Median-based resilient consensus over time-varying random networks

Yilun Shang

***Abstract*—This brief investigates the resilient consensus control for multiagent systems over a time-varying directed random network. We propose a median-based consensus strategy, which is purely distributed and, as opposed to the Weighted-Mean-Subsequence-Reduced approaches in the existing literature, shared estimate regarding the number of malicious agents in the neighborhood of each cooperative agent is not required. This offers more applicability and flexibility as seeking a shared estimate of surrounding threats is often difficult in practice. In addition to malicious agents, random availability of communication edges is accommodated in the random network framework. Sufficient conditions are derived for reaching almost sure consensus by using a martingale convergence theorem. Finally, the theoretical findings are illustrated by numerical simulations.**

***Index Terms*—resilient consensus; random networks; multiagent system; time-varying topology.**

I. Introduction

THE study of coordinated control in multiagent systems has experienced dramatic progresses in the past decades, where a common task of the group of agents is to seek consensus through a communication network using only local information exchange [1]–[3]. Towards this objective, an essential ingredient of consensus problems is to design distributed control protocols such that the states of agents converge asymptotically in some sense.

Due to the broad applications of large-scaled networked systems and sensor networks, the security and resilience of consensus against faults or attacks on nodes and edges have become a paramount issue. A topological property of network robustness, called r-robustness, is introduced in [4] to facilitate distributed consensus over networks with malicious agents. It is shown that if the number of malicious neighbors for each cooperative agent is no more than r, then consensus can be achieved with a $(2r + 1)$-robust communication network. The consensus protocol proposed in [4] has been extended to investigate higher-order [5], switched [6] and hybrid [7] dynamical systems. In [8], [9], trusted cooperative nodes are introduced to reduce the network connectivity requirement. Networks with trusted nodes have also been studied for resilient consensus in cyber-physical networks against deception attacks [10]. Resilient consensus in the presence of locally bounded malicious agents is realized by using impulsive control and event-driven methods in [11] and [12], respectively. Stochastic resilient consensus has been studied for networks under random link failure and channel noise [13] as well as hybrid random behavior [14]. In some real-world applications involving complicated environment such as smart homes, heuristic algorithms and metaheuristic algorithms have also been widely applied to tackle cooperative control problems in complex nonlinear systems against adversarial hazards or obstacles; see e.g. [15]–[17].

A main drawback in the existing line of research in distributed resilient consensus including those mentioned above is that the global information regarding the number of malicious agents is shared among all cooperative agents. For example, in a typical Weighted-Mean-Subsequence-Reduced (W-MSR) algorithm [4], [7], [8], [18], cooperative agents will need to scrap the r largest and r smallest neighbor states during each iteration, where r is the estimated upper bound of malicious neighbors. In heuristic algorithms mentioned above, on the other hand, optimization problems are often have to be solved by employing advanced techniques such as neural networks [16], [19]. In this brief, we propose a simple median-based resilient control protocol which does not require shared information about the number of malicious agents. The main contribution of this brief is summarized as follows. Firstly, compared to the classical mean-based resilient consensus protocols, we develop a novel median-based strategy which circumvents the requirement of shared estimate regarding local hazards. Secondly, building on martingale convergence theory, we show that almost sure convergence of states can be realized when the underlying communication network is a time-dependent directed random graph satisfying certain robustness conditions.

It is worth noting that the requirement of global information about the number of malicious agents has been tackled in the literature so far mainly by two philosophies. In [20], the author has applied a separate max-consensus process to estimate an upper bound of malicious agents followed by a W-MSR like algorithm. Another approach adopted by [21] circumvents this issue through updating the states of cooperative agents with the median of its neighbors. However, both approaches only work effectively for a deterministic communication network and time-invariant topology. Moreover, the median-based protocol considered here is fundamentally different from that in [21].

The rest of the brief is organized as follows. Section 2 presents some preliminaries and set up the system model. In Section 3, conditions are derived to achieve resilient consensus in the sense of almost sure convergence. Simulation results are delivered in Section 4 and conclusions are drawn in Section 5.

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II. Problem formulation

A. Graph theory

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{R}$ be the set of reals. For $t \in \mathbb{N}$, let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ represent a time-dependent weighted random digraph with a finite set of nodes (i.e. agents) $\mathcal{V} = \{1, 2, \cdots, N\}$, a set of edges $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency (random) matrix $\mathcal{A}(t) = (a_{ij}(t)) \in \mathbb{R}^{N\times N}$. Here, $a_{ij}(t) > 0$ if $(j, i) \in \mathcal{E}(t)$, namely, the node i can receive information from node j, and $a_{ij}(t) = 0$ otherwise. We assume $a_{ij}(t) > 0$ with probability $p_{ij}(t)$ and $a_{ij}(t) = 0$ with probability $1-p_{ij}(t)$. Note that the random network $\mathcal{G}(t)$ is general as we do not assume any independence for $a_{ij}(t)$ with respect to different i, j or t. The set of neighbors of $i \in \mathcal{V}$ is denoted by $\mathcal{N}_i(t) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}(t)\}$. The in-degree of i at time t is $d_i(t) = |\mathcal{N}_i(t)|$. For ease of presentation, We sometimes suppress the time t in notations when it is not essential.

A node set $\mathcal{S} \subseteq \mathcal{V}$ is said to be r-excess reachable if there is some $i \in \mathcal{S}$ such that it has at least r more neighbors outside $\mathcal{S}$ than inside $\mathcal{S}$, namely, $|\mathcal{N}_i \backslash \mathcal{S}| - |\mathcal{N}_i \cap \mathcal{S}| \geq r$. A graph $\mathcal{G}$ is called r-excess robust if for any two nonempty and disjoint sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$, at least one of them is r-excess reachable. By [21], an r-excess robust graph with $r \geq 2$ has in-degree at least r. Moreover, a set $\mathcal{S} \subseteq \mathcal{V}$ is called r-local [4] if any node in $\mathcal{V}\backslash\mathcal{S}$ has at most r neighbors in $\mathcal{S}$, namely, $|\mathcal{N}_i \cap \mathcal{S}| \leq r$ for any $i \notin \mathcal{S}$.

B. Model description

The node set $\mathcal{V}$ of the time-varying random graph $\mathcal{G}(t)$ is partitioned into two sets $\mathcal{V} = \mathcal{C} \cup \mathcal{M}$, where $\mathcal{C}$ contains all cooperative agents and $\mathcal{M}$ represents the set of malicious agents. The malicious agents here are also known as Byzantine nodes [4]–[6], [12], which can adopt unknown protocols and may have collision behaviors. We assume that the number and identity of them are not known to cooperative nodes, representing a huge threat to the consensus task. Cooperative nodes in $\mathcal{C}$, on the other hand, are under control and their consensus protocols are to be designed.

Formally, the dynamics of agent $i \in \mathcal{V}$ takes the following form

$$x_i(t + 1) = Ax_i(t) + Bu_i(t), \qquad t \in \mathbb{N}, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$ represent the state and control input of agent i, respectively. The control input $u_i(t)$ for $i \in \mathcal{C}$ is designed below in (5) while for any malicious agent $i \in \mathcal{M}$, $u_i(t)$ can take any value. Here, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ and let the characteristic polynomial of A be $\det(\lambda I_n - A) = \lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + \alpha_n$. If the matrix pair (A, B) is controllable, we can take

$$\begin{aligned} S =&(B, AB, A^2B, \cdots, A^{n-1}B) \\ &\cdot \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{n-2} \\ 0 & 0 & 1 & \cdots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n} \end{aligned} \tag{2}$$

and apply the similarity transform $y_i(t) = S^{-1}x_i(t)$ to convert (1) to the canonical form [22]:

$$y_i(t + 1) = \tilde{A}y_i(t) + \tilde{B}u_i(t), \qquad t \in \mathbb{N}, \tag{3}$$

where $\tilde{A} = S^{-1}AS = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$ and $\tilde{B} = S^{-1}B = (1, 0, 0, \cdots, 0)^{\mathrm{T}} \in \mathbb{R}^n$, where T means transpose.

Let $y_i(t) = (y_{i,n-1}(t), y_{i,n-2}(t), \cdots, y_{i,0}(t))^{\mathrm{T}}$. For each cooperative node $i \in \mathcal{C}$, we define a value

$$z_i(t) = y_{i,n-1}(t) + \sum_{l=1}^{n-1} \beta_l y_{i,n-1-l}(t), \tag{4}$$

where the coefficients $\beta_l \in \mathbb{R}$ $(1 \leq l \leq n-1)$ are chosen such that the polynomial $\lambda^{n-1} + \beta_1\lambda^{n-2} + \beta_2\lambda^{n-3} + \cdots + \beta_{n-1}$ is Schur stable. In the next subsection, we will design the consensus protocol for each cooperative node $i \in \mathcal{C}$, which receives the encoded state variables $z_j(t) \in \mathbb{R}$ $(j \in \mathcal{N}_i(t))$ instead of their neighbours' original vector states $\{x_j(t)\}$.

Remark 1. A simple sufficient condition for Schur stability is $1 > \beta_1 > \beta_2 > \cdots > \beta_{n-1} > 0$; see e.g. [23].

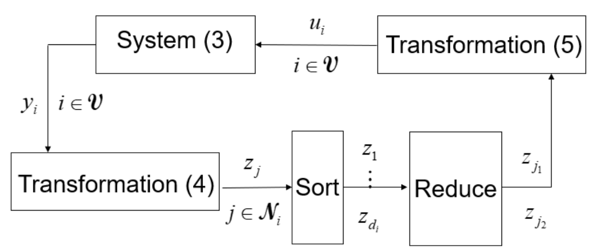

Fig. 1. A schematic of data flow for the median-based resilient consensus.

C. Median-based consensus protocol

Here, we present the median-based resilient consensus strategy for each cooperative node $i \in \mathcal{C}$ as follows (see Fig. 1 for a diagram of the flow).

I. At time step $t \in \mathbb{N}$, each cooperative node i receives the encoded state variable $z_j(t)$ $(j \in \mathcal{N}_i(t))$ from its neighbors and forms a sorted list $z_{(1)}(t) \geq z_{(2)}(t) \geq \cdots \geq z_{(d_i(t))}(t)$.

II. Let the two neighbors corresponding to $z_{\left(\left\lceil \frac{d_i(t)+1}{2} \right\rceil\right)}(t)$ and $z_{\left(\left\lfloor \frac{d_i(t)+1}{2} \right\rfloor\right)}(t)$ be denoted by j_1 and j_2, respectively. The control input in (1) is designed as

$$\begin{aligned} u_i(t) =& \sum_{l=1}^{n} \alpha_l y_{i,n-l}(t) - \sum_{l=1}^{n-1} \beta_l y_{i,n-l}(t) + z_i(t) \\ &+ \gamma_i(t) \cdot \big(a_{ij_1}(t)(z_{j_1}(t) - z_i(t)) \\ &+ a_{ij_2}(t)(z_{j_2}(t) - z_i(t))\big), \end{aligned} \tag{5}$$

where $0 < \gamma_i(t) < (a_{ij_1}(t) + a_{ij_2}(t))^{-1}$.

Remark 2. In the above notation, it is worth noting that j_1 and j_2 essentially depend on the node i and time t. Clearly, we have $j_1 = j_2$ if $d_i(t)$ is odd. Compared to the W-MSR algorithms [4]–[6], [13], we here involve the weighted median of neighbors' states instead of their (reduced) mean. Recall that the estimate of the number of total malicious agents in the neighborhood of each cooperative agent has to be shared among the network in order to implement a W-MSR type algorithm, which is often not realistic in practice. In the above proposed algorithm, such estimation of malicious neighbors is no longer needed. Nevertheless, our strategy is a distributed protocol and of low complexity similarly as W-MSR algorithms.

With the above strategy, we aim to show the cooperative agents can reach resilient consensus in the sense of almost sure convergence as t tends to infinity. In other words, almost sure resilient consensus for the multiagent system (1) is achieved if the following two conditions hold: (i) $x_i(t)$ is bounded for all $i \in \mathcal{C}$ and $t \in \mathbb{N}$, and (ii) $\lim_{t\to\infty} \mathrm{P}(x_i(t) - x_j(t) = 0_n) = 1$ for all $i, j \in \mathcal{C}$ and all initial conditions $\{x_i(0)\}_{i\in\mathcal{V}}$. Here, $0_n \in \mathbb{R}^n$ means a zero vector.

III. Resilient consensus analysis

In this section, we study the resilient consensus of the multiagent system (1) in the presence of malicious agents over the time-varying random network $\mathcal{G}(t)$. The main result reads as follows.

Theorem 1. *Consider a time-varying random network modelled by a weighted digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with $\mathcal{V} = \mathcal{C} \cup \mathcal{M}$, where cooperative nodes in $\mathcal{C}$ update their states following the strategy (1) and (5), and the set $\mathcal{M}$ of malicious nodes is r-local. If (A, B) is controllable and $\mathcal{G}(t)$ is $(2r+1)$-excess robust for $t \in \mathbb{N}$, then almost sure resilient consensus is achieved for the multiagent system as $t \to \infty$.*

Proof. To show the resilient consensus as defined in Section 2.3, we will proceed in two steps by showing first the boundedness of the states of cooperative agents and then the almost sure convergence.

(i) Recall that $y_i(t) = (y_{i,n-1}(t), y_{i,n-2}(t), \cdots, y_{i,0}(t))^{\mathrm{T}}$ for $t \in \mathbb{N}$, and it follows from (3) and (5) that for any node $i \in \mathcal{C}$,

$$\begin{aligned} y_{i,n-1}(t+1) =& -\sum_{l=1}^{n-1} \beta_l y_{i,n-l}(t) + z_i(t) \\ &+ \gamma_i(t) \cdot \big(a_{ij_1}(t)(z_{j_1}(t) - z_i(t)) \\ &+ a_{ij_2}(t)(z_{j_2}(t) - z_i(t))\big), \end{aligned} \tag{6}$$

and

$$\begin{aligned} &y_{i,n-2}(t+1) = y_{i,n-1}(t), \quad y_{i,n-3}(t+1) = y_{i,n-2}(t), \\ &\cdots, \quad y_{i,0}(t+1) = y_{i,1}(t). \end{aligned} \tag{7}$$

By (6) and the definition of $z_i(t)$ in (4), we obtain

$$\begin{aligned} z_i(t+1) =& z_i(t) + \gamma_i(t) \cdot \big(a_{ij_1}(t)(z_{j_1}(t) - z_i(t)) \\ &+ a_{ij_2}(t)(z_{j_2}(t) - z_i(t))\big), \end{aligned} \tag{8}$$

for $i \in \mathcal{C}$.

For any $t \in \mathbb{N}$, define $\overline{z}(t) = \max_{i\in\mathcal{C}} z_i(t)$ and $\underline{z}(t) = \min_{i\in\mathcal{C}} z_i(t)$. Fix any node $i_0 \in \mathcal{C}$. By assumption, $\mathcal{G}(t)$ is $(2r+1)$-excess robust; see Section II.A for the definition. If $r \geq 1$, the in-degree of $\mathcal{G}(t)$ is at least $2r+1$ as commented in Section 2.1. Since $\mathcal{M}$ is r-local, i_0 will receive no less than $r+1$ neighbors' states inside the range $[\underline{z}(t), \overline{z}(t)]$ at time $t+1$. Applying our median-based consensus protocol and using the fact that $\mathcal{M}$ is r-local again, we conclude that any value outside the range $[\underline{z}(t), \overline{z}(t)]$ will not be used in the update rule (5) for i_0 at step $t+1$. Namely, $z_{j_1}(t), z_{j_2}(t) \in [\underline{z}(t), \overline{z}(t)]$ here. In view of the assumption of $\gamma_{i_0}(t)$ in (5), we know from (8) that $z_{i_0}(t+1)$ is a convex combination of $z_{i_0}(t)$, $z_{j_1}(t)$, and $z_{j_2}(t)$. Hence, $z_{i_0}(t+1) \in [\underline{z}(t), \overline{z}(t)]$. On the other hand, if $r = 0$, some cooperative nodes in $\mathcal{C}$ may have zero in-degree. But in this case, the states of such nodes will still remain in $[\underline{z}(t), \overline{z}(t)]$ at time $t+1$. Hence, we have $\underline{z}(t) \leq \underline{z}(t+1) \leq \overline{z}(t+1) \leq \overline{z}(t)$. Note that although these states are random, this relationship always holds.

For $i \in \mathcal{C}$, let $y_i(t) = (y_{i,n-1}(t), w_i(t)^{\mathrm{T}})^{\mathrm{T}}$. Namely, $w_i(t) = (y_{i,n-2}(t), y_{i,n-3}(t), \cdots, y_{i,0}(t))^{\mathrm{T}}$. Involving (4) and (7), we have

$$w_i(t+1) = \hat{A} w_i(t) + \hat{B} z_i(t), \tag{9}$$

where $\hat{A} = \begin{pmatrix} -\beta_1 & -\beta_2 & -\beta_3 & \cdots & -\beta_{n-1} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\hat{B} = (1, 0, 0, \cdots, 0)^{\mathrm{T}} \in \mathbb{R}^{n-1}$. Since the characteristic polynomial $\det(\lambda I_{n-1} - \hat{A})$ of $\hat{A}$ is precisely the Schur stable polynomial defined below (4), we know that $\hat{A}$ is a Schur stable matrix. Using the input-to-state stability of the system (9), we know that $w_i(t)$ is bounded for all $t \in \mathbb{N}$; see e.g. [24, Example 3.4, Lemma 3.5]. Thanks to (6), $y_{i,n-1}(t)$ is also bounded. Consequently, $y_i(t)$ and hence $x_i(t) = S y_i(t)$ are also bounded for $t \in \mathbb{N}$.

(ii) To show the convergence, we will rely on the martingales convergence theorem; see, e. g. [25, p. 2]. From the step (i), we know that $\mathbb{E}(|\overline{z}(t) - \underline{z}(t)|) < \infty$ for $t \in \mathbb{N}$ and $\mathbb{E}(\overline{z}(t) - \underline{z}(t) | \mathcal{F}_{t-1}) \leq \overline{z}(t-1) - \underline{z}(t-1)$ for $t \geq 1$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{\{z_i(0)\}_{i\in\mathcal{V}}, \{z_i(1)\}_{i\in\mathcal{V}}, \cdots, \{z_i(t)\}_{i\in\mathcal{V}}\}$. Clearly, $\overline{z}(t) - \underline{z}(t)$ is a super-martingale with respect to the filtration $\mathcal{F}_t$. By the martingale convergence theorem, there exists some $z \geq 0$ such that

$$\overline{z}(t) - \underline{z}(t) \to z, \tag{10}$$

almost surely as $t \to \infty$.

We claim that the limit point z must be zero. In fact, if $z > 0$, we consider three sets of cooperative nodes partitioning $\mathcal{C}$ [13]: $\mathcal{Z}_1(t) = \{i \in \mathcal{C} : z_i(t) = \overline{z}(t)\}$, $\mathcal{Z}_2(t) = \{i \in \mathcal{C} : z_i(t) = \underline{z}(t)\}$, and $\mathcal{Z}_3(t) = \mathcal{C} \backslash (\mathcal{Z}_1(t) \cup \mathcal{Z}_2(t))$. Note that $\mathcal{Z}_1(t)$ and $\mathcal{Z}_2(t)$ are nonempty and disjoint. Since $\mathcal{G}(t)$ is $(2r+1)$-excess robust, one of the two sets $\mathcal{Z}_1(t)$ and $\mathcal{Z}_2(t)$ must be $(2r+1)$-excess reachable. In other words, there is a cooperative node i_0 in, for example $\mathcal{Z}_1(t)$, which has at least $2r+1$ more neighbors outside $\mathcal{Z}_1(t)$ than inside it. Since $\mathcal{M}$ is r-local, i_0

has at least $r+1$ more cooperative neighbors outside $\mathcal{Z}_1(t)$ than inside it. Recall that those cooperative neighbors outside must have value less than $z_{i_0}(t) = \overline{z}(t)$. By our median-based strategy, at least one cooperative neighbor outside will be used in the update of i_0 at time $t+1$, and no value greater than $z_{i_0}(t)$ will be used here. Therefore, by (6), the node i_0 will move toward $\mathcal{Z}_2(t)$ at time step $t+1$. With a similar argument as above, we know $\overline{z}(t) - \underline{z}(t)$ converges to 0 in probability as $t \to \infty$. However, this contradicts the assumption $z > 0$. Hence, we have $z = 0$ in (10).

Since $\overline{z}(t)$ and $\underline{z}(t)$ are monotonic, there exits $\hat{z} \in \mathbb{R}$ satisfying $\lim_{t\to\infty} z_i(t) = \hat{z}$ almost surely for all $i \in \mathcal{C}$. To show the convergence of $\{x_i(t)\}_{i\in\mathcal{C}}$ we will first examine the convergence of $\{w_i(t)\}_{i\in\mathcal{C}}$ through (9). To this end, for $i \in \mathcal{C}$ define the error $e_i(t) = w_i(t) - \hat{z}/(1+\sum_{l=1}^{n-1}\beta_l)1_{n-1}$, where $1_{n-1} \in \mathbb{R}^{n-1}$ is an all one vector. We have

$$\begin{aligned} e_i(t+1) =& \hat{A}w_i(t) + \hat{B}z_i(t) - \frac{\hat{z}}{1+\sum_{l=1}^{n-1}\beta_l}1_{n-1} \\ =& \hat{A}e_i(t) + \hat{A}\frac{\hat{z}}{1+\sum_{l=1}^{n-1}\beta_l}1_{n-1} + \hat{B}z_i(t) \\ & - \frac{\hat{z}}{1+\sum_{l=1}^{n-1}\beta_l}1_{n-1} \\ =& \hat{A}e_i(t) - \hat{z}\hat{B} + z_i(t)\hat{B}. \end{aligned} \tag{11}$$

Let $\delta_i(t) = z_i(t) - \hat{z}$ for $i \in \mathcal{C}$. Then $\lim_{t\to\infty}\delta_i(t) = 0$ almost surely and

$$e_i(t+1) = \hat{A}e_i(t) + \hat{B}\delta_i(t). \tag{12}$$

Since $\hat{A}$ is stable, for any positive definite matrix $Q \in \mathbb{R}^{(n-1)\times(n-1)}$ there exists a positive definite matrix $P \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying $Q = P - \hat{A}^{\mathrm{T}}P\hat{A}$ by the Lyapunov stability theory; see e.g. [26, Theorem 5.D5]. For $i \in \mathcal{C}$, define $V_i(t) = e_i(t)^{\mathrm{T}}Pe_i(t)$ and along the solution of (12) we obtain

$$\begin{aligned} V_i(t+1) - V_i(t) =& -e_i(t+1)^{\mathrm{T}}Pe_i(t+1) - e_i(t)^{\mathrm{T}}Pe_i(t) \\ =& -e_i(t)^{\mathrm{T}}Qe_i(t) + 2e_i(t)^{\mathrm{T}}\hat{A}^{\mathrm{T}}P\hat{B}\delta_i(t) \\ & + \delta_i(t)^2\hat{B}^{\mathrm{T}}P\hat{B}. \end{aligned} \tag{13}$$

As $\lim_{t\to\infty}\delta_i(t) = 0$ almost surely, we have $\lim_{t\to\infty}|2e_i(t)^{\mathrm{T}}\hat{A}^{\mathrm{T}}P\hat{B}\delta_i(t) + \delta_i(t)^2\hat{B}^{\mathrm{T}}P\hat{B}| = 0$ almost surely.

From (13) we know that for any $i \in \mathcal{C}$, $\lim_{t\to\infty}V_i(t) = 0$ almost surely. In fact, if this is not true, for any $t \in \mathbb{N}$ there exists $t' \geq t$ such that $e_i(t')^{\mathrm{T}}Qe_i(t') > 0$ and there exists $\rho > 0$ and $t'' \geq t'$ such that $V_i(t''+1) - V_i(t'') \leq -\rho$. Hence, $\limsup_{t\to\infty}V_i(t) = 0$ almost surely, which contradicts our assumption.

In view of the definition of $V_i(t)$, we have $\lim_{t\to\infty}e_i(t) = 0$ almost surely for $i \in \mathcal{C}$. This indicates $\lim_{t\to\infty}w_i(t) = \hat{z}/(1+\sum_{l=1}^{n-1}\beta_l)1_{n-1}$ almost surely. It follows from (7) that $\lim_{t\to\infty}y_i(t) = \hat{z}/(1+\sum_{l=1}^{n-1}\beta_l)1_n$ almost surely. Since $x_i(t) = Sy_i(t)$, we obtain $\lim_{t\to\infty}x_i(t) = \hat{z}/(1+\sum_{l=1}^{n-1}\beta_l)S1_n$ almost surely. As the limit holds for all $i \in \mathcal{C}$, the proof is complete. □

Remark 3. We have shown the convergence of states $x_i(t)$ for $i \in \mathcal{C}$, which is slightly stronger than the definition of resilient consensus defined at the end of Section 2 as a final constant consensus vector exists. This final status vector as well as the transient trajectories is influenced by the malicious agents because our strategy does not guarantee the removal of all malicious nodes at all times.

IV. SIMULATIONS

We consider a multiagent system with $N = 9$ agents, where $\mathcal{C} = \{1, 2, \cdots, 8\}$ and $\mathcal{M} = \{9\}$, over a random network $\mathcal{G}(t)$ with binary weights and $p_{i,i+1(\mod 9)}(t) = 0$, $p_{i+2(\mod 9),i}(t) = 0.5$, and any other $p_{ij}(t) = 1$ for $i \neq j$. For $t \in \mathbb{N}$ and $i \in \mathcal{V}$, the agents' dynamics are given by

$$x_i(t+1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x_i(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_i(t), \tag{14}$$

where $u_i(t)$ for $i \in \mathcal{C}$ is given by (5) with $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = 0.5$, $\gamma_i(t) = 0.3$, and $u_9(t) = \sin(t/10) + \ln((t+1)/10)$. Let $x_i(t) = (x_{i,1}(t), x_{i,2}(t))$ for $i \in \mathcal{C}$. It is straightforward to check that the conditions of Theorem 1 hold.

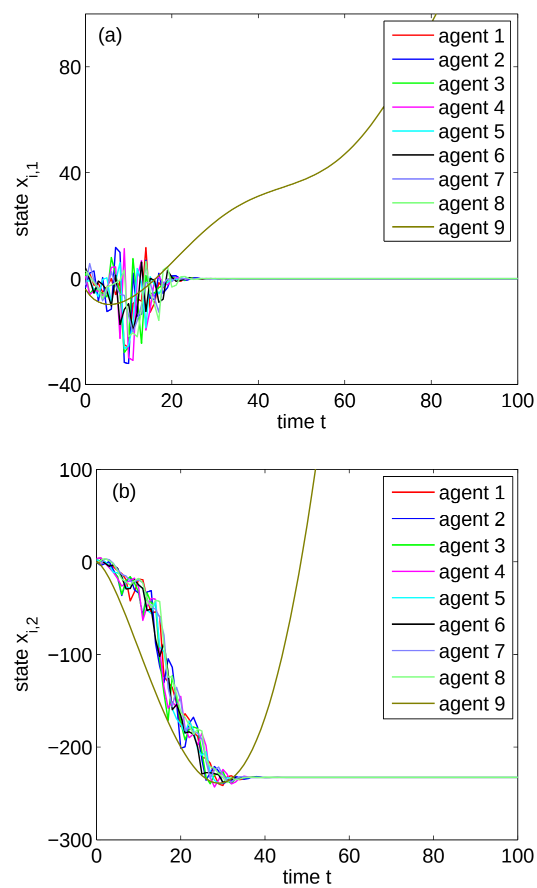

Fig. 2. Time evolution of the multiagent system (14) under the median-based resilient consensus strategy. The two state components are shown in (a) and (b), respectively.

By taking initial values randomly from $[-5, 5]$, we show the dynamical evolution of the system states in Fig. 2. We observe that for both components of the states in $\mathcal{C}$, the resilient consensus is achieved. The consensus can be better appreciated in Fig. 3, where the consensus error is defined as $\Delta(t) := \max_{i,j\in\mathcal{C}}\|x_i(t) - x_j(t)\|$, where $\|\cdot\|$ represents the

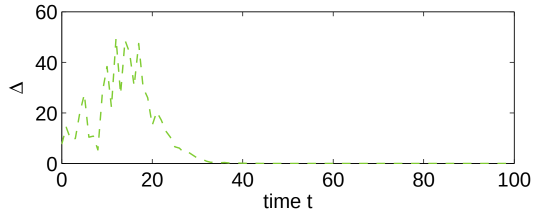

Fig. 3. The evolution of consensus error Δ with respect to time t for the system shown in Fig. 2.

Euclidean norm. The malicious agent 9 affects the evolution trajectories but fails to ruin the global asymptotic consensus reaching among cooperative agents.

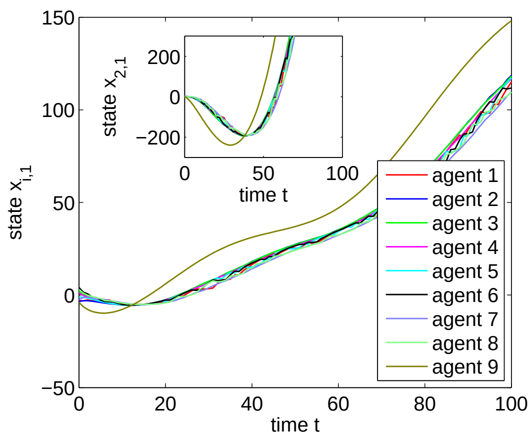

Fig. 4. Malicious agent prevents cooperative agents from reaching consensus under a simple averaging mechanism.

In the classical mean-based resilient strategies [4], [6], if a proper estimate of r is not agreed among all cooperative agents, resilient consensus may fail. We show in Fig. 4 the trajectories for the same system as above but using a simple averaging mechanism [1] without removing any neighbors. The malicious successfully prevents the global consensus in this case.

V. Conclusion

Motivated by security and uncertainties in realistic networked systems, we have addressed the resilient consensus problem for a group of dynamical cooperative agents over a time-varying random network in the presence of malicious or faulty agents. To achieve almost sure consensus, sufficient conditions are provided, which feature the importance of robust topology and bounded faults. The proposed median-based consensus protocol is a purely distributed strategy and no shared estimate regarding the number of malicious agents is needed for cooperative agents. Note that in our model (1) the update rate is taken as a unit, which means that limited information capacity is not taken into consideration. In reality, it is known that bandwidth and noises have noticeable influence on the multiagent system coordination [27]. It would be interesting to extend our framework in the direction involving information-theoretic concepts.

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