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Interval consensus of switched multiagent systems

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ABSTRACT

This paper studies interval consensus for switched multiagent systems over directed networks, which consist of a continuous-time subsystem and a discrete-time subsystem regulated by a switching rule. Interval consensus refers to a state constrained consensus, where each agent is allowed to propose an acceptable interval to saturate their expressed states and the final consensus state lies in the nonempty intersection of all these intervals. We establish conditions guaranteeing interval consensus for switched multiagent system under arbitrary switching rules. Furthermore, we introduce the scaled interval consensus notion, which allows both the convergence of states to a pre-assigned proportion and the scaled consensus values lying in a desired range. Simulation results are provided to verify the effectiveness of the theoretical results.

KEYWORDS

Switched multiagent system; constrained consensus; scaled consensus; directed network

1. Introduction

Distributed coordination in the control of multiagent systems has received considerable attention in control and system engineering. One of the fundamental topics is consensus problems (Cao et al., 2013; Olfati-Saber et al., 2007), which capitalize on the underlying network structure of complex systems to design distributed control protocols that facilitate agreement on certain global behavior of common interest. Many system constraints, including input saturation, measurement and communication, have been factored in recent research to accommodate realistic complexity in various applications (Ding et al., 2020; Fu et al., 2019; Silva et al., 2021; Yan et al., 2020).

Consensus in multiagent systems with state constraints has been challenging since the consensus state has to stay in the constraint set given all other restrictions such as system uncertainties and local information exchanges. For discrete-time systems, a projection-based method has been introduced in Nedić et al. (2010) to confine state trajectories in closed convex sets over networks with doubly stochastic adjacency matrices. This algorithm is generalized to cope with time delay (Lin et al., 2017), random

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noise (Li & Ren, 2021), and higher-order agent dynamics (Lin et al., 2020). Constrained consensus in continuous-time has been investigated in Lee & Mesbahi (2011) by using logarithmic barrier functions to push the states back to the constraint sets. The idea of projection has also been applied to continuous-time systems in the presence of malicious agents (Shang, 2020a). A discarded consensus protocol is proposed for both continuous- and discrete-time systems in Liu & Chen (2012) to discard a state of neighbor that falls outside the constraint set.

An interesting direction considered by a recent group of work is to limit the system excursion by employing the so-called interval consensus protocol (Fontan et al., 2020), where each agent proposes an admissible interval indicating a preferred operating range and only transmits its value saturated within this range. It is shown in Fontan et al. (2020) that the final consensus value lies in the intersection of all admissible intervals while the state trajectories are trespassable during the evolution. This idea is relevant in many applications. For example, a group of robots in a surveillance network may need to achieve a formation, where each robot has a certain target region; a processor may be able to share a workload only if the task is within the constraint of its allocated resource; an individual in social interactions may only feel comfortable to express their opinion if its not too extreme (Shang, 2021). The research of interval consensus has been conducted very recently in the case of uncertainty that is present in topology structure (Fu et al., 2020) and agent behavior (Shang, 2020b).

However, to our knowledge consensus problems with state constraints have only been investigated in multiagent systems consisting of only continuous-time subsystems or only discrete-time subsystems. In reality, a switched multi-agent system consisting of both continuous-time and discrete-time subsystems is more general and has many applications (Zheng & Wang, 2016). For instance, activating all agents in a discrete manner by a computer in an otherwise continuous-time multiagent system forms a switched multiagent system consisting of both continuous-time and discrete-time subsystems. Continuous-time plant can also be controlled by a digitally performed regulator or a physically performed one together with a switching protocol between them. Some recent applications of switched multiagent systems in communication networks with bumpless transfer control have been discussed in Ma et al. (2021). Based on the Lyapunov theory, consensus has been firstly proved for switched multiagent systems over connected undirected graphs in Zheng & Wang (2016). Since then switched multiagent systems have been deeply examined in a number of more advanced consensus algorithms including finite-time consensus (Lin & Zheng, 2017), resilient consensus (Shang, 2018), scaled consensus (Shang, 2019), controllability and observability (Tian et al., 2019), leader-follower containment control (Wang et al., 2020), and cluster consensus (Shang, in press). It is worth noting that asynchronously switched multiagent systems have been extensively studied in e.g. Xue et al. (2020); Yoo (2018), where heterogeneous dynamical agents do not share a common switching signal. However, in these works agents only follow continuous-time control laws.

In this paper we contribute to this line of research by considering interval consensus in switched multiagent systems over a directed network, where a continuous-time subsystem and a discrete-time subsystem alternate possibly obeying an arbitrary switching law. We propose an interval consensus protocol and show that consensus can be achieved with the final state converging inside the intersection of admissible intervals of all agents provided the underlying network is strongly connected. Previous works on switched systems (such as Lin & Zheng (2017); Shang (2018, 2019); Tian et al. (2019); Wang et al. (2020); Zheng & Wang (2016)) rely on the agent dynamics where state can evolve without restriction. The methods therein are not applicable in the constrained consensus problems considered here. Moreover, we extend the theory to scaled interval consensus by incorporating a scaling coefficient for each agent so that ratio of the states of any two agents asymptotically reach a prescribed value while their scaled states remain in the common intersection of their admissible intervals. As a byproduct, this result helps to demonstrate how ratio convergence is linked to state convergence (Roy, 2015), which have only been studied as two orthogonal directions so far. In fact, in the previous works, a domain of possible consensus state cannot be determined when ratio convergence is the control objective (Roy, 2015; Shang, 2019; Zhang et al., 2020). We mention that our framework is a two mode system, while the works Xue et al. (2020); Yoo (2018) admit general m-mode dynamics.

The result of the paper is organized as follows. Section 2 presents some preliminaries and the problem formulation. Section 3 is devoted to our main results. Numerical examples are provided in Section 4. We conclude the paper in Section 5.

2. Model formulation

Let $\mathbb{R}^+ = \{x : x \geq 0\}$ and $\mathbb{N}^+ = \{0, 1, 2, \cdots\}$ be the sets of non-negative reals and non-negative integers, respectively. Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ representing the communication topology of n agents in the node set $\mathcal{V} = \{1, 2, \cdots, n\}$, where $(i, j) \in \mathcal{E}$ is a directed edge from node i to node j indicating a communication link from i to j. The neighborhood of i is denoted by $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$ and the adjacency matrix associated with $\mathcal{G}$ is $(a_{ij}) \in (\mathbb{R}^+)^{n\times n}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$.

Inspired by the interval consensus framework, we assume each agent $i \in \mathcal{V}$ has an admissible interval $\mathcal{I}_i = [p_i, q_i]$ which the agent would like their final state to be in. The admissible interval is only assumed to be known to the agent i. Let $x_i(t)$ be the information state of agent i at time $t \geq 0$. We propose the following switched multiagent system, which consists of a continuous-time subsystem

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(\phi_j(x_j(t)) - x_i(t)), \quad i \in \mathcal{V} \tag{1}$$

and a discrete-time subsystem

$$x_i(t+1) = x_i(t) + h \sum_{j\in\mathcal{N}_i} a_{ij}(\phi_j(x_j(t)) - x_i(t)), \quad i \in \mathcal{V}, \tag{2}$$

where

$$\phi_j(z) = \begin{cases} p_j, & z < p_j \\ z, & p_j \leq z \leq q_j \\ q_j, & z > q_j, \end{cases} \tag{3}$$

is the saturated state that a neighbor received from agent j and $h > 0$ is the sampling period. Recall that the interval for agent i is given by $\mathcal{I}_i = [p_i, q_i]$ satisfying $p_i \leq q_i$ but the signs for p_i and q_i can be the same or different. Here, at each time instant t, the activated subsystem, whether (1) or (2), is controlled by the switching rule under consideration.

Remark 1. Note that the quantity $\phi_j(x_j)$ can be well interpreted as the 'expressed' opinion of agent j in a social network, where their opinions may have a discrepancy with their 'private' opinions due to, for example, peer pressure or conformity to certain norm (Shang, 2021). This mechanism is shown in Fontan et al. (2020) to have soft control over the state evolution and hence is different from many previous work in Li & Ren (2021); Lin et al. (2017); Liu & Chen (2012); Nedić et al. (2010), where private states are directly adjusted and taken on a par with expressed states.

Remark 2. The discrete-time subsystem and the continuous-time subsystem is activated alternatively in the switched multiagent system framework; c.f. Fig. 1 for a schematic of the data flow. Note that this framework is different from hybrid systems (Shang, 2020c; Zhao et al., 2020), where some nodes follow discrete-time protocols while others follow continuous-time ones. If the continuous-time system (1) is switched to the discrete-time system (2) at some time $t_0 \in \mathbb{R}^+$, the system (2) will follow as $x_i(t+1) = x_i(t) + h\sum_{j\in\mathcal{N}_i} a_{ij}(\phi_j(x_j(t)) - x_i(t))$ for $t = t_0,\ t_0+1,\ t_0+2, \cdots$. On the other hand, if the discrete-time system (2) is switched to the continuous-time system (1) at some time $t_0 \in \mathbb{R}^+$, then the system (1) will similarly follow starting from this time point.

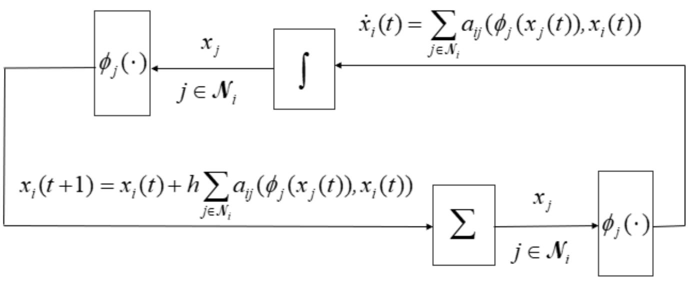

Figure 1. Data flow for the switched multiagent system.

Given a continuous and locally Lipschitz function $V(t) : \mathbb{R}^+ \to \mathbb{R}$, the Dini derivative is defined as $D^+V(t) = \limsup_{s\to 0^+}(V(t+s) - V(t))/s$. Let $V(t)$ be a continuous function on the interval (t_1, t_2). V is non-increasing over this interval if and only if $D^+V(t) \leq 0$ for any $t \in (t_1, t_2)$. Consider the differential equation

$$\dot{x}(t) = f(x(t)), \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $x(t)$ be a solution of (4). The upper right-hand derivative of V with respect to (4) is defined by $D_f^+V(x) = \limsup_{s\to 0^+}(V(x + sf(x)) - V(x))/s$. It is known that $D_f^+V(x)|_{x=x_0} = D^+V(x(t))|_{t=t_0}$, where $x(t_0) = x_0$ (Rouche et al., 1977). Moreover, we will use the following property of the Dini derivative.

Lemma 1. ((Danskin, 1966; Shi et al., 2013)) *For $i \in \mathcal{V}$, let $V_i(x) : \mathbb{R}^m \to \mathbb{R}$ be C^1 and $V(x) = \max_{i\in\mathcal{V}} V_i(x)$. Let $x(t)$ be an absolutely continuous function over some interval (t_1, t_2). Then $D^+V(x(t)) = \max_{i\in\Theta(t)} \dot{V}_i(x(t))$ for $t \in (t_1, t_2)$, where $\Theta(t) = \{i \in \mathcal{V} : V(x(t)) = V_i(x(t))\}$.*

As the state of agent $j \in \mathcal{V}$ is saturated via $\phi_j(x_j)$ following (1) and (2), the following results regarding robust consensus are useful. The continuous-time part stems from Shi & Johansson (2013, Thm 4.1) and the discrete-time part is a special case of Lin & Lei (2018, Thm 1).

Lemma 2. *Assume that $\mathcal{G}$ is a directed graph containing a directed spanning tree. Let $x(0) = (x_1(0), \cdots, x_n(0))$ be the initial condition.*

(i) Consider the continuous-time system over $\mathcal{G}$,

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) + \nu_i(t),$$
$$i \in \mathcal{V}, t \in \mathbb{R}^+, \tag{5}$$

where $\nu_i(t)$ is piecewise continuous. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that the following holds: If $\max_{i\in\mathcal{V}} \sup_{t\in\mathbb{R}^+} |\nu_i(t)| \leq \delta$, then $\limsup_{t\to\infty} \max_{i,j\in\mathcal{V}} |x_i(t) - x_j(t)| \leq \varepsilon$ for any $x(0)$.

(ii) Consider the discrete-time system over $\mathcal{G}$,

$$x_i(t+1) = x_i(t) + h \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) + \nu_i(t),$$
$$i \in \mathcal{V}, t \in \mathbb{N}^+, \tag{6}$$

where $0 < h < 1/\max_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} a_{ij}$ and $\nu_i(t)$ is bounded. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that the following holds: If $\max_{i\in\mathcal{V}} \sup_{t\in\mathbb{N}^+} |\nu_i(t)| \leq \delta$, then $\limsup_{t\to\infty} \max_{i,j\in\mathcal{V}} |x_i(t) - x_j(t)| \leq \varepsilon$ for any $x(0)$.

Remark 3. The quantity $\nu_i(t)$ in (5) and (6) presents noise or perturbation incurred to the system. Lemma 2 indicates that consensus error can be made sufficiently small if the noise can be appropriately controlled. We will show that the constructed noise term in our switched multiagent system (1) and (2) is vanishing, and hence by letting $\varepsilon \to 0$ in Lemma 1 we will derive the desired consensus behavior.

3. Main results

3.1. Interval consensus analysis

In this section, we investigate the interval consensus of switched multiagent system (1) and (2), where each agent can independently nominate an admissible interval and the system dynamics switch between continuous-time and discrete-time following a switching rule. Formally,

Definition 1. (Fontan et al., 2020) For each node $i \in \mathcal{V}$, we say that the multiagent system under consideration reaches interval consensus if $\lim_{t\to\infty} x_i(t)$ exits and the limit belongs to $\cap_{i\in\mathcal{V}}\mathcal{I}_i$ for any initial condition $x(0)$ and $i \in \mathcal{V}$.

Recall that ϕ_j in (3) is a non-decreasing function and we set $[\hat{p}, \hat{q}] = \cap_{i\in\mathcal{V}}\mathcal{I}_i$ if the intersection is non-empty. Clearly, $\hat{p} = \max_{i\in\mathcal{V}} p_i$ and $\hat{q} = \min_{i\in\mathcal{V}} q_i$. Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$.

Our first result regarding interval consensus of switched systems is the following.

Theorem 1. *Consider the switched multiagent system (1) and (2) over the directed graph $\mathcal{G}$ under an arbitrary switching rule. Assume that $\mathcal{G}$ is strongly connected and $[\hat{p}, \hat{q}]$ is non-empty. If $h < 1/\max_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} a_{ij}$, then for any $x(0)$, there exists $\hat{c} \in [\hat{p}, \hat{q}]$ such that $\lim_{t\to\infty} x_i(t) = \hat{c}$ for all $i \in \mathcal{V}$.*

Before proving the theorem, several remarks are in order.

Remark 4. It is worth noting that merely having a directed spanning tree in $\mathcal{G}$ does not work here. Suppose there is a root node $r \in \mathcal{V}$ such that $\mathcal{N}_r = \emptyset$. If $x_r(0) \notin [\hat{p}, \hat{q}]$, then obviously the above final consensus value $\hat{c}$ can not be found.

Remark 5. We do not impose any restriction such as $x_i(0) \in \mathcal{I}_i$ $(i \in \mathcal{V})$, on the initial condition $x(0)$. As mentioned in Fontan et al. (2020), this feature of interval consensus provides more flexibility compared with other consensus algorithms with hard-wired constraints; see e.g. Li & Ren (2021); Lin et al. (2017); Nedić et al. (2010); Shang (2020a).

Remark 6. The switching law here between continuous-time subsystem and discrete-time subsystem can be arbitrary, which is highly desirable as the switching law usually can not be pre-determined. However, previous works on switched multiagent systems often impose restrictive conditions such as lower bounds on the activation time of continuous-time subsystems; see e.g. Lin & Zheng (2017); Tian et al. (2019); Wang et al. (2020).

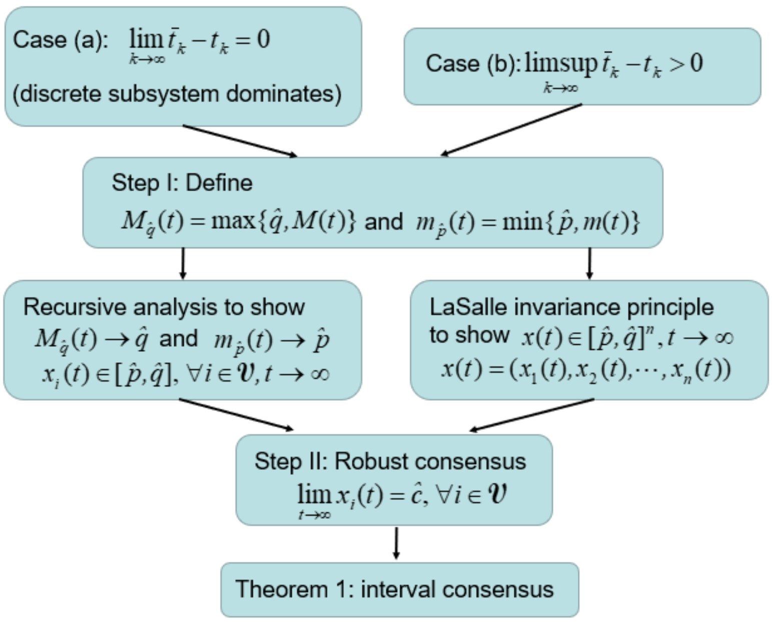

Figure 2. Schematic illustration of the proof structure of Theorem 1. It is divided into two situations: Case (a) the discrete-time subsystem dominates and Case (b) the continuous-time subsystem dominates. In each case, the procedure is divided into two steps: Step I shows that the convergence into the common interval and Step II focuses on the robust consensus of the switched system.

Proof of Theorem 1. Let $M(t) = \max_{i\in\mathcal{V}} x_i(t)$ and $m(t) = \min_{i\in\mathcal{V}} x_i(t)$ be the maximum and minimum of the states at time $t \geq 0$. Without loss of generality, suppose that there exists a sequence of time instants $0 \leq t_1 \leq \bar{t}_1 \leq t_2 \leq \bar{t}_2 \leq \cdots \leq \bar{t}_{k-1} \leq t_k \leq \bar{t}_k \leq \cdots$ such that the continuous-time subsystem (1) is activated during $t \in (t_k, \bar{t}_k]$ and the discrete-time subsystem (2) is activated during $t \in (\bar{t}_{k-1}, t_k]$. Two complementary scenarios will be considered below: (a) $\lim_{k\to\infty} \bar{t}_k - t_k = 0$; and (b) there exists $\Delta > 0$ such that for any $k_1 \in \mathbb{N}^+$ we have some $k \geq k_1$ satisfying $\bar{t}_k - t_k \geq \Delta$. A schematic diagram for the proof structure is shown in Fig. 2.

(a). We divide the proof in this scenario into two steps. In Step I, we show $x_i(t) \in [\hat{p}, \hat{q}]$ for any node $i \in \mathcal{V}$ as $t \to \infty$. In Step II, we show the existence of $\hat{c}$.

Step I. For $t \geq 0$, we define two functions $M_{\hat{q}}(t) = \max\{\hat{q}, M(t)\}$ and $m_{\hat{p}}(t) = \min\{\hat{p}, m(t)\}$. When the discrete-time subsystem (2) is activated over $[t, t+1]$, using the assumption $h < 1/\max_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} a_{ij}$ and (3), we have

$$\begin{aligned} M_{\hat{q}}(t+1) =& \max\Big\{\hat{q}, \max_{i\in\mathcal{V}}\Big\{\Big(1-h\sum_{j\in\mathcal{N}_i} a_{ij}\Big)x_i(t) \\ &+ h\sum_{j\in\mathcal{N}_i} a_{ij}\phi(x_j(t))\Big\}\Big\} \\ \leq& \max\Big\{\hat{q}, \max_{i\in\mathcal{V}}\Big\{\Big(1-h\sum_{j\in\mathcal{N}_i} a_{ij}\Big)M_{\hat{q}}(t) \\ &+ h\sum_{j\in\mathcal{N}_i} a_{ij}M_{\hat{q}}(t)\Big\}\Big\} = M_{\hat{q}}(t). \end{aligned} \tag{7}$$

Hence, there exists a constant $\hat{M}$ such that $\lim_{t\to\infty} M_{\hat{q}}(t) = \hat{M}$. On the other hand, when the continuous-time subsystem (1) is activated over some interval $I \subseteq [t, t+1]$, the above limit still holds since $\lim_{t\to\infty} |I| = 0$ and $x_i(t)$ is continuous when (1) is in action. Similarly, we have a constant $\hat{m}$ such that $\lim_{t\to\infty} m_{\hat{p}}(t) = \hat{m}$, and $\hat{m} \leq \hat{p} \leq \hat{q} \leq \hat{M}$.

We next show $\hat{m} = \hat{p}$ by using a contradiction argument. Assume that $\hat{m} < \hat{p}$. For any time s, there exists some node $i_0 \in \mathcal{V}$ satisfying $x_{i_0}(s) = m(s) \leq \hat{m} < \hat{p}$. Since $\mathcal{G}$ is strongly connected, consider a node i_1 with $i_0 \in \mathcal{N}_{i_1}$. When the discrete-time subsystem (2) is activated on $[s, s+1]$, we obtain

$$\begin{aligned} & x_{i_1}(s+1) \\ =& \Big(1-h\sum_{j\in\mathcal{N}_{i_1}} a_{i_1 j}\Big)x_{i_1}(s) + h\sum_{j\in\mathcal{N}_{i_1}, j\neq i_0} a_{i_1 j}\phi_j(x_j(s)) \\ & + h a_{i_1 i_0}\phi_{i_0}(x_{i_0}(s)) \\ \leq& \Big(1-h\sum_{j\in\mathcal{N}_{i_1}} a_{i_1 j}\Big)M_{\hat{q}}(s) + h\sum_{j\in\mathcal{N}_{i_1}, j\neq i_0} a_{i_1 j}M_{\hat{q}}(s) \\ & + h a_{i_1 i_0}\hat{p} \\ =& (1-h a_{i_1 i_0})M_{\hat{q}}(s) + h a_{i_1 i_0}\hat{p} \\ \leq& (1-\theta_1)M_{\hat{q}}(s) + \theta_1\hat{p}, \end{aligned} \tag{8}$$

where we take $\theta_1 = h\min\{a_{ij} : a_{ij} > 0, (i,j) \in \mathcal{E}\}/2 > 0$, and note that $\hat{p} \leq M_{\hat{q}}(s)$. When the continuous-time subsystem (1) is activated inside $[s, s+1]$, there exists a number $\delta(s) \geq 0$ satisfying $\lim_{s\to\infty} \delta(s) = 0$ such that

$$\begin{aligned} x_{i_1}(s+1) \leq& (1-h a_{i_1 i_0})(M_{\hat{q}}(s) + \delta(s)) + h a_{i_1 i_0}(\hat{p} + \delta(s)) \\ \leq& (1-\theta_1)M_{\hat{q}}(s) + \theta_1\hat{p}, \end{aligned} \tag{9}$$

where we choose s sufficiently large and hence $\delta(s)$ can be sufficiently close to zero. Likewise, we examine the state of node i_0 at time $s+1$ as follows. When the discrete-time system (2) is activated on $[s, s+1]$, we have

$$\begin{aligned} x_{i_0}(s+1) \leq & \Big(1 - h \sum_{j\in\mathcal{N}_{i_0}} a_{i_0 j}\Big)\hat{p} + \Big(h \sum_{j\in\mathcal{N}_{i_0}} a_{i_0 j}\Big) M_{\hat{q}}(s) \\ \leq & \theta_2 \hat{p} + (1-\theta_2) M_{\hat{q}}(s), \end{aligned} \tag{10}$$

where we take $\theta_2 = (1 - h \max_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} a_{ij})/2 > 0$. When the continuous-time subsystem (1) is activated inside $[s, s+1]$, similarly arguing as (9) we can derive (10) again. Hence, we have for $k = i_0, i_1$ and sufficiently large s,

$$x_k(s+1) \leq \theta\hat{p} + (1-\theta) M_{\hat{q}}(s), \tag{11}$$

where $\theta := \min\{\theta_1, \theta_2\}$.

Consider the contiguous time interval $[s+1, s+2]$. When the discrete-time subsystem is activated on it, we have for $k = i_0, i_1$,

$$\begin{aligned} & x_k(s+2) \\ = & \Big(1 - h \sum_{j\in\mathcal{N}_k} a_{kj}\Big) x_k(s+1) + h \sum_{j\in\mathcal{N}_k} a_{kj} \phi_j(x_j(s+1)) \\ \leq & \theta\hat{p} + \theta\Big(h \sum_{j\in\mathcal{N}_k} a_{kj}\Big)(M_{\hat{q}}(s) - \hat{p}) + (1-\theta) M_{\hat{q}}(s) \\ \leq & \theta\hat{p} + \theta(1-2\theta)(M_{\hat{q}}(s) - \hat{p}) + (1-\theta) M_{\hat{q}}(s) \\ = & 2\theta^2 \hat{p} + (1-2\theta^2) M_{\hat{q}}(s), \end{aligned} \tag{12}$$

where we have used the definition of θ and (7). When the continuous-time subsystem (1) is activated inside $[s+1, s+2]$, we can take s sufficiently large such that

$$\begin{aligned} x_k(s+2) \leq & \theta(\hat{p} + \delta(s)) + \theta(1-2\theta)(M_{\hat{q}}(s) - \hat{p} + \delta(s)) \\ & + (1-\theta)(M_{\hat{q}}(s) + \delta(s)) \\ \leq & \theta^2 \hat{p} + (1-\theta^2) M_{\hat{q}}(s), \end{aligned} \tag{13}$$

where, again as in (9), $\delta(s) \geq 0$ is a number satisfying $\lim_{s\to\infty} \delta(s) = 0$. Combining (12) and (13) we have for $k = i_0, i_1$ and sufficiently large s, $x_k(s+2) \leq \theta^2\hat{p} + (1-\theta^2) M_{\hat{q}}(s)$. Repeating the above arguments, we obtain for $k = i_0, i_1$,

$$x_k(s+\tau) \leq \theta^\tau \hat{p} + (1-\theta^\tau) M_{\hat{q}}(s), \tag{14}$$

where $\tau = 1, 2, \cdots, n-1$. As $\mathcal{G}$ is strongly connected, we next consider a node i_2 such that $i_0 \in \mathcal{N}_{i_2}$ or $i_1 \in \mathcal{N}_{i_2}$. Without loss of generality, we assume the latter. When the discrete-time subsystem (2) is activated on $[s+1, s+2]$, by using (7) and (14) we have

$$\begin{aligned} x_{i_2}(s+2) \leq & (1 - h a_{i_2 i_1}) M_{\hat{q}}(s) + h a_{i_2 i_1}((1-\theta) M_{\hat{q}}(s) + \theta\hat{p}) \\ \leq & (1-2\theta) M_{\hat{q}}(s) + 2\theta((1-\theta) M_{\hat{q}}(s) + \theta\hat{p}) \\ = & 2\theta^2 \hat{p} + (1-2\theta^2) M_{\hat{q}}(s). \end{aligned} \tag{15}$$

When the continuous-time subsystem (1) is activated inside $[s+1, s+2]$, similarly as above there exists $\delta(s) \geq 0$ and $\lim_{s\to\infty} \delta(s) = 0$ such that

$$\begin{aligned}
&x_{i_2}(s+2) \\
\leq &(1-2\theta)(M_{\hat{q}}(s)+\delta(s)) + 2\theta((1-\theta)M_{\hat{q}}(s)+\theta\hat{p}) \\
\leq &(1-\theta^2)M_{\hat{q}}(s)+\theta^2\hat{p}, \qquad (16)
\end{aligned}$$

where we take s sufficiently large. Hence, combining (15) and (16) we know that $x_{i_2}(s+2) \leq (1-\theta^2)M_{\hat{q}}(s)+\theta^2\hat{p}$ always holds. Repeating the above arguments, we obtain

$$x_{i_2}(s+\tau) \leq (1-\theta^\tau)M_{\hat{q}}(s)+\theta^\tau\hat{p} \qquad (17)$$

for $\tau = 2, 3, \cdots, n-1$. Since the graph is strongly connected, we can then consider a node i_3 so that $i_0 \in \mathcal{N}_{i_3}$ or $i_1 \in \mathcal{N}_{i_3}$ or $i_2 \in \mathcal{N}_{i_3}$. We eventually can visit all nodes in $\mathcal{G}$ and a recursion gives rise to

$$x_k(s+n-1) \leq (1-\theta^{n-1})M_{\hat{q}}(s)+\theta^{n-1}\hat{p} \qquad (18)$$

for $k = i_0, i_1, \cdots, i_{n-1}$ and sufficiently large s.

We examine two situations: (i) $\hat{p} < \hat{M}$ and (ii) $\hat{p} = \hat{M}$. We will show

$$\limsup_{t\to\infty} M(t) \leq \hat{p} \qquad (19)$$

is always true. In fact, in the case of (i), by the definition of $\hat{M}$ we have $(1-\theta^{n-1})M_{\hat{q}}(s)+\theta^{n-1}\hat{p} < \hat{M}$ for any sufficiently large s. By (18) we have $x_k(s+n-1) < \hat{M}$ for all $k \in \mathcal{V}$. This means $\hat{M} = \hat{q}$. Therefore, there is $T > 0$ such that $\phi_i(x_i(t)) \leq M_{\hat{p}}(t)$ for all $i \in \mathcal{V}$ and $t \geq T$ regardless of the specific subsystems. Hence, we can prove inequalities (8)-(18) again by replacing $M_{\hat{q}}(s)$ with $M_{\hat{p}}(s)$ therein employing the same argument for sufficiently large s satisfying $s \geq T$. In particular, the bound (18) can be recast as

$$M(s+n-1) \leq (1-\theta^{n-1})M_{\hat{p}}(s)+\theta^{n-1}\hat{p} \qquad (20)$$

for sufficiently large s. Let $s \to \infty$ and we obtain (19). In the case of (ii), the statement $\limsup_{t\to\infty} M(t) > \hat{p} = \hat{M}$ would lead to a contradiction against the definition of $\hat{M}$. Hence, (19) is true.

Recall we have assumed that $\hat{m} < \hat{p}$. This means $\lim_{t\to\infty} m(t) = \hat{m}$. We next claim that $\limsup_{t\to\infty} M(t) = \hat{p}$. In fact, if, on the contrary, $\limsup_{t\to\infty} M(t) < \hat{p}$ holds, then there must exist some node $j_0 \in \mathcal{V}$ and $T > 0$ such that for any $t \geq T$ we have $\phi_{j_0}(x_{j_0}(t)) = \hat{p}$. Since $\mathcal{G}$ is strongly connected, we consider j_1 with $j_0 \in \mathcal{N}_{j_1}$. Since $\limsup_{t\to\infty} M(t) < \hat{p}$, $x_{j_1}(t) < \hat{p} - \varepsilon$ for some $\varepsilon > 0$ and all t sufficiently large. If the discrete-time subsystem (2) is activated, then $x_{j_1}(t+1) - x_{j_1}(t) \geq \delta > 0$ for some δ; if the continuous-time subsystem (1) is activated, then we have $\dot{x}_{j_1}(t) \geq \delta > 0$. In either case, this contracts $x_{j_1}(t) < \hat{p}$. Therefore, we conclude $\limsup_{t\to\infty} M(t) = \hat{p}$.

Next, we claim that $\lim_{t\to\infty} m(t) = \hat{p}$. In fact, if this is not the case, then there exists $\varepsilon > 0$ satisfying $\lim_{t\to\infty} x_{j_0}(t) = \lim_{t\to\infty} m(t) = \hat{p} - \varepsilon$ and $\limsup_{t\to\infty} x_{j_1}(t) = \limsup_{t\to\infty} M(t) = \hat{p}$. Hence, for any T there exists $t \geq T$ such that $x_{j_0}(t) < \hat{p} - \varepsilon/2 < \hat{p} - \varepsilon/4 < x_{j_1}(t)$. If the discrete-time subsystem (2) is activated, $x_{j_0}(t+1) - x_{j_0}(t) \to 0$

as $t \to \infty$. If the continuous-time subsystem (1) is activated, $\dot{x}_{j_0}(t) \to 0$ as $t \to \infty$. Therefore, for any $k \in \mathcal{N}_{j_0}$, $x_k(t)$ is sufficiently close to $\hat{p} - \varepsilon$ as t tends to infinity. Likewise, any node $k \in \mathcal{N}_{j_1}$, $x_k(t)$ is sufficiently close to $\hat{p}$ as t tends to infinity. Since $\mathcal{G}$ is strongly connected and finite, there exists some time T such that (i) there is a directed spanning tree with root node r and two directed paths linking r to j_0 and r to j_1 respectively, and (ii) $x_r(T) < \hat{p} - \varepsilon/2 < \hat{p} - \varepsilon/4 < x_r(T)$. We obtain a contradiction in (ii). Hence, we proved $\lim_{t\to\infty} m(t) = \hat{p}$.

This contracts our assumption $\hat{m} < \hat{p}$. Thus, we proved $\hat{m} = \hat{p}$. Analogously, we can show $\hat{M} = \hat{q}$. This indicates $x_i(t) \in [\hat{p}, \hat{q}]$ for any node $i \in \mathcal{V}$ as $t \to \infty$.

Step II. If the discrete-time subsystem (2) is activated on $[t, t+1]$, we can rewrite the system (2) as follows:

$$x_i(t+1) = x_i(t) + h \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) + \nu_i(t), \quad i \in \mathcal{V}, \tag{21}$$

where $\nu_i(t) = h \sum_{j\in\mathcal{N}_i} a_{ij}(\phi_j(x_j(t)) - x_j(t))$. If the continuous-time subsystem (1) is activated at some t instead, it can be recast as (1):

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) + \nu_i(t), \quad i \in \mathcal{V}, \tag{22}$$

where $\nu_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(\phi_j(x_j(t)) - x_j(t))$. By the result of Step 1 and our definition of the saturation function (3), we conclude that for any $\delta > 0$, there exists a time $T > 0$ such that $|\nu_i(t)| \le \delta$ for all $i \in \mathcal{V}$ and $t \ge T$ regardless of the specific subsystem in action. We can then use the robust consensus result Lemma 2 and let ε tend to zero to derive

$$\lim_{t\to\infty} \max_{i,j\in\mathcal{V}} |x_i(t) - x_j(t)| = 0. \tag{23}$$

Fix a node $i \in \mathcal{V}$ and denote a limit point of it by $\hat{c}$. We will show all nodes converge to $\hat{c}$, which will conclude our proof in Case (a). Note that $\hat{c} \in [\hat{p}, \hat{q}]$. We only need to consider the situation $\hat{p} < \hat{q}$. In view of (23), for any $\varepsilon > 0$, there is a sufficiently large t_0 satisfying (i) the discrete-time subsystem (2) is activated at t_0, and (ii) $|x_j(t_0) - \hat{c}| \le \varepsilon$ for any $j \in \mathcal{V}$.

If $\hat{p} < \hat{c} < \hat{q}$, then we can choose a sufficiently small ε so that $\hat{p} < x_j(t_0) < \hat{q}$ also holds true for any node $j \in \mathcal{V}$. The system starting from time t_0 then becomes a standard switched multiagent system (Zheng & Wang, 2016), and hence the final consensus state is $\hat{c}$ for all nodes. Next, if $\hat{c} = \hat{q}$, then we choose a sufficiently small ε satisfying $\hat{p} < x_j(t_0) \le \hat{q} + \varepsilon$ for any $j \in \mathcal{V}$. Applying the argument in the beginning of Step 1, we know that $M(t)$ converges to a limit M since ε is vanishing. It follows from (23) that $m(t)$ converges to the same limit, and hence we have $M = \hat{q} = \hat{c}$. Finally, if $\hat{c} = \hat{q}$, we can proceed analogously as above by showing $m(t)$ converges to some limit m and $m = \hat{p} = \hat{c}$. This concludes the proof of Case (a).

(b). In this scenario, we will similarly approach through two steps. In Step 1, we show $x_i(t) \in [\hat{p}, \hat{q}]$ for any node $i \in \mathcal{V}$ as $t \to \infty$. In Step 2, we show the existence of $\hat{c}$.

Step I. For $t \ge 0$, recall the definitions of $M_{\hat{q}}(t) = \max\{\hat{q}, M(t)\}$ and $m_{\hat{p}}(t) = \min\{\hat{p}, m(t)\}$. When $t \in (t_k, \bar{t}_k)$ with $\bar{t}_k - t_k \ge \Delta$ for some large k, we know that $M(t) > \hat{q}$ over an interval $[t, t+\varepsilon)$ with $t + \varepsilon < \bar{t}_k$ as long as $M_{\hat{q}}(t) > \hat{q}$. Define a non-empty set $\Theta(t) = \{i \in \mathcal{V} : M(t) = x_i(t)\}$. It follows from Lemma 1 and the continuous-time subsystem (1) that the Dini derivative takes the form

$$D^{+}M_{\hat{q}}(t)=\max_{i\in\Theta(t)}\Big(\sum_{j\in\mathcal{N}_i}a_{ij}(\phi_j(x_j(t)))-x_i(t)\Big). \tag{24}$$

Fix a node $i_0\in\Theta(t)$, and we have $x_{i_0}(t)\geq x_j(t)$ for any $j\in\mathcal{V}$. Clearly, if $x_j(t)>\hat{q}$ then $\phi_j(x_j(t))\leq x_j(t)$; if $x_j(t)\leq\hat{q}$ then $\phi_j(x_j(t))\leq\hat{q}$. Therefore, under the assumption of $M_{\hat{q}}(t)>\hat{q}$, we obtain $\phi_j(x_j(t))\leq x_{i_0}(t)$ and furthermore $D^{+}M_{\hat{q}}(t)\leq 0$ by using (24). A direct proof by contradiction shows that if $M_{\hat{q}}(t_0)=\hat{q}$ and $t_0\in(t_k,\bar{t}_k]$, then $M_{\hat{q}}(t)=\hat{q}$ for all $t\in[t_0,\bar{t}_k]$. On the other hand, from the proof of Case (a), we see that $M_{\hat{q}}(t+1)\leq M_{\hat{q}}(t)$ when $t\in(\bar{t}_k,t_{k+1}]$ and $t+1\in(\bar{t}_k,t_{k+1}]$.

Likewise, we have

$$D^{+}m_{\hat{p}}(t)=-\min_{i\in\Gamma(t)}\Big(\sum_{j\in\mathcal{N}_i}a_{ij}(\phi_j(x_j(t)))-x_i(t)\Big), \tag{25}$$

where $\Gamma(t)=\{i\in\mathcal{V}:m(t)=x_i(t)\}$ is non-empty. We can show $D^{+}m_{\hat{p}}(t)\geq 0$ when $m_{\hat{p}}(t)<\hat{p}$. If $m_{\hat{p}}(t_0)=\hat{p}$ and $t_0\in(t_k,\bar{t}_k]$, then $m_{\hat{p}}(t)=\hat{p}$ for all $t\in[t_0,\bar{t}_k]$. Similarly, from the proof of Case (a), we have $m_{\hat{p}}(t+1)\geq m_{\hat{p}}(t)$ when $t\in(\bar{t}_k,t_{k+1}]$ and $t+1\in(\bar{t}_k,t_{k+1}]$.

Define $V(t)=M_{\hat{q}}(t)-m_{\hat{p}}(t)$. We have $D^{+}V(t)=D^{+}M_{\hat{q}}(t)-D^{+}m_{\hat{p}}(t)\leq 0$ for any $t\geq t_k$. Define a set $\Xi=\{x\in\mathbb{R}^n:D^{+}V(x)=0\}$. We claim that Ξ is a subset of $[\hat{p},\hat{q}]^n$. In fact, suppose that there is a point $\hat{x}=(\hat{x}_1,\cdots,\hat{x}_n)\in\Xi$ and $\hat{x}\notin[\hat{p},\hat{q}]^n$. Without loss of generality, we can assume $\hat{x}_i=\max_{j\in\mathcal{V}}\hat{x}_j>\hat{q}$. Consider a solution of the continuous-time subsystem (1) with $x(t_k)=\hat{x}$. Define a non-empty set $\Lambda=\{j\in\mathcal{V}:\hat{x}_j=\hat{x}_i\}$. Since $\mathcal{G}$ is strongly connected, along either the continuous-time subsystem (1) or the discrete-time subsystem (2), any node in Λ is eventually dragged down by the other nodes in $\mathcal{V}\backslash\Lambda$ or by $\hat{q}$. Thus, there exists some time τ with $\tau\in(t_{k_0},\bar{t}_{k_0}]$ for some $k_0\geq k$ satisfying $x_j(\tau)<\hat{x}_i$ for all $j\in\mathcal{V}$. Hence, we have $M_{\hat{q}}(\tau)<M_{\hat{q}}(t_k)$ and accordingly $V(x(\tau))<V(x(t_k))$. This contradicts $D^{+}V(x)=0$, i.e., the trajectory is not inside Ξ. Thus, we see that the claim $\Xi\subseteq[\hat{p},\hat{q}]^n$ holds true.

Thanks to the LaSalle invariance principle (Rouche et al., 1977, Thm 3.2), the set of ω-limit points of $x(t)$ in $(t_k,\bar{t}_k]$ for any $k\in\mathbb{N}^{+}$ is contained in Ξ and hence in $[\hat{p},\hat{q}]^n$. By the results in Rouche et al. (1977, p.364-p.365), $x(t)\in[\hat{p},\hat{q}]^n$ for $t\in(t_k,\bar{t}_k]$ with all large enough k. In view of the proof of Case (a), we know that for any sufficiently large t, $x(t)\in[\hat{p},\hat{q}]^n$.

Step II. To prove the existence of $\hat{c}$, we can proceed the same way as Step 2 in Case (a). This concludes the proof of Theorem 1. □

3.2. Scaled interval consensus analysis

Next, we introduce the following scaled interval consensus problem, which can be viewed as a generalization of standard interval consensus if all scaling coefficients of agents equal to one.

Definition 2. For each node $i\in\mathcal{V}$, let $\alpha_i\neq 0$ be its scaling coefficient. We say that the multiagent system under consideration reaches scaled interval consensus with respect to $(\alpha_1,\cdots,\alpha_n)$ if (i) $\lim_{t\to\infty}\alpha_i x_i(t)-\alpha_j x_j(t)=0$ for all $i,j\in\mathcal{V}$ and all initial condition $x(0)$, and (ii) $\alpha_i x_i(t)\in[\hat{p},\hat{q}]=\cap_{i\in\mathcal{V}}\mathcal{I}_i$.

To achieve scaled interval consensus for a switched multiagent system with both continuous-time and discrete-tine subsystems, we propose the following agent dynamics. Given the scaling coefficients $(\alpha_1, \cdots, \alpha_n)$, we have a continuous-time subsystem:

$$\dot{x}_i(t) = \mathrm{sgn}(\alpha_i) \sum_{j \in \mathcal{N}_i} a_{ij}(\phi_j(\alpha_j x_j(t)) - \alpha_i x_i(t)), \quad i \in \mathcal{V} \tag{26}$$

and a discrete-time subsystem:

$$\begin{aligned} x_i(t+1) =& x_i(t) + \mathrm{sgn}(\alpha_i) h \\ &\cdot \sum_{j \in \mathcal{N}_i} a_{ij}(\phi_j(\alpha_j x_j(t)) - \alpha_i x_i(t)), \quad i \in \mathcal{V}, \end{aligned} \tag{27}$$

where $\mathrm{sgn}(\cdot)$ is the signum function with $\mathrm{sgn}(z) = 1$ if $z > 0$ and $\mathrm{sgn}(z) = -1$ if $z < 0$, the saturation function $\phi_j(\cdot)$ is defined in (3) and $h > 0$ is the sampling period.
Theorem 2. *Consider the switched multiagent system (26) and (27) over the directed graph $\mathcal{G}$ under an arbitrary switching rule. Assume that $\mathcal{G}$ is strongly connected and $[\hat{p}, \hat{q}]$ is non-empty. Given the scaling coefficients $(\alpha_1, \cdots, \alpha_n)$, if $h < 1/\max_{i \in \mathcal{V}} |\alpha_i| \sum_{j \in \mathcal{N}_i} a_{ij}$, then for any $x(0)$, there exists $\hat{c} \in [\hat{p}, \hat{q}]$ such that $\lim_{t \to \infty} \alpha_i x_i(t) = \hat{c}$ for all $i \in \mathcal{V}$.*

Clearly, from Theorem 2 we know that scaled interval consensus can be reached for the switched multiagent system (26) and (27) under any switching rule. Theorem 1 is a special case for $\alpha_i \equiv 1$ $(i \in \mathcal{V})$. In the previous study of scaled consensus, e.g., Roy (2015); Shang (2019); Zhang et al. (2020), the final consensus value (or even the convergence of the states) is generally not required. Our result here reveals a mechanism of regulating both the convergence value for each agent and the ratio between each pair of agents at the same time.

Proof of Theorem 2. Let $y_i(t) = \alpha_i x_i(t)$ for each $i \in \mathcal{V}$. The continuous-time subsystem (26) and the discrete-time subsystem (27) can be rewritten as

$$\dot{y}_i(t) = |\alpha_i| \sum_{j \in \mathcal{N}_i} a_{ij}(\phi_j(y_j(t)) - y_i(t)), \quad i \in \mathcal{V} \tag{28}$$

and

$$y_i(t+1) = y_i(t) + |\alpha_i| h \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t)) - y_i(t)), \quad i \in \mathcal{V}, \tag{29}$$

respectively.

By re-defining $M_{\hat{q}}(t) = \max\{\hat{q}, \max_{i \in \mathcal{V}} y_i(t)\}$ and $m_{\hat{p}}(t) = \min\{\hat{p}, \min_{i \in \mathcal{V}} y_i(t)\}$, we can proceed as in the proof of Theorem 1 and obtain the desired result. □

4. Simulation results

In this section, we consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, 3, 4\}$ as shown in Fig. 3. This directed graph is strongly connected with a binary adjacency matrix, namely, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and zero otherwise. For the nodes in $\mathcal{V}$, we propose their admissible intervals as $\mathcal{I}_1 = [0, 3]$, $\mathcal{I}_2 = [-2, 1]$, $\mathcal{I}_3 = [-1, 4]$, $\mathcal{I}_4 = [-2, 2]$. Hence, their intersection is $\cap_{i=1}^4 \mathcal{I}_i = [\hat{p}, \hat{q}] = [0, 1]$; see Fig. 4.

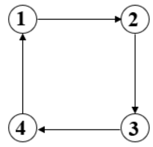

Figure 3. Strongly connected network topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ over $n = 4$ nodes.

Example 1. In this example, we set the scaling coefficients $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = -1$, we consider the scaled interval consensus for the switched multiagent system characterized by (26) and (27). We specify an alternating switching rule as shown in Fig. 5(a) and set the control gain $h = 0.5$, which satisfies the condition of Theorem 2. In addition, we use the following initial condition $x(0) = (-1, 2, 0, 3)$.

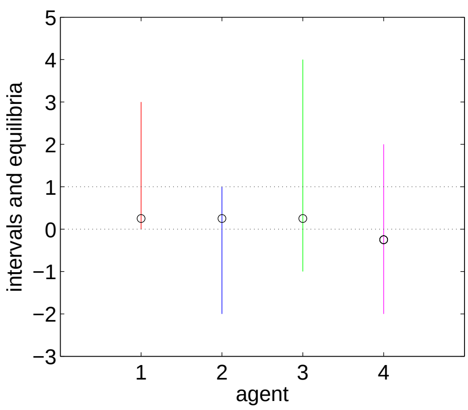

Figure 4. Admissible intervals $\mathcal{I}_i = [p_i, q_i]$ for each agent $i \in \mathcal{V}$. The two horizontal lines indicate the common interval $[\hat{p}, \hat{q}] = [0, 1]$. The limit states for the switched multiagent system of Example 1 are shown as black circles.

We perform simulations for the switched multiagent system (26) and (27) and the evolution of agents' trajectories is shown in Fig. 5(b). We observe that agents 1, 2 and 3 converges to $\hat{c} \approx 0.25 \in [0, 1]$ and the agent 4 tends to $-\hat{c}$, which is consistent with our prediction in Theorem 2. The proposed scaling coefficients here mean that agent 4 will reach the opposite value of the consensus state of the other three agents. The final asymptotic states are also displayed in Fig. 4. It is worth noting that both the initial states and the trajectories of the agents are not entirely within their respective admissible intervals. Nevertheless, the final (scaled) consensus resides in the common interval, demonstrating the very feature of soft constraint in interval consensus.

Example 2. As a further example, we set the scaling coefficients $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ in this example, and consider the interval consensus for the switched multiagent system described by (1) and (2). Here, we take the alternating switching rule as an aperiodic general switching rule shown in Fig. 6(a). We set $h = 0.5$ and initial configuration $x(0) = (3, -1, -2, 1)$. The state evolution result is shown in Fig. 6(b), where all four agents converge to the common consensus value around 0.34 sitting in the intersection of admissible interval $[\hat{p}, \hat{q}] = [0, 1]$. The agrees with our theoretical prediction in Theorem 1.

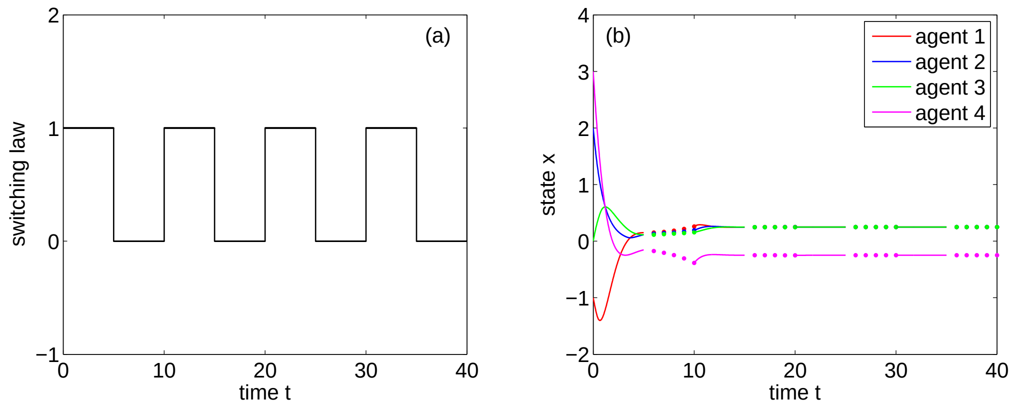

Figure 5. (a) Switching rule of the switched multiagent system, where the signal 1 means the activation of continuous-time subsystem (26) and 0 means the activation of the discrete-time subsystem (27). (b) Time evolution of the agents' states for Example 1.

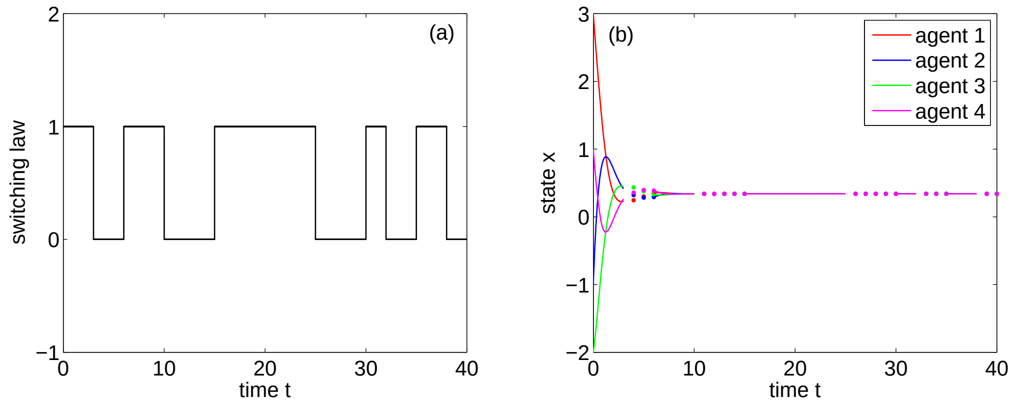

Figure 6. (a) Switching rule of the switched multiagent system, where the signal 1 means the activation of continuous-time subsystem (1) and 0 means the activation of the discrete-time subsystem (2). (b) Time evolution of the agents' states for Example 2.

5. Conclusion

In this paper, we have studied the interval consensus problem over directed networks with switched agent dynamics. The system under consideration has a continuous-time subsystem and and a discrete-time subsystem, regulated by a switching rule. We established the condition that guarantees interval consensus for switched multiagent system under an arbitrary switching rule. Interval consensus here features a soft constraint on the agents' state by offering each agent an individual admissible interval, where the agent would like its ultimate consensus value to sit. Moreover, we introduce the scaled interval consensus problem and extend the result to accommodate constant scaling coefficients. This further allows flexibility that is appealing for many transcale control systems such as compartmental mass-action systems and spacecraft robotic simulations (Roy, 2015). The interval consensus is studied in a fixed directed graph here. It would be interesting to consider switching topologies or a general m-mode system by extending the analysis of robustness consensus, in which time-dependent topologies are combined with transition of the agent dynamics.

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