Eulerian Derivation of the Conservation Equation for Energy in a Non-Inertial Frame of Reference in Arbitrary Motion

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Abstract

The standard inertial Navier-Stokes equations consisting of the conservation equations for mass, momentum and energy are often used to investigate the motion of a compressible fluid around an object that is in arbitrary motion. The non-inertial form of the Navier-Stokes equations can be used to accurately capture the acceleration effects that arise from the unsteady motion. The acceleration source terms that arise in the conservation equation for momentum have been extensively documented. In this paper, an Eulerian approach for deriving the apparent forces is presented to transform the governing conservation equation for energy into a non-inertial reference frame that is in arbitrary motion. The Eulerian approach is based on successive Galilean transformations between an inertial frame, an orientation-preserving non-inertial frame and a rotating non-inertial frame. The paper demonstrates that for an object in arbitrary motion, the rate of work done due to fictitious forces affects the rate of change of the total energy. The fictitious work arises in the kinetic energy equation while the internal energy and enthalpy equations remain invariant in the non-inertial frame. The present derivation is a step towards quantifying the contribution of the fictitious work terms to the heat transfer of a body that is accelerating/decelerating.

Keywords: Accelerating flow, Non-Inertial Frame, Navier-Stokes, Fictitious acceleration, Fictitious work

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1. Introduction

Acceleration and deceleration are often encountered in a multitude of engineering applications including, helicopter rotors in hover and forward flight [1], missile aerodynamics [2], manoeuvring fighter planes during acceleration and deceleration [3], aerodynamics around a decelerating Formula 1 and various high performance cars [4] and aero-ballistic ordinance [5]. Accelerating/decelerating motion is considered here to combine unsteady translation and unsteady rotation of an object immersed in a compressible fluid [2, 3, 6, 7, 8, 9]. This can also include combined oscillating translation and rotation [10, 11, 12].

In all the aerodynamic applications mentioned, the unsteady flow field due to the acceleration/deceleration can have a significant impact on the aerodynamics and flight dynamics of the object under consideration. For example, the unsteady flow field around accelerating objects in a compressible fluid were investigated by Roohani [13], Roohani and Skews [14], Mahomed et. al [15], Roohani et. al [16]. It was found that the unsteady flow field was significantly different from the steady state flow field at the same Mach number for subsonic and transonic flows. Subsonic lift was lower and subsonic drag was higher than the steady state profile during acceleration while the reverse was observed for deceleration. The boundary layer characteristics over bodies moving in arbitrary motion were investigated numerically in Combrinck [17]. Steeper velocity profiles normal to the wall were observed in acceleration and flow separation was observed in deceleration. The skin friction, momentum thickness and displacement thickness under subsonic flow were shown also to increase in the near wall region. In the above studies, the effects observed increased with an increase in the acceleration magnitude. Thus in applications where the acceleration/deceleration is significant, it becomes important to not only investigate the transient effects but also the acceleration effects that result from the acceleration of the objects centre of mass [3].

In the inertial frame, an object accelerating/decelerating in a steady compressible fluid can be observed to undergo deflections. These deflections can be explained from the non-inertial frame. In this frame, the Navier-Stokes equations includes fictitious acceleration source terms [18, 19, 20] which capture the acceleration effects that arise from the unsteady motion. A Lagrangian fluid parcel concept is often used to obtain the acceleration in the non-inertial reference frame [18, 20]. An alternative approach based on the Eulerian formulation for deriving the apparent forces was introduced by Kageyama and Hyodo [21].

The conservation equation for mass remains unchanged when transformed from an inertial frame to a non-inertial frame of reference both for incompressible [6, 21, 22] and a compressible fluid [3, 5, 7, 8, 9, 11, 12]. The type of motion is insignificant when considering the conservation equation for mass as can be seen by the various studies that have looked at various cases, pure steady rotation [21, 22], unsteady pure rotation [7] and arbitrary motion [3, 5, 6, 8, 9].

The conservation equation for momentum in the non-inertial frame consists of additional fictitious acceleration terms [3, 5, 6, 7, 8, 9, 11, 12, 21]. These additional acceleration source terms are dependent on the type of motion considered. In pure steady rotation for both an incompressible and a compressible fluid, the acceleration source terms are the Coriolis and Centrifugal terms [21, 22]. Studies that have used an Eulerian approach to transform the governing conservation equations for momentum into a rotational frame include [21, 22]. In pure unsteady rotation, the fictitious acceleration source terms in the conservation equation for momentum are the Coriolis, Centrifugal and Euler or unsteady rotation term [2, 3, 7]. In full arbitrary motion, the fictitious acceleration source terms are the Coriolis, Centrifugal, Euler, acceleration due to translation [2, 3, 6, 10, 12] and in the context of aero-ballistic motion, two additional terms were obtained by Combrinck et. al [5], these are the Magnus acceleration term and the acceleration term due to an unsteady rotation axes.

The conservation equation for energy in a non-inertial frame has been considered in the case of pure steady rotation [23], pure unsteady rotation [24] and arbitrary motion [3, 9, 10, 11]. The work-energy theorem for pure steady rotation was shown by Diaz et. al [23] to remain covariant under Galilean transformation if the work done by fictitious forces is included correctly. Not all the acceleration source terms that appear in the conservation equation for momentum contribute to the work done in the non-inertial frame. It was found that the Coriolis force does not contribute to the work done by fictitious forces in the non-inertial frame thus care is to be taken when including fictitious work terms in the work-energy theorem.

Extension to non-uniform rotation of the reference frame was done by Manjarres et. al [24]. The studies of Diaz et. al. [23] and Manjarres et. al. [24] were based on a system of point particles and did not consider the work energy theorem in situations where there is combined translation and rotation. In arbitrary motion, the total energy in the non-inertial frame was determined using a Lagrangian approach in the studies of Cariglino [11] and Gardi [10]. The scalar product of the conservation equation for momentum with velocity in the non-inertial arbitrarily moving reference frame was used. The Eulerian formulation was used by Combrinck et. al. [5] to show that the non-inertial frame internal energy equation remained Galilean invariant under transformation from the inertial frame. The study of Combrinck et. al. [5] made a claim to inconsistencies with the total energy equation in [10, 11]. The total energy equation and rothalpy form were derived by Gledhill et. al. [3] using general transforms. Fictitious works were found to appear in the non-inertial frame and similar to Manjarres [24] the Coriolis was found to not contribute to the work done in the non-inertial frame.

While the conservation equation for momentum has been extensively documented in the literature, studies that formulate the conservation equation for energy in the non-inertial frame are few. Given that fictitious acceleration source terms arise as a consequence of the arbitrary motion of the reference frame in the conservation equation for momentum, there is a need to determine in the most general form the conservation equation for energy in an arbitrarily moving frame of reference. To cover all forms of energy, there is a need to verify whether the kinetic energy and enthalpy forms of the energy equation contain any fictitious work terms.

In this paper, an Eulerian approach for deriving the apparent forces is presented to transform the governing conservation equation for energy into a non-inertial reference frame that is in arbitrary motion. The Eulerian approach is based on successive Galilean transformations between an inertial frame, an orientation-preserving non-inertial frame and a rotating non-inertial frame. The Eulerian formulation introduced by Kageyama and Hyodo [21] provides the Eulerian relationships between inertial frame scalars, vectors and vector time derivatives to their counterparts in the non-inertial frame. These relationships can be used to formulate the equations of other physical systems other than fluids in a non-inertial frame. Kageyama and Hyodo demonstrated the general nature of their Eulerian formulation by transforming the induction equation for the magnetic field of Maxwell's equations to a rotating frame [21]. The physical meaning of the derivation is clear, the equations in the rotating reference frame are obtained by applying a local Galilean transformation of inertial frame quantities followed by a projection of the quantities to the rotating frame. The method is mathematically rigorous. In Sect. 2, the transformation between reference frames is introduced. The conservation equation for energy, kinetic, enthalpy and internal energy equations are transformed into the non-inertial frame in Sect. 3. The limiting cases of the conservation equation of energy are discussed in Sect. 4.

2. Reference Frame Transformations

Consider three reference frames O, O' and $\hat{O}$ as depicted in Figure 1

1. Frame O is stationary and considered to be the inertial frame
2. Frame O' is an orientation preserving frame. Its unit vectors are parallel to those of frame O. Frame O' is moving relative to the origin of frame O with velocity $\mathbf{V}_{rel}$ and is free to translate in all three coordinate directions. This reference frame can either be inertial or non-inertial depending on whether the translation velocity of O' is steady or unsteady.
3. Frame $\hat{O}$ is the non-inertial rotational frame and shares an origin with frame O'. It translates relative to frame O with velocity$\mathbf{V}_{rel}$ and rotates about frame O' with angular velocity Ω. This reference frame exhibits full 6 degrees of freedom motion.

At an initial time $t = t_i$, both frames O' and $\hat{O}$ are translating with a steady velocity $\mathbf{V}_{rel}$ relative to frame O. At a subsequent time $t = t_i + \Delta t$, frame O' and $\hat{O}$ are accelerating relative to frame O with a relative acceleration $\mathbf{a}_{rel}$ and are at a distance $\mathbf{x}_{rel}$ relative to frame O. Frame $\hat{O}$ has also undergone arbitrary rotation about the origin of frame O'. Consider a point P accelerating/decelerating in both rotation and translation in frame

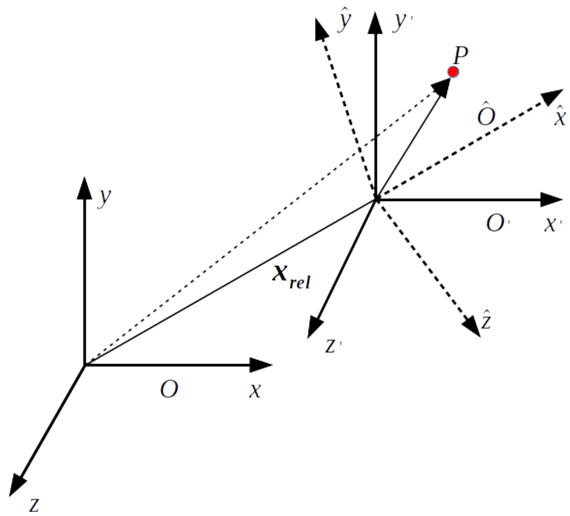

Figure 1: Coordinate position of point P in the non-inertial reference frames O' and $\hat{O}$ relative to the stationary frame O

$\hat{O}$. Point P rotates about the origin of frame O'. The coordinate position of point P can be described in all three reference frames. Therefore the equations governing the imposed motion on point P can be established in all three reference frames.

Let $\mathbf{x}, \mathbf{x}'$ and $\hat{\mathbf{x}}$ denote the position vectors in the inertial frame O, orientation preserving frame O' and the non-inertial frame $\hat{O}$ respectively. In the inertial frame, the conservation equations for mass, momentum and energy hold. The form of these governing equations in frame $\hat{O}$ is sought. Two transformations between the inertial frame and the non-inertial frame are done to resolve the non-inertial effects experienced by point P. Firstly, a local Galilean transformation modified for unsteady motion is used to transform the governing equations from the inertial frame O to the orientation preserving frame O'. This transformation considers the relative unsteady translation between frames O and O'. Secondly, a rotational transform is used to project the equations in frame O' to frame $\hat{O}$. This second transformation considers the arbitrary rotation that frame $\hat{O}$ undergoes.

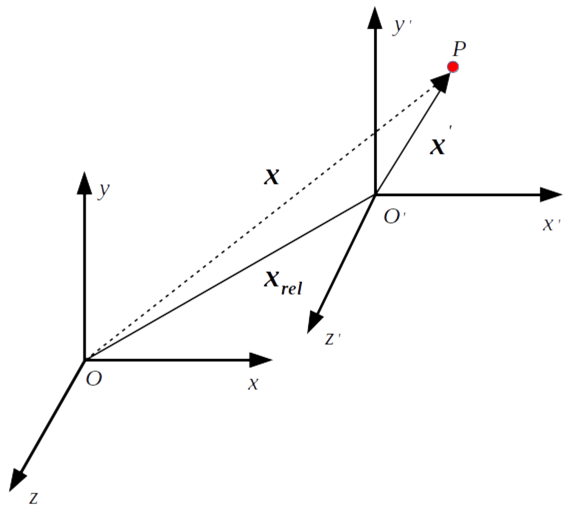

Figure 2: Coordinate position of point P in frame O' relative to the stationary frame O

2.1. Local Modified Galilean Transformation

The non-inertial coordinate positions of point P can be obtained by transforming the inertial coordinates firstly from frame O to frame O' using a local Galilean transformation. A rotational transform is then used to project the coordinates from frame O' to the non-inertial frame $\hat{O}$. Firstly, we seek the inertial position of point P relative to frame O as depicted in Figure (2);

$$\mathbf{x} = \mathbf{x}_{rel} + \mathbf{x}'. \tag{1}$$

Here, $\mathbf{x}_{rel}$ is the relative position between frames O and O', for constant acceleration this is given by the equation of motion,

$$\mathbf{x}_{rel} = \mathbf{V}_{rel}\Delta t + \frac{1}{2}\mathbf{a}_{rel}\Delta t^2. \tag{2}$$

Here $\mathbf{V}_{rel}$ is the relative velocity between frames O and O'. For unsteady combined translation and rotation, the relative velocity is composed of a time dependent translation and a rotation component,

$$\mathbf{V}_{rel} = \mathbf{V}(t) + \Omega \times \mathbf{x}'. \tag{3}$$

Here $\mathbf{V}(t)$ is the translation velocity of frame O' relative to frame O and Ω is the time dependent angular velocity of frame $\hat{O}$ about the origin of frame O'. The relative acceleration is obtained by taking the time derivative of the relative velocity

$$\begin{aligned} \mathbf{a}_{rel} &= \frac{\partial \mathbf{V}_{rel}}{\partial t}, \\ &= \underbrace{\frac{\partial \mathbf{V}(t)}{\partial t}}_{\text{Translation}} + \underbrace{\dot{\Omega} \times \mathbf{x}'}_{\text{Euler}} + \underbrace{\Omega \times \dot{\mathbf{x}}'}_{\text{Unsteady Axes}} \quad . \end{aligned} \tag{4}$$

The relative acceleration consists of the acceleration due to translation, Euler or rotational acceleration and the acceleration of the centre of rotation [5]. In the derivation by Gledhill et. al. [3], the relative acceleration is given by the translation acceleration and the Euler acceleration term, the acceleration of the centre of rotation is not considered.

Substituting the expressions for the relative velocity (3) and relative acceleration (4), the relative position of frame O' from equation (2) is given by,

$$\mathbf{x}_{rel} = \left(\mathbf{V}(t) + \Omega \times \mathbf{x}'\right)\Delta t + \frac{1}{2}\left(\frac{\partial \mathbf{V}(t)}{\partial t} + \dot{\Omega} \times \mathbf{x}' + \Omega \times \dot{\mathbf{x}}'\right)\Delta t^2. \tag{5}$$

With the relative position, the coordinates of the origin of frame O' from equation (1) are given by

$$\mathbf{x}' = \mathbf{x} - \left(\mathbf{V}(t) + \Omega \times \mathbf{x}' + \frac{1}{2}\left(\frac{\partial \mathbf{V}(t)}{\partial t} + \dot{\Omega} \times \mathbf{x}' + \Omega \times \dot{\mathbf{x}}'\right)\Delta t\right)\Delta t. \tag{6}$$

The modified local Galilean transform is used to transform the velocity from frame O to frame O' from equation (6). The modified local Galilean transform between the inertial frame O and the orientation preserving frame O' moving at relative velocity $\mathbf{V}_{rel}$ is given by

$$\mathbf{u}'(\mathbf{x}',t) = \mathbf{G}^{\mathbf{V}_{rel}}\mathbf{u}(\mathbf{x},t). \tag{7}$$

The velocity of the origin of frame O' is thus

$$\begin{aligned} \mathbf{u}'(\mathbf{x}',t) &= \mathbf{G}^{\mathbf{V}_{rel}}\mathbf{u}(\mathbf{x},t), \\ &= \mathbf{u}(\mathbf{x},t) - \mathbf{V}(t) + \mathbf{x}' \times \Omega - \left(\frac{\partial \mathbf{V}(t)}{\partial t} + \dot{\Omega} \times \mathbf{x}' + \Omega \times \dot{\mathbf{x}}'\right)\Delta t. \end{aligned} \tag{8}$$

Equation (8) gives the velocity of the orientation preserving frame in-terms of the inertial frame velocity and the positions in the orientation preserving frame.

2.2. Rotational Transformation

The velocity of the origin of frame O' is projected onto the rotational frame $\hat{O}$ with an angle θ. The angle of projection is determined by taking into account contributions from the the angular velocity Ω and the angular acceleration $\dot{\Omega}$ at a time t

$$\theta = \Omega t + \dot{\Omega} t^2. \tag{9}$$

The matrix representations of the rotational transform between the orientation preserving frame O' and the non-inertial frame $\hat{O}$ in the three coordinate directions x, y, z are given by

$$\mathbf{R}_x^\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix}, \tag{10}$$

$$\mathbf{R}_y^\theta = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}, \tag{11}$$

$$\mathbf{R}_z^\theta = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

The rotation $\mathbf{R}^\theta$ in multiple directions is given by a product of the individual matrices depending on the order of transformation. The velocity of the origin of frame $\hat{O}$ is thus given by

$$\begin{aligned} \hat{\mathbf{u}}(\hat{\mathbf{x}}, t) &= \mathbf{R}^\theta \mathbf{u}'(\mathbf{x}', t), \\ &= \mathbf{R}^\theta \mathbf{G}^{\mathbf{V}_{rel}} \mathbf{u}(\mathbf{x}, t), \\ &= \mathbf{R}^\theta \left(\mathbf{u}(\mathbf{x}, t) - \mathbf{V}(t) + \mathbf{x}' \times \Omega - \left(\frac{\partial \mathbf{V}(t)}{\partial t} + \dot{\Omega} \times \mathbf{x}' + \Omega \times \dot{\mathbf{x}}' \right) \Delta t \right). \end{aligned} \tag{13}$$

2.3. Transformation of Operations on Vector and Scalar Fields

The equations of fluid motion consist of differential operations on vectors and scalars. These relationships have been derived rigorously and used in the following works [3, 5, 7, 12, 21]. Let $\phi(\mathbf{x}, t)$ be a scalar and $\Phi(\mathbf{x}, t)$ be a vector, the following relations hold;

1. Local Galilean invariance of a scalar field,

$$\hat{\phi}(\hat{\mathbf{x}},t) = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\phi(\mathbf{x},t) = \mathbf{R}^{\theta}\phi(\mathbf{x},t). \tag{14}$$

2. Local Galilean transformation of the time derivative of a scalar field,

$$\left(\frac{\partial\hat{\phi}(\hat{\mathbf{x}},t)}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial\phi(\mathbf{x},t)}{\partial t} + \mathbf{V}_{rel}\cdot\nabla\phi(\mathbf{x},t)\right). \tag{15}$$

3. Local Galilean transform of the time derivative of a vector field,

$$\left(\frac{\partial\hat{\Phi}(\hat{\mathbf{x}},t)}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial}{\partial t} + \mathbf{V}_{rel}\cdot\nabla - \Omega\times\right)\Phi'(\mathbf{x}',t). \tag{16}$$

4. Galilean invariance of the nabla operator,

$$\hat{\nabla} = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\nabla = \mathbf{R}^{\theta}\nabla, \tag{17}$$

$$\hat{\nabla} = \frac{\partial}{\partial\hat{x}}\hat{\mathbf{i}} + \frac{\partial}{\partial\hat{y}}\hat{\mathbf{j}} + \frac{\partial}{\partial\hat{z}}\hat{\mathbf{k}}. \tag{18}$$

3. Conservation Equation for Energy

The non-inertial total energy, kinetic energy, enthalpy and internal energy equations for laminar compressible fluid flow are derived in this section using the Eulerian approach introduced in [21] and expanded upon in [5, 7]. This is archived by transformation of the individual terms of the equations in the inertial frame using the modified local Galilean transformation and Rotational transformation between the inertial frame and the non-inertial arbitrarily moving frame as discussed in the previous section.

3.1. Conservation Equation for Energy in Inertial Stationary Frame

The conservative form of the conservation equation for energy in an inertial frame is given by

$$\frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\mathbf{u}) = \nabla\cdot(k\nabla T) - \nabla\cdot(p\mathbf{u}) + \nabla\cdot(\tau\cdot\mathbf{u}) + \rho\mathbf{f}\cdot\mathbf{u}. \tag{19}$$

Here, E is the total energy given here in-terms of the internal energy e

$$E = e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}, \tag{20}$$

ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure of the fluid, k is the thermal conductivity, T is the temperature, τ is the viscous stress tensor and $\mathbf{f}$ is the body force. Substituting for E into equation (19)

$$\frac{\partial \rho e}{\partial t}+\frac{1}{2}\frac{\partial\left(\rho\mathbf{u}\cdot\mathbf{u}\right)}{\partial t}+\nabla\cdot\left(\rho e\mathbf{u}\right)+\frac{1}{2}\nabla\cdot\left(\rho\mathbf{u}\left(\mathbf{u}\cdot\mathbf{u}\right)\right)=\nabla\cdot\left(k\nabla T\right)-\nabla\cdot\left(p\mathbf{u}\right) +\nabla\cdot\left(\tau\cdot\mathbf{u}\right)+\rho\mathbf{f}\cdot\mathbf{u}. \tag{21}$$

The fourth term on the left hand side of the energy equation can be written in the form of a matrix vector expression using matrix products,

$$\nabla\cdot\left(\rho\mathbf{u}\left(\mathbf{u}\cdot\mathbf{u}\right)\right)=\nabla\cdot\left(\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right). \tag{22}$$

Similarly, the seventh term with the viscous stress tensor can be decomposed into two terms using the properties of the divergence operator,

$$\nabla\cdot\left(\tau\cdot\mathbf{u}\right)=\left(\nabla\cdot\tau\right)\cdot\mathbf{u}+\tau:\,\mathrm{tr}\left(\nabla\mathbf{u}\right). \tag{23}$$

Here the second term of equation (23) is the positive definite dissipation function Φ formed from a double inner product of the viscous stress tensor and the trace $tr()$ of the gradient of the velocity,

$$\Phi=\tau:\,\mathrm{tr}\left(\nabla\mathbf{u}\right). \tag{24}$$

The total energy equation in the inertial frame can thus be written as

$$\frac{\partial \rho e}{\partial t}+\frac{1}{2}\frac{\partial\left(\rho\mathbf{u}\cdot\mathbf{u}\right)}{\partial t}+\nabla\cdot\left(\rho e\mathbf{u}\right)+\frac{1}{2}\nabla\cdot\left(\rho\mathbf{u}\cdot\mathbf{u}\otimes\mathbf{u}\right)=\nabla\cdot\left(k\nabla T\right)-\nabla\cdot\left(p\mathbf{u}\right) +\left(\nabla\cdot\tau\right)\cdot\mathbf{u}+\Phi+\rho\mathbf{f}\cdot\mathbf{u}. \tag{25}$$

The time rate of change of the total energy is due here to heat addition and the rate of work done. Heat addition is due to thermal conduction. The pressure, viscous stresses and body forces contribute to the rate of work done.

3.2. Energy Conservation Equation in Non-Inertial Frame

The individual terms of the total energy equation (25) will be transformed in the sections that follow.

3.2.1. Non-Inertial Total Energy

Equations (13) and (14) can be used to transform equation (20) to the non-inertial frame

$$\begin{aligned}
\hat{E} &= \hat{e} + \frac{1}{2}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}, \\
&= \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right), \\
&= \mathbf{R}^{\theta}\left(e + \frac{1}{2}\left(\mathbf{u} - \mathbf{V}_{rel} - \mathbf{a}_{rel}\Delta t\right)\cdot\left(\mathbf{u} - \mathbf{V}_{rel} - \mathbf{a}_{rel}\Delta t\right)\right). \qquad (26)
\end{aligned}$$

In the present derivation, the assumption is that Δt is small. With this assumption the higher order acceleration terms become negligible in the limit as $\Delta t \to 0$. In the limit of $\Delta t \to 0$ equation (26) reduces to,

$$\hat{E} = \mathbf{R}^{\theta}\left(E + \frac{1}{2}\left(-\mathbf{u}\cdot\mathbf{V}_{rel} - \mathbf{V}_{rel}\cdot\mathbf{u} + \mathbf{V}_{rel}\cdot\mathbf{V}_{rel}\right)\right). \qquad (27)$$

The total energy in the non-inertial frame (27) is a sum of the total energy in the inertial frame E with the kinetic energy in the non-inertial frame $\mathbf{V}_{rel}\cdot\mathbf{V}_{rel}$. The other two terms in (27) take into account the movement of frame $\hat{O}$ relative to frame O. The two terms are as a result of observing the motion either from frame O or from frame O', thus taking into account the inter frame kinetic energy.

3.2.2. Temporal Terms

The two temporal terms give the rate of change of the total energy. The first term can be transformed using the product rule for partial derivatives and the modified local Galilean transformation of the time derivative of a scalar product,

$$\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\mathbf{G}^{\mathbf{V}_{rel}}e\frac{\partial\rho}{\partial t} + \mathbf{G}^{\mathbf{V}_{rel}}\rho\frac{\partial e}{\partial t}\right). \qquad (28)$$

Using equations (14) and (15) and the product rule, equation (28) transforms into

$$\begin{aligned}
\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} &= \mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t} + \mathbf{V}_{rel}\cdot\nabla\rho e\right), \qquad (29) \\
&= \mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t} + \nabla\cdot\left(\rho e\mathbf{V}_{rel}\right)\right). \qquad (30)
\end{aligned}$$

Equation (29) can be reduced to equation (30) as a consequence of the divergence of the relative velocity being zero as shown by Combrinck and Gledhill [3, 5],

$$\nabla\cdot\mathbf{V}_{rel} = 0. \qquad (31)$$

It is evident that in the non-inertial frame, the time derivative of a scalar is given by the convective derivative of inertial frame quantities with the convective velocity $\mathbf{V}_{rel}$. This implies that if the non-inertial frame is rotating or translating, the time derivative has to be corrected for the convection of the flow that results due to the translation or rotation.

Using the product rule the second temporal term in the non-inertial frame can be simplified to

$$\frac{1}{2}\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \frac{1}{2}\mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\frac{\partial \rho\mathbf{u}}{\partial t}\cdot\mathbf{u} + \rho\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial t}\right). \tag{32}$$

Equation (32) requires the evaluation of the time derivative of the velocity,

$$\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\frac{\partial \mathbf{u}}{\partial t}\right). \tag{33}$$

Applying the local Galilean transformation using equations (13) and (16)

$$\begin{aligned}\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = {} & \mathbf{R}^{\theta}\left(\frac{\partial}{\partial t} + \mathbf{V}_{rel}\cdot\nabla - \Omega\times\right)\mathbf{u} \\ & - \mathbf{R}^{\theta}\left(\frac{\partial}{\partial t} + \mathbf{V}_{rel}\cdot\nabla - \Omega\times\right)\mathbf{V}_{rel} \\ & - \mathbf{R}^{\theta}\left(\frac{\partial}{\partial t} + \mathbf{V}_{rel}\cdot\nabla - \Omega\times\right)\mathbf{a}_{rel}\Delta t. \end{aligned} \tag{34}$$

It has been shown in [3, 5, 21] that for any vector $\mathbf{b}$,

$$\mathbf{b}\cdot\nabla\mathbf{V}_{rel} = -\mathbf{b}\times\Omega. \tag{35}$$

The result (35) is used here to simplify equation (34). Thus in the limit as $\Delta t \to 0$ equation (34) reduces to

$$\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V}_{rel}\cdot\nabla\mathbf{u} - \frac{\partial \mathbf{V}_{rel}}{\partial t} - \Omega\times\mathbf{u}\right). \tag{36}$$

The last part of equation (36) is part of the Coriolis acceleration term, this can be rewritten using equation (35) as

$$\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V}_{rel}\cdot\nabla\mathbf{u} - \frac{\partial \mathbf{V}_{rel}}{\partial t} - \mathbf{u}\cdot\nabla\mathbf{V}_{rel}\right). \tag{37}$$

The time derivative of the product of the density and velocity is now sought, by the product rule

$$\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \left(\frac{\partial \hat{\rho}}{\partial t}\right)_{\hat{O}}\hat{\mathbf{u}} + \hat{\rho}\left(\frac{\partial \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}. \tag{38}$$

Using equations (13), (14), (15) and (37) in equation (38) and taking the limit as $\Delta t \to 0$

$$\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial \rho \mathbf{u}}{\partial t} - \frac{\partial \rho \mathbf{V}_{rel}}{\partial t} + \left(\nabla \cdot \rho \mathbf{V}_{rel}\right)\left(\mathbf{u} - \mathbf{V}_{rel}\right) + \rho \mathbf{V}_{rel} \cdot \nabla \mathbf{u} - \rho \mathbf{u} \cdot \nabla \mathbf{V}_{rel}\right). \tag{39}$$

The third term in (39) can be expanded and the equation re-written in the form

$$\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = \mathbf{R}^{\theta}\left(\frac{\partial \rho \mathbf{u}}{\partial t} - \frac{\partial \rho \mathbf{V}_{rel}}{\partial t} + \nabla \cdot \left(\rho \mathbf{V}_{rel} \otimes \left(\mathbf{u} - \mathbf{V}_{rel}\right)\right) - \rho\left(\mathbf{u} - \mathbf{V}_{rel}\right) \cdot \nabla \mathbf{V}_{rel}\right). \tag{40}$$

The second complete temporal term can now be evaluated by combining the results of equation (35), (36), (40) and collecting terms using the product rule

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = & \frac{1}{2}\mathbf{R}^{\theta}\left(\frac{\partial \rho \mathbf{u} \cdot \mathbf{u}}{\partial t} - \frac{\partial \rho \mathbf{u} \cdot \mathbf{V}_{rel}}{\partial t} - \frac{\partial \rho \mathbf{V}_{rel} \cdot \mathbf{u}}{\partial t} + \frac{\partial \rho \mathbf{V}_{rel} \cdot \mathbf{V}_{rel}}{\partial t}\right) \\
& + \frac{1}{2}\mathbf{R}^{\theta}\left(\left(\nabla \cdot \rho \mathbf{V}_{rel} \otimes \mathbf{u}\right) \cdot \left(\mathbf{u} - \mathbf{V}_{rel}\right) - \left(\nabla \cdot \rho \mathbf{V}_{rel} \otimes \mathbf{V}_{rel}\right) \cdot \left(\mathbf{u} - \mathbf{V}_{rel}\right)\right) \\
& + \frac{1}{2}\mathbf{R}^{\theta}\left(\rho\left(\mathbf{u} - \mathbf{V}_{rel}\right) \cdot \left(\mathbf{V}_{rel} \cdot \nabla \mathbf{u}\right)\right) \\
& - \frac{1}{2}\mathbf{R}^{\theta}\left(\rho \mathbf{V}_{rel} \cdot \mathbf{u} \times \Omega\right).
\end{aligned} \tag{41}$$

The two temporal terms transformed into the non-inertial frame are given by

$$\begin{aligned}
\left(\frac{\partial \hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} + \frac{1}{2}\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} = & \mathbf{R}^{\theta}\left(\frac{\partial \rho e}{\partial t} + \nabla \cdot \left(\rho e \mathbf{V}_{rel}\right)\right) \\
& + \frac{1}{2}\mathbf{R}^{\theta}\left(\frac{\partial \rho \mathbf{u} \cdot \mathbf{u}}{\partial t} - \frac{\partial \rho \mathbf{u} \cdot \mathbf{V}_{rel}}{\partial t} - \frac{\partial \rho \mathbf{V}_{rel} \cdot \mathbf{u}}{\partial t} + \frac{\partial \rho \mathbf{V}_{rel} \cdot \mathbf{V}_{rel}}{\partial t}\right) \\
& + \frac{1}{2}\mathbf{R}^{\theta}\left(\left(\nabla \cdot \rho \mathbf{V}_{rel} \otimes \mathbf{u}\right) \cdot \left(\mathbf{u} - \mathbf{V}_{rel}\right) - \left(\nabla \cdot \rho \mathbf{V}_{rel} \otimes \mathbf{V}_{rel}\right) \cdot \left(\mathbf{u} - \mathbf{V}_{rel}\right)\right) \\
& + \frac{1}{2}\mathbf{R}^{\theta}\left(\rho\left(\mathbf{u} - \mathbf{V}_{rel}\right) \cdot \left(\mathbf{V}_{rel} \cdot \nabla \mathbf{u}\right)\right) \\
& - \frac{1}{2}\mathbf{R}^{\theta}\left(\rho \mathbf{V}_{rel} \cdot \mathbf{u} \times \Omega\right).
\end{aligned} \tag{42}$$

Both temporal terms have additional terms when they are transformed from the inertial frame O to the non-inertial frame $\hat{O}$. The rate of change of the internal energy has an additional term that accounts for the convection of the internal energy by the relative velocity. The time rate of change of the kinetic energy term in the non-inertial frame consists of the time rate of change of the non-inertial kinetic energy corrected for the convection between the frames that is due to the relative velocity $\mathbf{V}_{rel}$.

3.2.3. Diffusion Terms

In this section the two diffusion terms on the left hand side of equation (25) will be transformed. The first diffusion term is transformed into the non-inertial frame using (13) and (14)

$$\begin{aligned}\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} &= \mathbf{R}^{\theta}\left(\mathbf{G}^{\mathbf{V}_{rel}}\nabla\cdot(\rho e\mathbf{u})\right),\\ &= \mathbf{R}^{\theta}\left(\nabla\cdot\rho e\mathbf{u}-\nabla\cdot\rho e\mathbf{V}_{rel}-\nabla\cdot\rho e\mathbf{a}_{rel}\Delta t\right).\end{aligned} \tag{43}$$

Taking the limit as $\Delta t\to 0$, equation (43) simplifies to

$$\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} = \mathbf{R}^{\theta}\left(\nabla\cdot\rho e\mathbf{u}-\nabla\cdot\rho e\mathbf{V}_{rel}\right). \tag{44}$$

The internal energy flux in the non-inertial frame is given by the difference between the internal energy flux in the inertial frame to the internal energy flux due to the relative velocity. The transformation of the second diffusion term to the non-inertial frame is given by

$$\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) = \frac{1}{2}\mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u})\right). \tag{45}$$

Since the outer product $\mathbf{u}\otimes\mathbf{u}$ is a matrix, equation (45) can be decomposed using the properties of the divergence operator

$$\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) = \frac{1}{2}\mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left((\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{u}+\rho\mathbf{u}\otimes\mathbf{u}\colon\operatorname{tr}(\nabla\mathbf{u})\right). \tag{46}$$

Evaluating the local Galilean transform of equation (46) in the limit that $\Delta t\to 0$

$$\begin{aligned}\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) &= \frac{1}{2}\mathbf{R}^{\theta}\left((\nabla\cdot\rho(\mathbf{u}-\mathbf{V}_{rel})\otimes(\mathbf{u}-\mathbf{V}_{rel}))\cdot(\mathbf{u}-\mathbf{V}_{rel})\right)\\ &\quad+\frac{1}{2}\mathbf{R}^{\theta}\left(\rho(\mathbf{u}-\mathbf{V}_{rel})\otimes(\mathbf{u}-\mathbf{V}_{rel}):\operatorname{tr}(\nabla(\mathbf{u}-\mathbf{V}_{rel}))\right).\end{aligned} \tag{47}$$

Expanding out the matrix products in equation (47) and taking note that for any symmetric matrix $\mathbf{T}$ and antisymmetric matrix $\nabla\mathbf{V}_{rel}$, $\mathbf{T}\colon\operatorname{tr}(\nabla\mathbf{V}_{rel})=0$,

$$\begin{aligned}\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) &= \frac{1}{2}\mathbf{R}^{\theta}\left((\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{u}-(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{V}_{rel}\right)\\ &\quad-\frac{1}{2}\mathbf{R}^{\theta}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{V}_{rel}+\nabla\cdot\rho\mathbf{V}_{rel}\otimes\mathbf{u}-\nabla\cdot\rho\mathbf{V}_{rel}\otimes\mathbf{V}_{rel}\right)\cdot(\mathbf{u}-\mathbf{V}_{rel})\\ &\quad+\frac{1}{2}\mathbf{R}^{\theta}\rho\left(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{V}_{rel}-\mathbf{V}_{rel}\otimes\mathbf{u}+\mathbf{V}_{rel}\otimes\mathbf{V}_{rel}\right):\operatorname{tr}(\nabla\mathbf{u}).\end{aligned} \tag{48}$$

Equation (46) is used to re-combine the two terms of the diffusion term,

$$\begin{aligned}
\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) &= \frac{1}{2}\mathbf{R}^{\theta}\left(\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)-\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\right)\cdot\mathbf{V}_{rel}\right)\\
&-\frac{1}{2}\mathbf{R}^{\theta}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{V}_{rel}+\nabla\cdot\rho\mathbf{V}_{rel}\otimes\mathbf{u}-\nabla\cdot\rho\mathbf{V}_{rel}\otimes\mathbf{V}_{rel}\right)\cdot\left(\mathbf{u}-\mathbf{V}_{rel}\right)\\
&+\frac{1}{2}\mathbf{R}^{\theta}\rho\left(-\mathbf{u}\otimes\mathbf{V}_{rel}-\mathbf{V}_{rel}\otimes\mathbf{u}+\mathbf{V}_{rel}\otimes\mathbf{V}_{rel}\right):\operatorname{tr}\left(\nabla\mathbf{u}\right).
\end{aligned}\tag{49}$$

At this stage the left hand side of the total energy equation in the non-inertial frame can be evaluated using equations (42), (44), (49) and taking into consideration that for any vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ we can write $\mathbf{a}\otimes\mathbf{b}:\operatorname{tr}\left(\nabla\mathbf{c}\right)=\mathbf{a}\cdot\left(\mathbf{b}\cdot\nabla\mathbf{c}\right)$,

$$\begin{aligned}
&\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}}+\frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}+\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}}+\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right)=\\
&\mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t}+\nabla\cdot\rho e\mathbf{u}+\frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right)\\
&+\frac{1}{2}\mathbf{R}^{\theta}\left(-\frac{\partial\rho\mathbf{u}\cdot\mathbf{V}_{rel}}{\partial t}-\frac{\partial\rho\mathbf{V}_{rel}\cdot\mathbf{u}}{\partial t}+\frac{\partial\rho\mathbf{V}_{rel}\cdot\mathbf{V}_{rel}}{\partial t}\right)\\
&+\frac{1}{2}\mathbf{R}^{\theta}\left(-\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\right)\cdot\mathbf{V}_{rel}-\rho\mathbf{V}_{rel}\cdot\mathbf{u}\times\Omega\right)\\
&+\frac{1}{2}\mathbf{R}^{\theta}\left(-\rho\left(\mathbf{V}_{rel}\otimes\mathbf{u}\right):\operatorname{tr}\left(\nabla\mathbf{u}\right)-\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{V}_{rel}\right)\cdot\left(\mathbf{u}-\mathbf{V}_{rel}\right)\right).
\end{aligned}\tag{50}$$

The product rule of partial derivatives is used to simplify the terms of equation (50),

$$\begin{aligned}
&\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}}+\frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}+\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}}+\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right)=\\
&\mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t}+\nabla\cdot\rho e\mathbf{u}+\frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right)\\
&-\mathbf{R}^{\theta}\mathbf{V}_{rel}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)-\mathbf{R}^{\theta}\rho\left(\mathbf{u}-\mathbf{V}_{rel}\right)\cdot\left(\frac{\partial\mathbf{V}_{rel}}{\partial t}-\mathbf{u}\times\Omega\right)\\
&-\mathbf{R}^{\theta}\left(\mathbf{u}\cdot\mathbf{V}_{rel}+\frac{1}{2}\mathbf{V}_{rel}\cdot\mathbf{V}_{rel}\right)\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{u}\right).
\end{aligned}\tag{51}$$

The terms on the right hand side will be explained below. The first group of terms in (51) gives the left hand side of the conservation equation for energy in the inertial frame. The second group of terms in (51) gives the scalar product of the relative velocity with left hand side of the conservation equation for momentum in the inertial frame. The third group of terms gives the scalar product of the non-inertial velocity with material derivative of the relative velocity. The last group of terms give the product of the velocity with the conservation equation for mass in the inertial frame. This term sums to zero. The complete simplified left hand side of the conservation equation for energy in the non-inertial frame is given by

$$\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}}+\frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}+\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}}+\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right)= \mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t}+\nabla\cdot\rho e\mathbf{u}+\frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right) -\mathbf{R}^{\theta}\mathbf{V}_{rel}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)-\mathbf{R}^{\theta}\rho\left(\mathbf{u}-\mathbf{V}_{rel}\right)\cdot\left(\frac{\partial\mathbf{V}_{rel}}{\partial t}-\mathbf{u}\times\Omega\right). \tag{52}$$

The rotational transform is now applied to the last term on the right hand side of equation (52) using equation (13)

$$\begin{aligned}-\mathbf{R}^{\theta}\rho\left(\mathbf{u}-\mathbf{V}_{rel}\right)\cdot\left(\frac{\partial\mathbf{V}_{rel}}{\partial t}-\mathbf{u}\times\Omega\right)&=-\hat{\rho}\hat{\mathbf{u}}\cdot\left(\mathbf{R}^{\theta}\frac{\partial\mathbf{V}_{rel}}{\partial t}-\hat{\mathbf{u}}\times\Omega-\mathbf{R}^{\theta}\mathbf{V}_{rel}\times\Omega\right),\\ &=-\hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\left(\frac{\partial\mathbf{V}_{rel}}{\partial t}+\mathbf{V}_{rel}\cdot\nabla\mathbf{V}_{rel}\right),\\ &=-\hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}.\end{aligned}\tag{53}$$

Here $\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{V}_{rel}\cdot\nabla$. Substituting the result (53) back into (52)

$$\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}}+\frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}+\hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}}+\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right)= \mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t}+\nabla\cdot\rho e\mathbf{u}+\frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right) -\mathbf{R}^{\theta}\mathbf{V}_{rel}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)-\hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}. \tag{54}$$

The last term of equation (54) is the rate of work done by the convective relative acceleration. We can substitute the expression for V_{rel} into the last term of equation (54),

$$\hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}=\hat{\rho}\hat{\mathbf{u}}\cdot\left(\underbrace{\mathbf{R}^{\theta}\frac{\partial\mathbf{V}(t)}{\partial t}}_{\text{Translation}}+\underbrace{\dot{\Omega}\times\hat{\mathbf{x}}}_{\text{Euler}}+\underbrace{\Omega\times\dot{\hat{x}}}_{\text{Unsteady Axes}}-\underbrace{\mathbf{R}^{\theta}\mathbf{V}(t)\times\Omega}_{\text{Magnus}}+\underbrace{\hat{\mathbf{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right). \tag{55}$$

3.2.4. *Work due to forces*

The forces that are considered here are the pressure, viscous and body forces. The work due to pressure is transformed to the non-inertial frame as

$$\hat{\nabla}\cdot\left(\hat{p}\hat{\mathbf{u}}\right)=\mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\nabla\cdot p\mathbf{u}\right). \tag{56}$$

The divergence term in the inertial frame (56) can be decomposed using the property of the divergence operator into

$$\hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\mathbf{u}\cdot\nabla p + p\nabla\cdot\mathbf{u}\right). \tag{57}$$

Applying the modified Galilean transform to equation (57) using equations (14) -(17) in the limit as $\Delta t \to 0$

$$\begin{aligned}\hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) &= \mathbf{R}^{\theta}\left((\mathbf{u}-\mathbf{V}_{rel}-\mathbf{a}_{rel}\Delta t)\cdot\nabla p + p\nabla\cdot(\mathbf{u}-\mathbf{V}_{rel}-\mathbf{a}_{rel}\Delta t)\right),\\ &= \mathbf{R}^{\theta}\left(\mathbf{u}\cdot\nabla p - \mathbf{V}_{rel}\cdot\nabla p + p\nabla\cdot\mathbf{u} - p\nabla\cdot\mathbf{V}_{rel}\right).\end{aligned} \tag{58}$$

Using equation (31) and the associative property of the divergence operator, equation (58) simplifies to

$$\hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) = \mathbf{R}^{\theta}\left(\nabla\cdot p\mathbf{u} - \mathbf{V}_{rel}\cdot\nabla p\right). \tag{59}$$

The work due to viscous forces is transformed to the non-inertial frame as

$$\hat{\nabla}\cdot(\hat{\tau}\cdot\hat{\mathbf{u}}) = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left((\nabla\cdot\tau)\cdot\mathbf{u} + \Phi\right). \tag{60}$$

The dissipation function Φ is a scalar and is thus Galilean invariant under transformation to the non-inertial frame. The divergence of the viscous stress tensor has been shown in [3, 5] to be invariant under transformation by the local Galilean transformation. Applying the modified local Galilean transform to equation (60) in the limit that $\Delta t \to 0$

$$\hat{\nabla}\cdot(\hat{\tau}\cdot\hat{\mathbf{u}}) = \mathbf{R}^{\theta}\left((\nabla\cdot\tau)\cdot\mathbf{u} - (\nabla\cdot\tau)\cdot\mathbf{V}_{rel} + \Phi\right). \tag{61}$$

It can be assumed that the work done by body forces is due to the gravitational acceleration,

$$\hat{\rho}\hat{\mathbf{f}}\cdot\hat{\mathbf{u}} = \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}}, \tag{62}$$

$$= \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\rho\mathbf{g}\cdot\mathbf{u}\right). \tag{63}$$

Applying the modified Galilean transform in equation (63) in the limit as $\Delta t \to 0$ taking note that the gravitational acceleration vector is invariant under transformation of the local Galilean transformation,

$$\hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} = \mathbf{R}^{\theta}\left(\rho\mathbf{g}\cdot\left(\mathbf{u}-\mathbf{V}_{rel}-\mathbf{a}_{rel}\Delta t\right)\right), \tag{64}$$

$$= \mathbf{R}^{\theta}\left(\rho\mathbf{g}\cdot\mathbf{u}-\rho\mathbf{g}\cdot\mathbf{V}_{rel}\right). \tag{65}$$

3.2.5. Heat addition

The transformation of the term related to heat addition is given by

$$\hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\nabla\cdot k\nabla T\right). \tag{66}$$

The temperature and thermal conductivity are scalars that are Galilean invariant under transformations with the local modified Galilean transform. Equation (66) reduces to

$$\hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) = \mathbf{R}^{\theta}\left(\nabla\cdot k\nabla T\right). \tag{67}$$

3.3. Non-Inertial Conservation Equation for Energy

The conservation equation for energy in an arbitrarily moving non-inertial frame of reference is obtained by combining the results of equations (54), (59), (61), (65) and (67)

$$\begin{aligned}\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} + \frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} + \frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right)\\ -\hat{\nabla}\cdot\left(\hat{p}\hat{\mathbf{u}}\right) + \hat{\nabla}\cdot\left(\hat{\tau}\cdot\hat{\mathbf{u}}\right) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} + \mathbf{R}^{\theta}\left(\frac{\partial\rho e}{\partial t} + \frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t} + \nabla\cdot\rho e\mathbf{u} + \frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right)\\ +\mathbf{R}^{\theta}\left(-\nabla\cdot k\nabla T + \nabla\cdot p\mathbf{u} - \left(\nabla\cdot\tau\right)\cdot\mathbf{u} - \Phi - \rho\mathbf{g}\cdot\mathbf{u}\right)\\ -\mathbf{R}^{\theta}\mathbf{V}_{rel}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \left(\nabla\cdot\tau\right) - \rho\mathbf{g}\right) - \hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}.\end{aligned} \tag{68}$$

The first group of terms on the right hand side of equation (68) give the conservation equation for energy in the inertial frame which sums up to zero by equation (19). The second group of terms on the right hand side of equation (68) is the conservation equation for momentum in the inertial frame which sums up to zero. The conservation equation for energy in an arbitrarily moving non-inertial frame is thus given by

$$\begin{aligned}\left(\frac{\partial\hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} + \frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} + \frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right)\\ -\hat{\nabla}\cdot\left(\hat{p}\hat{\mathbf{u}}\right) + \hat{\nabla}\cdot\left(\hat{\tau}\cdot\hat{\mathbf{u}}\right) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} - \hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}.\end{aligned} \tag{69}$$

Equation (69) can be written as

$$
\begin{aligned}
&\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\hat{\mathbf{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot\left(\hat{p}\hat{\mathbf{u}}\right) + \hat{\nabla}\cdot\left(\hat{\tau}\cdot\hat{\mathbf{u}}\right) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} \\
&-\hat{\rho}\hat{\mathbf{u}}\cdot\left(\underbrace{\mathbf{R}^{\theta}\frac{\partial \mathbf{V}(t)}{\partial t}}_{\text{Translation}} + \underbrace{\dot{\Omega}\times\hat{\mathbf{x}}}_{\text{Euler}} + \underbrace{\Omega\times\dot{\hat{x}}}_{\text{Unsteady Axes}} \underbrace{-\mathbf{R}^{\theta}\mathbf{V}(t)\times\Omega}_{\text{Magnus}} + \underbrace{\hat{\mathbf{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right).
\end{aligned}
\tag{70}
$$

The conservation equation for energy in an arbitrarily moving reference frame consists of fictitious work done by the relative convective acceleration between the non-inertial and inertial stationary frame. The rate of work done is due to the translation acceleration, the Euler term of unsteady rotation, the acceleration term due to the unsteady motion of the rotational axes, the Magnus acceleration and the Centrifugal acceleration. It is important to note that the Coriolis acceleration source term does not do any work in the non-inertial arbitrarily moving reference frame. This is in agreement with the work of [3, 23, 24].

3.4. Kinetic Energy Equation in Non-Inertial Frame

The kinetic energy in the inertial frame K is given by

$$K = \frac{1}{2}\mathbf{u}\cdot\mathbf{u}. \tag{71}$$

The rate of change of the kinetic energy is determined by the rate of work done by the pressure, viscous and body forces. This is obtained by taking the scalar product of the momentum equation in the inertial frame with the inertial velocity,

$$\frac{1}{2}\frac{\partial \rho\mathbf{u}}{\partial t}\cdot\mathbf{u} + \frac{1}{2}\left(\nabla\cdot\left(\rho\mathbf{u}\otimes\mathbf{u}\right)\right)\cdot\mathbf{u} = -\mathbf{u}\cdot\nabla p + \left(\nabla\cdot\tau\right)\cdot\mathbf{u} + \rho\mathbf{f}\cdot\mathbf{u}. \tag{72}$$

For consistency with the rest of the text the kinetic energy in the inertial frame (72) is given in conservative form as

$$\frac{1}{2}\frac{\partial \rho\mathbf{u}\cdot\mathbf{u}}{\partial t} + \frac{1}{2}\left(\nabla\cdot\left(\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right) = -\mathbf{u}\cdot\nabla p + \left(\nabla\cdot\tau\right)\cdot\mathbf{u} + \rho\mathbf{f}\cdot\mathbf{u}. \tag{73}$$

Equation (73) can be transformed into the non-inertial frame by applying the local Galilean and Rotational transformations for each of the individual terms in a similar way to the conservation equation for energy. Firstly, the non-inertial kinetic energy eqiation is obtained by tranforming (71) using equations (13) and (14)

$$\begin{aligned}\hat{K} &= \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right),\\ &= \mathbf{R}^{\theta}\left(\frac{1}{2}\left(\mathbf{u}-\mathbf{V}_{rel}-\mathbf{a}_{rel}\Delta t\right)\cdot\left(\mathbf{u}-\mathbf{V}_{rel}-\mathbf{a}_{rel}\Delta t\right)\right).\end{aligned} \tag{74}$$

In the limit that $\Delta t \rightarrow 0$ equation (74) reduces to,

$$\hat{K} = \mathbf{R}^{\theta}\left(K+\frac{1}{2}\left(-\mathbf{u}\cdot\mathbf{V}_{rel}-\mathbf{V}_{rel}\cdot\mathbf{u}+\mathbf{V}_{rel}\cdot\mathbf{V}_{rel}\right)\right). \tag{75}$$

The kinetic energy in the non-inertial frame (75) is a sum of the kinetic energy in the inertial frame K with the kinetic energy in the non-inertial frame $\mathbf{V}_{rel}\cdot\mathbf{V}_{rel}$. The other two terms in (75) take into account the movement of frame $\hat{O}$ relative to frame O, thus the two terms take into account the inter frame kinetic energy.

3.4.1. Temporal and Diffusion terms

The temporal and diffusion terms in equation (73) are similar to the terms in the conservation equation for energy given by equations (41) and (49). It has been shown that in the non-inertial frame these terms transforms into

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial\hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}}+\frac{1}{2}\left(\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) &= \mathbf{R}^{\theta}\left(\frac{1}{2}\frac{\partial\rho\mathbf{u}\cdot\mathbf{u}}{\partial t}+\frac{1}{2}\left(\nabla\cdot\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}\right)\right)\\ &-\mathbf{R}^{\theta}\mathbf{V}_{rel}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)-\mathbf{R}^{\theta}\rho\left(\mathbf{u}-\mathbf{V}_{rel}\right)\cdot\left(\frac{\partial\mathbf{V}_{rel}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{V}_{rel}\right).\end{aligned} \tag{76}$$

The first group of terms on the right hand side of equation (76) gives the left hand side of the kinetic energy equation in the inertial frame (73). The second group of terms of equation (76) give the left hand side of the conservation equation for momentum in the inertial frame. The last group of terms give the rate of work done as obtained with the conservation equation for energy above.

3.4.2. Rate of Work Done

Applying the local Galilean and Rotational transformations to the rate of work terms on the RHS of equation (72),

$$-\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{p}+\left(\hat{\nabla}\cdot\hat{\tau}\right)\cdot\hat{\mathbf{u}}+\hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} = \mathbf{R}^{\theta}\mathbf{G}^{\mathbf{V}_{rel}}\left(-\mathbf{u}\cdot\nabla p+\left(\nabla\cdot\tau\right)\cdot\mathbf{u}+\rho\mathbf{g}\cdot\mathbf{u}\right). \tag{77}$$

In the above equation, the divergence of the viscous stress tensor, pressure and density are Galilean invariant. Substituting for the Galilean transformation of the velocity in the limit that $\Delta t \to 0$,

$$-\hat{\mathbf{u}} \cdot \hat{\nabla} \hat{p} + \left(\hat{\nabla} \cdot \hat{\tau} \right) \cdot \hat{\mathbf{u}} + \hat{\rho} \hat{\mathbf{g}} \cdot \hat{\mathbf{u}} = \mathbf{R}^{\theta} \left(-\mathbf{u} \cdot \nabla p + (\nabla \cdot \tau) \cdot \mathbf{u} + \rho \mathbf{g} \cdot \mathbf{u} \right) + \mathbf{R}^{\theta} \left(\mathbf{V}_{rel} \cdot \nabla p - (\nabla \cdot \tau) \cdot \mathbf{V}_{rel} - \rho \mathbf{g} \cdot \mathbf{V}_{rel} \right). \tag{78}$$

Combination of equations (76) and (78) results in the complete kinetic energy equation in the non-inertial frame,

$$\frac{1}{2} \left(\frac{\partial \hat{\rho} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}{\partial t} \right)_{\hat{O}} + \frac{1}{2} \hat{\nabla} \cdot \left(\hat{\rho} \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right) = -\hat{\mathbf{u}} \cdot \hat{\nabla} \hat{p} + \left(\hat{\nabla} \cdot \hat{\tau} \right) \cdot \hat{\mathbf{u}} + \hat{\rho} \hat{\mathbf{g}} \cdot \hat{\mathbf{u}} - \mathbf{R}^{\theta} \rho \left(\mathbf{u} - \mathbf{V}_{rel} \right) \cdot \left(\frac{\partial \mathbf{V}_{rel}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{V}_{rel} \right). \tag{79}$$

Similar to the conservation for energy, the last term of equation (79) can be simplified by applying the rotational transform to obtain

$$\frac{1}{2} \left(\frac{\partial \hat{\rho} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}{\partial t} \right)_{\hat{O}} + \frac{1}{2} \hat{\nabla} \cdot \left(\hat{\rho} \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right) = -\hat{\mathbf{u}} \cdot \hat{\nabla} \hat{p} + \left(\hat{\nabla} \cdot \hat{\tau} \right) \cdot \hat{\mathbf{u}} + \hat{\rho} \hat{\mathbf{g}} \cdot \hat{\mathbf{u}} \quad - \hat{\rho} \hat{\mathbf{u}} \cdot \mathbf{R}^{\theta} \frac{D \mathbf{V}_{rel}}{Dt}. \tag{80}$$

The kinetic energy equation in the non-inertial frame for an object undergoing arbitrary motion is given by (80). The last term in equation (80) is the rate of work done by the fictitious convective relative acceleration. It is evident from this result that the non-inertial rate of work terms obtained in the conservation equation for energy are due to the kinetic energy. This is consistent since the kinetic energy is a form of mechanical energy which should take into account the state of motion that an object undergoes.

3.5. Internal Energy Equation in Non-Inertial Frame

The internal energy equation is obtained by subtracting the mechanical kinetic energy (73) from the conservation equation for energy (19). The resulting equation for the internal energy in the inertial frame is given by

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot \rho e \mathbf{u} = \nabla \cdot (k \nabla T) - p \left(\nabla \cdot \mathbf{u} \right) + \tau \colon \nabla \mathbf{u}. \tag{81}$$

In the non-inertial frame the same approach is used. The internal energy equation in the non-inertial frame is obtained by subtracting the mechanical kinetic energy from the conservation equation for energy in the non-inertial frame. The conservation equation for energy in the non-inertial frame was given by,

$$\left(\frac{\partial \hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} + \frac{1}{2}\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} + \frac{1}{2}\hat{\nabla}\cdot\left(\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) =$$
$$\hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot\left(\hat{p}\hat{\mathbf{u}}\right) + \hat{\nabla}\cdot\left(\hat{\tau}\cdot\hat{\mathbf{u}}\right) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} - \hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}. \tag{82}$$

and the kinetic energy equation in the non-inertial frame was given by,

$$\frac{1}{2}\left(\frac{\partial \hat{\rho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} + \frac{1}{2}\hat{\nabla}\cdot\left(\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right) = -\hat{\mathbf{u}}\cdot\hat{\nabla}\hat{p} + \left(\hat{\nabla}\cdot\hat{\tau}\right)\cdot\hat{\mathbf{u}} + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}}$$
$$- \hat{\rho}\hat{\mathbf{u}}\cdot\mathbf{R}^{\theta}\frac{D\mathbf{V}_{rel}}{Dt}. \tag{83}$$

The difference between equation (82) and equation (83) gives the internal energy in the non-inertial frame

$$\left(\frac{\partial \hat{\rho}\hat{e}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{e}\hat{\mathbf{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{p}\left(\hat{\nabla}\cdot\hat{\mathbf{u}}\right) + \hat{\tau}:\hat{\nabla}\hat{\mathbf{u}}. \tag{84}$$

From equation (84), the internal energy in the non-inertial frame remains invariant under transformation as found by [5, 7, 22].

3.6. Enthalpy Equation in Non-inertial Frame

The conservation equation for energy was given in terms of the internal energy. The equation can also be given in terms of the enthalpy. As a function of enthalpy, the conservation equation for energy is

$$E = h - \frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}. \tag{85}$$

Here, h is the enthalpy which can be written as a function of the internal energy

$$h = e + \frac{p}{\rho}, \tag{86}$$

and p is the pressure. Substituting for E in terms of enthalpy in the conservation equation for energy in an inertial frame (19)

$$\frac{\partial \rho h}{\partial t} + \frac{1}{2}\frac{\partial \rho\mathbf{u}\cdot\mathbf{u}}{\partial t} + \nabla\cdot(\rho h\mathbf{u}) + \frac{1}{2}\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{u}) - \frac{\partial p}{\partial t} = \nabla\cdot(k\nabla T)$$
$$+\nabla\cdot(\tau\cdot\mathbf{u}) + \rho\mathbf{g}\cdot\mathbf{u}. \tag{87}$$

Similar to the internal energy, the enthalpy equation in the inertial frame is obtained by subtracting the kinetic energy equation from the conservation equation for energy. In the inertial frame the enthalpy equation is given by,

$$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho h \mathbf{u}) - \frac{\partial p}{\partial t} = \nabla \cdot (k \nabla T) + \mathbf{u} \cdot \nabla p + \tau \colon \nabla \mathbf{u}. \tag{88}$$

The conservation equation for energy written in terms of the enthalpy in the non-inertial frame is similar to that written in terms of internal energy. The difference lies in the time rate of change of the pressure term. This term will be transformed to the non-inertial frame in what follows. The pressure is a scalar and is Galilean invariant under transformations from inertial to non-inertial frame. The time derivative term by equations (14) and (15) becomes

$$\begin{aligned} -\left(\frac{\partial \hat{p}}{\partial t}\right)_{\hat{O}} &= -\mathbf{R}^{\theta} \mathbf{G}^{\mathbf{V}_{rel}} \frac{\partial p}{\partial t}, \\ &= \mathbf{R}^{\theta} \left(-\frac{\partial p}{\partial t} - \mathbf{V}_{rel} \cdot \nabla p \right). \end{aligned} \tag{89}$$

All the other terms can be transformed in a similar manner as was done with the terms that involve the internal energy since the enthalpy h is a scalar that is Galilean invariant. Thus the total energy equation in the non-inertial frame written in terms of the enthalpy is given by

$$\begin{aligned} &\left(\frac{\partial \hat{\rho} \hat{h}}{\partial t}\right)_{\hat{O}} + \frac{1}{2} \left(\frac{\partial \hat{\rho} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}{\partial t}\right)_{\hat{O}} + \hat{\nabla} \cdot \left(\hat{\rho} \hat{h} \hat{\mathbf{u}}\right) + \frac{1}{2} \hat{\nabla} \cdot \left(\hat{\rho} \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}\right) - \left(\frac{\partial \hat{p}}{\partial t}\right)_{\hat{O}} \\ &\qquad = \hat{\nabla} \cdot \left(\hat{k} \hat{\nabla} \hat{T}\right) + \hat{\nabla} \cdot (\hat{\tau} \cdot \hat{\mathbf{u}}) + \hat{\rho} \hat{\mathbf{g}} \cdot \hat{\mathbf{u}} - \hat{\rho} \hat{\mathbf{u}} \cdot \mathbf{R}^{\theta} \frac{D \mathbf{V}_{rel}}{Dt}. \end{aligned} \tag{90}$$

Subtracting the kinetic energy equation in the non-inertial frame from (90),

$$\left(\frac{\partial \hat{\rho} \hat{h}}{\partial t}\right)_{\hat{O}} + \hat{\nabla} \cdot \left(\hat{\rho} \hat{h} \hat{\mathbf{u}}\right) - \left(\frac{\partial \hat{p}}{\partial t}\right)_{\hat{O}} = \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{p} + \hat{\nabla} \cdot \left(\hat{k} \hat{\nabla} \hat{T}\right) + \hat{\tau} \colon \hat{\nabla} \hat{\mathbf{u}}. \tag{91}$$

The enthalpy equation in the non-inertial frame does not contain any fictitious work terms. It is further more evident that the fictitious work terms in a non-inertial frame that undergoes arbitrary motion are only present in the kinetic energy equation. Since the conservation equation for energy is a sum of either the enthalpy or internal energy with the kinetic energy, the fictitious work terms appear in the conservation equation for energy.

4. Discussion

The results of the present derivation show which of the different forms of the energy equations contain fictitious work. The conservation equation for energy can also be considered for various types of motion. The following limiting cases arise for the conservation equation for energy. These results also apply to the kinetic energy equation while the internal energy and enthalpy equations remain invariant for all cases.

Case I : Pure Steady Translation

$$\frac{\partial \mathbf{V}(t)}{\partial t} = 0, \qquad \Omega = 0, \qquad \dot{\Omega} = 0, \qquad \dot{\hat{x}} = 0, \tag{92}$$

$$\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\hat{\mathbf{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\hat{\mathbf{u}}) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}}. \tag{93}$$

Case II: Pure Steady Rotation

$$\frac{\partial \mathbf{V}(t)}{\partial t} = 0, \qquad \mathbf{V}(t) = 0, \qquad \dot{\Omega} = 0, \qquad \dot{\hat{\mathbf{x}}} = 0, \tag{94}$$

$$\begin{aligned}\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\hat{\mathbf{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\hat{\mathbf{u}}) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} \\ -\hat{\rho}\hat{\mathbf{u}}\cdot\underbrace{\hat{\mathbf{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}.\end{aligned} \tag{95}$$

Case III: Pure Unsteady Translation

$$\Omega = 0, \qquad \dot{\Omega} = 0, \qquad \dot{\hat{x}} = 0, \qquad \mathbf{R}^{\theta} = \mathbf{I}, \tag{96}$$

$$\begin{aligned}\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\hat{\mathbf{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\hat{\mathbf{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\hat{\mathbf{u}}) + \hat{\rho}\hat{\mathbf{g}}\cdot\hat{\mathbf{u}} \\ -\hat{\rho}\hat{\mathbf{u}}\cdot\underbrace{\frac{\partial \mathbf{V}(t)}{\partial t}}_{\text{Translation}}.\end{aligned} \tag{97}$$

Case IV: Pure Unsteady Rotation

$$\frac{\partial \mathbf{V}(t)}{\partial t} = 0, \qquad \mathbf{V}(t) = 0, \qquad \dot{\hat{x}} = 0, \tag{98}$$

$$\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\mathbf{\hat{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\mathbf{\hat{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\mathbf{\hat{u}}) + \hat{\rho}\mathbf{\hat{g}}\cdot\mathbf{\hat{u}}$$
$$-\hat{\rho}\mathbf{\hat{u}}\cdot\left(\underbrace{\dot{\Omega}\times\mathbf{\hat{x}}}_{\text{Euler}} + \underbrace{\mathbf{\hat{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right). \tag{99}$$

Case V: Steady Translation, Unsteady Rotation

$$\frac{\partial \mathbf{V}(t)}{\partial t} = 0, \qquad \hat{\dot{x}} = 0, \tag{100}$$

$$\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\mathbf{\hat{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\mathbf{\hat{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\mathbf{\hat{u}}) + \hat{\rho}\mathbf{\hat{g}}\cdot\mathbf{\hat{u}}$$
$$-\hat{\rho}\mathbf{\hat{u}}\cdot\left(\underbrace{\dot{\Omega}\times\mathbf{\hat{x}}}_{\text{Euler}} - \underbrace{\mathbf{R}^{\theta}\mathbf{V}(t)\times\Omega}_{\text{Magnus}} + \underbrace{\mathbf{\hat{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right). \tag{101}$$

Case VI: Unsteady Translation, Steady Rotation

$$\dot{\Omega} = 0, \qquad \hat{\dot{x}} = 0, \tag{102}$$

$$\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\mathbf{\hat{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\mathbf{\hat{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\mathbf{\hat{u}}) + \hat{\rho}\mathbf{\hat{g}}\cdot\mathbf{\hat{u}}$$
$$-\hat{\rho}\mathbf{\hat{u}}\cdot\left(\underbrace{\mathbf{R}^{\theta}\frac{\partial \mathbf{V}(t)}{\partial t}}_{\text{Translation}} - \underbrace{\mathbf{R}^{\theta}\mathbf{V}(t)\times\Omega}_{\text{Magnus}} + \underbrace{\mathbf{\hat{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right). \tag{103}$$

Case VII: Full Arbitrary Motion

$$\left(\frac{\partial \hat{\rho}\hat{E}}{\partial t}\right)_{\hat{O}} + \hat{\nabla}\cdot\hat{\rho}\hat{E}\mathbf{\hat{u}} = \hat{\nabla}\cdot\left(\hat{k}\hat{\nabla}\hat{T}\right) - \hat{\nabla}\cdot(\hat{p}\mathbf{\hat{u}}) + \hat{\nabla}\cdot(\hat{\tau}\cdot\mathbf{\hat{u}}) + \hat{\rho}\mathbf{\hat{g}}\cdot\mathbf{\hat{u}}$$
$$-\hat{\rho}\mathbf{\hat{u}}\cdot\left(\underbrace{\mathbf{R}^{\theta}\frac{\partial \mathbf{V}(t)}{\partial t}}_{\text{Translation}} + \underbrace{\dot{\Omega}\times\mathbf{\hat{x}}}_{\text{Euler}} + \underbrace{\Omega\times\hat{\dot{x}}}_{\text{Unsteady Axes}} - \underbrace{\mathbf{R}^{\theta}\mathbf{V}(t)\times\Omega}_{\text{Magnus}} + \underbrace{\mathbf{\hat{x}}\times\Omega\times\Omega}_{\text{Centrifugal}}\right). \tag{104}$$

The fictitious accelerations that do work in the non-inertial frame are dependent on the type of motion considered, similar to the conservation equation for momentum in a non-inertial frame of reference, this is demonstrated by equations (93)-(104).

The studies of [10, 11] give a form of the total energy equation in a non-inertial arbitrarily moving frame that contains the work done by the Coriolis acceleration source term which is inconsistent with other studies. The study of [9] does not include any fictitious work terms in their non-inertial total energy equation even though the type of motion considered is arbitrary motion. This derivation was motivated by the claims in Combrinck [5, 7] that there were instances in the literature were the energy equation had been observed with additional acceleration source terms. It is important to differentiate which of the energy equations one is using when working in the non-inertial frame to avoid confusion. When working with various forms of the energy equations in an arbitrary frame of reference, the present derivation can be used as a tool to determine whether fictitious acceleration source terms should be taken into account.

5. Conclusions

In this paper, an Eulerian approach for deriving the apparent forces is presented to transform the governing conservation equation for energy, kinetic energy, internal energy and enthalpy equations into a non-inertial reference frame in arbitrary motion. The Eulerian approach is based on successive Galilean transformations between an inertial frame, an orientation-preserving non-inertial frame and a rotating non-inertial frame. The paper demonstrates that for an object undergoing arbitrary motion fictitious work arises in the kinetic energy equation and conservation equation for energy while the internal energy and enthalpy equations remain invariant. For an object in arbitrary motion, the rate of work done due to fictitious forces affects the rate of change of the total energy. The fictitious work terms in the non-inertial frame are due to the translation acceleration, Euler rotational acceleration term, acceleration due to the movement of the axes of rotation, Centrifugal acceleration and the Magnus acceleration. Similar to the conservation equation for momentum in the non-inertial frame, the fictitious work terms are dependent on the type of motion under consideration. The present derivation is a step towards quantifying the contribution of the fictitious force terms to the heat transfer of an accelerating object. This has implications to enhance the understanding of the observed behaviour in the inertial frame.

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