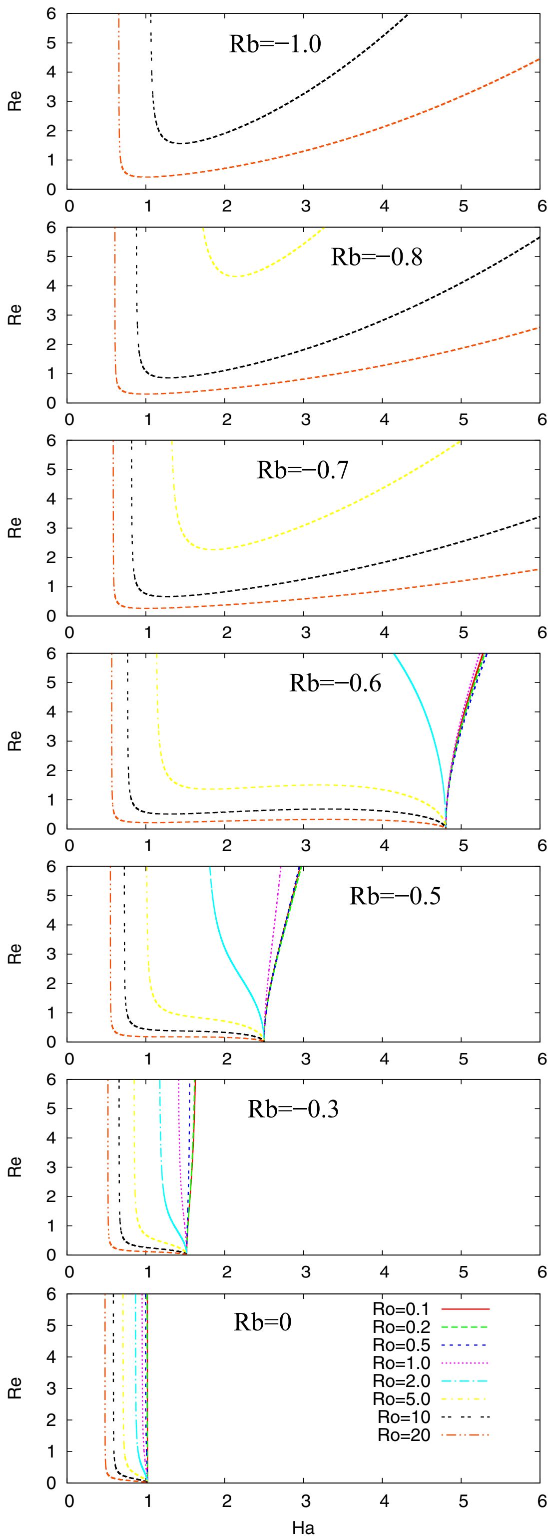

FIG. 2. (Color online) Marginal curves for $n = 1.2$ and various values of Rb, in dependence on Ro. From top to bottom, the instability changes its character from a (magnetically triggered) shear-driven instability to a (rotationally influenced) current-driven TI. For $n = 1.2$, TI appears first for $\mathrm{Rb} = n^2/4 - 1 = -0.64$.

In the limit $\mathrm{Ro} \to \infty$ the limit values of Ha converge slowly to zero according to $\mathrm{Ha}_{(\mathrm{Re,Ro})\to\infty} \simeq n^{-1/2}\,\mathrm{Ro}^{-1/4}$.

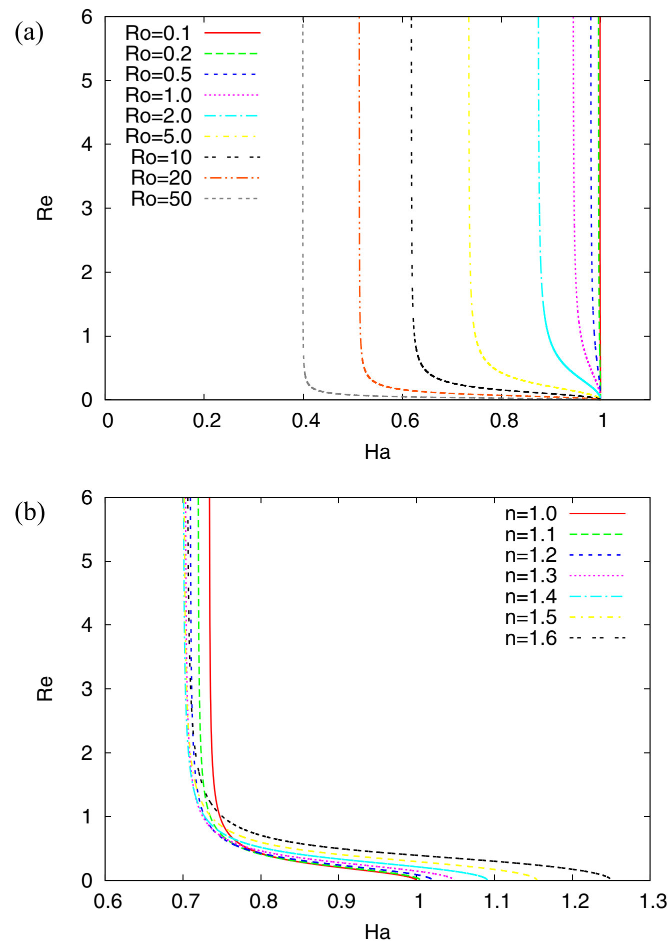

FIG. 3. (Color online) Marginal curves for $\mathrm{Rb} = 0$. (a) Dependence on Ro for $n = 1.0$. (b) Dependence on n for $\mathrm{Ro} = 5$.

In the following, we compare our WKB results with recent findings [19] obtained for a TC flow with inner and outer radii r_i and r_o rotating with the angular velocities Ω_i and Ω_o, respectively. The corresponding ratios are defined as $\hat{\eta} = r_i/r_o$, and $\hat{\mu} = \Omega_o/\Omega_i$. For this TC configuration, the following modified definitions of the Reynolds and Hartmann number were used: $\widehat{\mathrm{Re}} = \Omega_o r_i (r_o - r_i)/\nu$, $\widehat{\mathrm{Ha}} = B_\phi(r_i)[r_i(r_o - r_i)]^{1/2}/(\mu_0 \rho \nu \eta)^{1/2}$. The nontrivial point is now how to translate the $\hat{\mu}$ of a TC flow, characterized by $\Omega(r) = a + b/r^2$, to the Ro of a flow with $\Omega(r) \sim r^{2\,\mathrm{Ro}}$. An often used correspondence, based on equalizing the corresponding angular velocities at r_i and r_o [20], leads to

$$\mathrm{Ro}^* \simeq -1/2 \log_{\hat{\eta}} \hat{\mu}, \tag{8}$$

while an alternative, more shear-oriented version leads to

$$\mathrm{Ro}^{**} \simeq \frac{1}{2}\frac{(1+\hat{\eta})(\hat{\mu}-1)}{(1-\hat{\eta})(\hat{\mu}+1)}. \tag{9}$$

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In Ref. [19], the destabilizing effect of positive shear had been studied for TC flows (with $\mathrm{Rb} = 0$ only), both for a wide gap with $\hat{\eta} = 0.5$ as well as a narrow gap with $\hat{\eta} = 0.95$. In either case, for

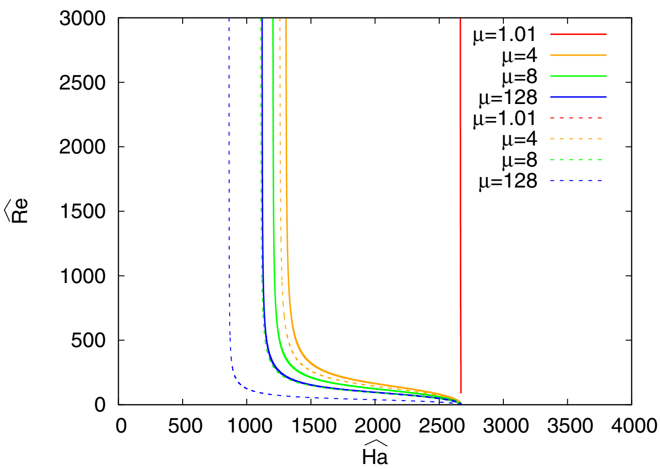

FIG. 4. (Color online) Marginal curve for $\mathrm{Rb} = 0$ and $n = 1.41$, as scaled according to Ref. [19]. The solid lines correspond to the translation of $\hat{\mu}$ to Ro**, and the dashed lines to Ro*.

large values of $\hat{\mu}$, the critical Ha converged to some nonzero constant, which is not compatible with the translation to Ro* since the latter should lead to a zero critical Ha (according to $\mathrm{Ha}_{(\mathrm{Re,Ro})\to\infty} \simeq n^{-1/2}\,\mathrm{Ro}^{-1/4}$—see above). It turns out that the translation to Ro** is physically more adequate.

With the reasonable choice $k_z = k_r = \pi/(r_i - r_o)$ we obtain the translations $\widehat{\mathrm{Re}} = \pi^2 2^{5/2}\hat{\mu}\hat{\eta}/[(1+\hat{\mu})(1-\hat{\eta})]\mathrm{Re}$ and $\widehat{\mathrm{Ha}} = \pi^2(1+\hat{\eta})^2/[(2\hat{\eta})^{1/2}(1-\hat{\eta})^{3/2}]\mathrm{Ha}$. For $\hat{\eta} = 0.95$ this amounts to $\widehat{\mathrm{Re}} = 1061/(1+1/\hat{\mu})\mathrm{Re}$ and $\widehat{\mathrm{Ha}} = 2435\,\mathrm{Ha}$. Figure 4 shows the corresponding WKB results, both for assuming a translation to Ro* (dashed lines) and to Ro** (solid lines). For $\widehat{\mathrm{Re}} = 0$ our result $\widehat{\mathrm{Ha}} = 2670$ agrees reasonably well with the exact value $\widehat{\mathrm{Ha}} = 3060$ of the modal stability analysis [19]. What is more, the typical bend of the marginal curve to the left for increasing $\widehat{\mathrm{Re}}$, and the limit values of $\widehat{\mathrm{Ha}}$ for large $\widehat{\mathrm{Re}}$, are also confirmed. Yet, subtle differences show up for the two ways of translation: The use of Ro** confirms the existence of a finite limit value for the critical $\widehat{\mathrm{Ha}}$, as typical for TC flows, while the use of Ro* would ultimately lead to a zero limit value.

This encouraging consistency of the local approximation and the modal stability analysis, evidenced for $\mathrm{Rb} = 0$, brings us back to the point whether, for $\mathrm{Rb} = -1$, the ULL can be confirmed in a TC experiment. Assuming Ro** as more physical than Ro*, in the limit $\hat{\mu} \to \infty$ we obtain $\mathrm{Ro}^{**}_{\hat{\mu}\to\infty} = 1/2(1+\hat{\eta})/(1-\hat{\eta})$. This means, in turn, that to emulate some Ro in a TC flow, $\hat{\eta}$ has to fulfill the relation $\hat{\eta} = (2\,\mathrm{Ro} - 1)/(2\,\mathrm{Ro} + 1)$. With a view on the ULL, this implies that for $\mathrm{Ro} = 6$, say, a minimum value of $\hat{\eta} = 11/13 = 0.846$ is needed. For TC flows with wider gaps, such as $\hat{\eta} = 1/2$, the necessary shear could simply not be realized.

What are, then, the prospects for a corresponding experiment? Evidently, we need a rather narrow gap flow. Let us stick, for a first estimate, to the safe value $\hat{\eta} = 0.95$, and take the typical values $\mathrm{Ro} = 6$, $\mathrm{Ha} = 2$, and $\mathrm{Re} = 12$ as read off from Fig. 1(a). This translates to $\hat{\mu} = 1.89$, $\widehat{\mathrm{Re}} = 8324$, and $\widehat{\mathrm{Ha}} = 4870$. For a prospective TC experiment with Na at 150 °C, with $\rho = 910\ \mathrm{kg/m^3}$, $\nu = 5.94 \times 10^{-7}\ \mathrm{m^2/s}$, $\sigma = 9 \times 10^6$ S/m, and an outer diameter of $r_o = 0.25$ m, this would amount to a rather moderate rotation frequency of $\Omega_o/(2\pi) = 0.26$ Hz, yet a huge magnetic field $B_\phi(r_i) = 0.69$ T that requires a central current of $I = 8.6 \times 10^5$ A. Exhausting the shear resources, by choosing $\hat{\mu} \to \infty$ and $\hat{\eta} = 0.85 \approx 11/13$, those values would drop to $\widehat{\mathrm{Re}} = 3796$, $\widehat{\mathrm{Ha}} = 892$, or, physically, to $\Omega_o/(2\pi) = 0.044$ Hz, $B_\phi(r_i) = 77$ mT, and $I = 8.2 \times 10^4$ A. Any real TC experiment, however, would need more detailed simulations with a 1D marginal stability code to confirm and optimize the parameters.

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