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Common Neighborhood Energy of Commuting Graphs of Finite Groups

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Abstract: The commuting graph of a finite non-abelian group G with center $Z(G)$, denoted by $\Gamma_c(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. Alwardi et al. (Bulletin, 2011, 36, 49-59) defined the common neighborhood matrix $CN(\mathcal{G})$ and the common neighborhood energy $E_{cn}(\mathcal{G})$ of a simple graph $\mathcal{G}$. A graph $\mathcal{G}$ is called CN-hyperenergetic if $E_{cn}(\mathcal{G}) > E_{cn}(K_n)$, where $n = |V(\mathcal{G})|$ and K_n denotes the complete graph on n vertices. Two graphs $\mathcal{G}$ and $\mathcal{H}$ with equal number of vertices are called CN-equienergetic if $E_{cn}(\mathcal{G}) = E_{cn}(\mathcal{H})$. In this paper we compute the common neighborhood energy of $\Gamma_c(G)$ for several classes of finite non-abelian groups, including the class of groups such that the central quotient is isomorphic to group of symmetries of a regular polygon, and conclude that these graphs are not CN-hyperenergetic. We shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic.

Keywords: commuting graph; CN-energy; finite group

MSC: 20D99; 05C50; 15A18; 05C25

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1. Introduction

Let $\mathcal{G}$ be a simple graph whose vertex set is $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$. The common neighborhood of two distinct vertices v_i and v_j, denoted by $C(v_i, v_j)$, is the set of vertices adjacent to both v_i and v_j other than v_i and v_j. The common neighborhood matrix of $\mathcal{G}$, denoted by $CN(\mathcal{G})$, is a matrix of size n whose (i,j)th entry is 0 or $|C(v_i, v_j)|$ according as $i = j$ or $i \neq j$. The common neighborhood matrix is a symmetric matrix, hence all its eigenvalues are real. The common neighborhood eigenvalues are symmetric with respect to the origin for some special class of graphs. There is a nice relation between $CN(\mathcal{G})$ and $A(\mathcal{G})$, the adjacency matrix of $\mathcal{G}$. More precisely, if $i \neq j$ then the (i,j)th entry of $CN(\mathcal{G})$ is same as the (i,j)th entry of $A(\mathcal{G})^2$, which is the number of 2-walks between the vertices v_i and v_j. Further, the (i,i)th entry of $CN(\mathcal{G})$ is equal to the degree of v_i. Hence, $CN(\mathcal{G}) = A(\mathcal{G})^2 - D(\mathcal{G})$, where $D(\mathcal{G})$ is the degree matrix of $\mathcal{G}$. Let CN-spec($\mathcal{G}$) be the spectrum of $CN(\mathcal{G})$. Then CN-spec($\mathcal{G}$) is the set of all the eigenvalues of $CN(\mathcal{G})$ with multiplicities. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct eigenvalues of $CN(\mathcal{G})$ with multiplicities $a_1, a_2, \ldots, a_k$, respectively, then we write CN-spec($\mathcal{G}$) $= \{\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots, \alpha_k^{a_k}\}$. The common neighborhood energy (abbreviated as CN-energy) of the graph $\mathcal{G}$ is given by

$$E_{cn}(\mathcal{G}) = \sum_{i=1}^{k} a_i |\alpha_i|.$$

The study of CN-energy of graphs was introduced by Alwardi et al. in [1]. Various properties of CN-energy of a graph can also be found in [1,2]. The motivation of studying $E_{cn}(\mathcal{G})$ comes from the study of $E(\mathcal{G})$, which is well-known as energy of $\mathcal{G}$, a notion introduced by Gutman [3]. Many results on $E(\mathcal{G})$, including some bounds and chemical applications, can be found in [4–15]. It is worth recalling that $E(\mathcal{G})$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of $\mathcal{G}$. It is also interesting to note that $E(\mathcal{G})$ can be obtained if $E_{cn}(\mathcal{G})$ is known for some classes of graphs. For instance, $E(K_n) = E_{cn}(K_n)/(n-2)$ and $E(K_{m,n}) = \sqrt{E_{cn}(K_{m,n}) + 2(n+n)}$, where K_n is the complete graph on n vertices and $K_{m,n}$ is the complete bipartite graph on $(m+n)$ vertices. A graph $\mathcal{G}$ is called CN-hyperenergetic if $E_{cn}(\mathcal{G}) > E_{cn}(K_n)$, where $n = |V(\mathcal{G})|$. It is still an open problem to produce a CN-hyperenergetic graph or to prove the non-existence of such graph (see [1] (Open problem 1)). In this paper we give an attempt to answer this problem by considering commuting graphs of finite groups.

The commuting graph of a finite non-abelian group G with center $Z(G)$ is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy = yx$. We write $\Gamma_c(G)$ to denote this graph. In [16–23], various aspects of $\Gamma_c(G)$ are studied. In Section 2 of this paper, we derive an expression for computing CN-energy of a particular class of graphs and list a few already known results. In Section 3, we compute CN-energy of commuting graph of certain metacyclic group, dihedral group (which is the group of symmetries of a regular polygon), quasidihedral group, generalized quarternion group, Hanaki group etc. We also consider some generalizations of dihedral group and generalized quarternion group. Two graphs $\mathcal{G}$ and $\mathcal{H}$ with equal number of vertices are called CN-equienergetic if $E_{cn}(\mathcal{G}) = E_{cn}(\mathcal{H})$. In Section 3, we shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic. As consequences of our results, in Section 4, we show that $\Gamma_c(G)$ for all G considered in Section 3 are not CN-hyperenergetic. We also identify some positive integers n such that $\Gamma_c(G)$ is not CN-hyperenergetic if G is an n-centralizer group. It is worth mentioning that CN-spectrums of $\Gamma_c(G)$ for certain classes of finite groups have been computed in [24] recently. However, the method adopted here, in computing CN-energy of $\Gamma_c(G)$ for various families of finite groups, is independent of CN-spec($\Gamma_c(G)$).

Recall that an n-centralizer group G is a group such that $|\text{Cent}(G)| = n$, where $\text{Cent}(G) = \{C_G(w) : w \in G\}$ and $C_G(w) = \{v \in G : vw = wv\}$ is the centralizer of w (see [25,26]). We also identify some $r \in \mathbb{Q}^{>0}$ such that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Pr(G) = r$. Also recall that the commutativity degree of G, denoted by $\Pr(G)$, is the probability that a randomly chosen pair of elements of G commute.

Readers may review [27–32] for the background and various results regarding this notion. Further, we show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is not planar or toroidal. Note that a graph is planar or toroidal according as its genus is zero or one respectively. Finally, we conclude the paper with a few conjectures.

2. A Useful Formula and Prerequisites

We write $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ to denote that $\mathcal{G}$ has two components namely $\mathcal{G}_1$ and $\mathcal{G}_2$. Also, lK_m denotes the disjoint union of l copies of the complete graph K_m on m vertices. We begin this section with the following two key results of Alwardi et al. [1].

Theorem 1 ([1] Proposition 2.4)**.** *If $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \cdots \sqcup \mathcal{G}_m$ then $E_{cn}(\mathcal{G}) = \sum_{i=1}^{m} E_{cn}(\mathcal{G}_i)$.*

Lemma 1 ([1] Example 2.1)**.** *If K_n denotes the complete graph on n vertices then*

$$E_{cn}(K_n) = 2(n-1)(n-2).$$

Now we derive a formula for CN-energy of graphs which are disjoint unions of some complete graphs. The following theorem is very useful in order to compute CN-energy of commuting graphs of finite groups.

Theorem 2. *Let $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $1 \leq i \leq k$. Then*

$$E_{cn}(\mathcal{G}) = 2\sum_{i=1}^{k} l_i(m_i - 1)(m_i - 2).$$

Proof. By Theorem 1 we have

$$E_{cn}(\mathcal{G}) = \sum_{i=1}^{k} l_i E_{cn}(K_{m_i}).$$

Therefore, the result follows from Lemma 1. □

We conclude this section with the following useful results from [17,18].

Lemma 2. *Let G be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to*

1. *The Suzuki group $Sz(2)$, presented by $\langle u, v : u^5 = v^4 = 1, v^{-1}uv = u^2\rangle$, then $\Gamma_c(G) = 5K_{3|Z(G)|} \sqcup K_{4|Z(G)|}$.*
2. *$\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime p, then $\Gamma_c(G) = (p+1)K_{(p-1)|Z(G)|}$.*
3. *The dihedral group D_{2m} $(m \geq 2)$, presented by $\langle u, v : u^m = v^2 = 1, vuv^{-1} = u^{-1}\rangle$, then $\Gamma_c(G) = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$.*

Lemma 3. *Let G be a non-abelian group. If G is isomorphic to*

1. *A group of order pq, where p and q are primes with $p \mid (q-1)$, then $\Gamma_c(G) = K_{q-1} \sqcup qK_{p-1}$.*
2. *The quasidihedral group QD_{2^n} $(n \geq 4)$, presented by $\langle u, v : u^{2^{n-1}} = v^2 = 1, vuv^{-1} = u^{2^{n-2}-1}\rangle$, then $\Gamma_c(G) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$.*
3. *$PSL(2, 2^k)$, the projective special linear group for $k \geq 2$, then $\Gamma_c(G) = 2^{k-1}(2^k-1)K_{2^k} \sqcup (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2}$.*
4. *$GL(2, q)$, the general linear group where $q = p^n > 2$ and p is a prime, then $\Gamma_c(G) = \frac{q(q-1)}{2}K_{q^2-q} \sqcup \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup (q+1)K_{q^2-2q+1}$.*

Lemma 4. *Let G be a non-abelian group. If G is isomorphic to*

1. *The Hanaki group $A(n, \sigma)$ $(n \geq 2)$ of order 2^{2n} given by*

$$\left\{ U(x,y) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & \sigma(x) & 1 \end{bmatrix} : x, y \in F \right\},$$

under matrix multiplication where $F = GF(2^n)$ and $\sigma \in Aut(F)$ given by $\sigma(u) = u^2$, then $\Gamma_c(G) = (2^n - 1)K_{2^n}$.

2. *The Hanaki group $A(n, p)$ of order p^{3n} given by*

$$\left\{ V(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} : x, y, z \in F \right\},$$

under matrix multiplication where $F = GF(p^n)$ and p is a prime, then $\Gamma_c(G) = (p^n + 1)K_{p^{2n}-p^n}$.

3. CN-Energy of Commuting Graphs

In this section, we compute $E_{cn}(\Gamma_c(G))$ for several classes of finite non-abelian groups.

Theorem 3. *Let G be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to*

1. *The Suzuki group* $Sz(2)$, *then*

$$E_{cn}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12).$$

2. $\mathbb{Z}_p \times \mathbb{Z}_p$, *then*

$$E_{cn}(\Gamma_c(G)) = 2(p+1)((p-1)|Z(G)| - 1)((p-1)|Z(G)| - 2).$$

3. *The dihedral group* D_{2m} $(m \geq 2)$, *then*

$$E_{cn}(\Gamma_c(G)) = 2((m^2 - m + 1)|Z(G)|^2 - (6m-3)|Z(G)| + 2m + 2).$$

Proof. By Lemma 2 and Theorem 2 we have

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2(4|Z(G)|-1)(4|Z(G)|-2) + 10(3|Z(G)|-1)(3|Z(G)|-2), & \text{if } \frac{G}{Z(G)} \cong Sz(2) \\ 2(p+1)((p-1)|Z(G)|-1)((p-1)|Z(G)|-2), & \text{if } \frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ 2((m-1)|Z(G)|-1)((m-1)|Z(G)|-2) \\ \quad +2m(|Z(G)|-1)(|Z(G)|-2), & \text{if } \frac{G}{Z(G)} \cong D_{2m}. \end{cases}$$

Hence, the result follows on simplification. □

We have the following two corollaries of Theorem 3.

Corollary 1. *Let G be isomorphic to one of the following groups*

1. $\mathbb{Z}_2 \times Q_8$,
2. $\mathbb{Z}_2 \times D_8$,
3. $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle u, v : u^4 = v^4 = 1, vuv^{-1} = u^{-1}\rangle$,
4. $\mathcal{M}_{16} = \langle u, v : u^8 = v^2 = 1, vuv = u^5\rangle$,
5. $SG(16,3) = \langle u, v : u^4 = v^4 = 1, uv = v^{-1}u^{-1}, uv^{-1} = vu^{-1}\rangle$,
6. $D_8 * \mathbb{Z}_4 = \langle u, v, w : u^4 = v^2 = w^2 = 1, uv = vu, uw = wu, vw = u^2wv\rangle$.

Then $E_{cn}(\Gamma_c(G)) = 36$.

Proof. If G is isomorphic to one of the above listed group then it is of order 16. Therefore, $|Z(G)| = 4$ and so $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, putting $p = 2$ in Theorem 3 (2) we get the required result. □

Corollary 2. *Let G be a non-abelian group.*

1. *If G is of order* p^3, *for any prime p, then*

$$E_{cn}(\Gamma_c(G)) = 2(p+1)(p^2 - p - 1)(p^2 - p - 2).$$

2. *If G is the metacyclic group* M_{2mn} $(m \geq 3)$, *presented by* $\langle u, v : u^m = v^{2n} = 1, vuv^{-1} = u^{-1}\rangle$, *then*

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2((m^2 - m + 1)n^2 - (6m-3)n + 2m + 2), & \text{if } m \text{ is odd} \\ 2((m^2 - 2m + 4)n^2 - (6m-6)n + m + 2), & \text{if } m \text{ is even.} \end{cases}$$

3. *If G is the dihedral group* D_{2m} $(m \geq 3)$, *then*

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2(m-2)(m-3), & \text{if } m \text{ is odd} \\ 2(m-3)(m-4), & \text{if } m \text{ is even.} \end{cases}$$

4. *If G is the generalized quaternion group Q_{4n} $(n \geq 2)$, presented by $\langle u, v : v^{2n} = 1, u^2 = v^n, uvu^{-1} = v^{-1} \rangle$, then*

$$E_{cn}(\Gamma_c(G)) = 2(2n-3)(2n-4).$$

Proof. (1) If G is of order p^3 then $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 3 (2).

(2) We have

$$|Z(M_{2mn})| = \begin{cases} n, & \text{if } m \text{ is odd} \\ 2n, & \text{if } m \text{ is even} \end{cases} \text{ and } \frac{M_{2mn}}{Z(M_{2mn})} \cong \begin{cases} D_{2m}, & \text{if } m \text{ is odd} \\ D_m, & \text{if } m \text{ is even.} \end{cases}$$

Hence, the result follows from Theorem 3 (3).

(3) Follows from part (2), considering $n = 1$.

(4) Follows from Theorem 3 (3), since $|Z(Q_{4n})| = 2$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. □

In the following theorems we compute $E_{cn}(\Gamma_c(G))$ for more families of groups.

Theorem 4. *Let G be a non-abelian group.*

1. *If G is of order pq, where p and q are primes with $p \mid (q-1)$, then*

$$E_{cn}(\Gamma_c(G)) = 2(q^2 + p^2q - 5pq + q + 6).$$

2. *If G is the quasidihedral group QD_{2^n} $(n \geq 4)$, then*

$$E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4).$$

3. *If $G = PSL(2, 2^k)$ then*

$$E_{cn}(\Gamma_c(G)) = 2^{4k+1} - 4 \cdot 2^{3k+1} + 2^{2k+1} + 6 \cdot 2^{k+1} + 12.$$

4. *If $G = GL(2, q)$ then*

$$E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q.$$

Proof. (1) If G is of order pq then, by Lemma 3 (1) and Theorem 2, we have

$$E_{cn}(\Gamma_c(G)) = 2((q-2)(q-3) + q(p-2)(p-3)).$$

This gives the required result on simplification.

(2) Follows from Lemma 3 (2) and Theorem 2.

(3) By Lemma 3 (3) and Theorem 2 we have

$$\frac{E_{cn}(\Gamma_c(G))}{2} = (2^k+1)(2^k-2)(2^k-3) + 2^{k-1}(2^k+1)(2^k-3)(2^k-4) + 2^{k-1}(2^k-1)(2^k-1)(2^k-2),$$

which gives the required result.

(4) By Lemma 3 (4) and Theorem 2 we have

$$E_{cn}(\Gamma_c(G)) = q(q+1)(q^2-3q+1)(q^2-3q) + q(q-1)(q^2-q-1)(q^2-q-2) + 2(q+1)(q^2-2q)(q^2-2q-1),$$

which gives the required result on simplification. □

Theorem 5. *Let G be a non-abelian group.*

1. *If G is the Hanaki group* $A(n, \sigma)$ *then*

$$E_{cn}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2).$$

2. *If G is the Hanaki group* $A(n, p)$ *then*

$$E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2).$$

Proof. The result follows from Lemma 4 and Theorem 2. □

Note that all the groups considered above are abelian centralizer group (in short, AC-group). Now we present a result on $E_{cn}(\Gamma_c(G))$ if G is a finite AC-group.

Theorem 6. *Consider that an AC-group G has distinct centralizers* $X_1, \dots, X_n$ *of non-central elements of G. Then* $E_{cn}(\Gamma_c(G)) = 2\sum_{i=1}^{n}(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2)$.

Proof. We have $\Gamma_c(G) = \bigsqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}$, by [17] (Lemma 1). Therefore, by Theorem 2, the result follows. □

Corollary 3. *Let K be a finite abelian group and H be a finite non-abelian AC-group. If* $G \cong H \times K$ *then*

$$E_{cn}(\Gamma_c(G)) = 2\sum_{i=1}^{n}(|Y_i||K| - |Z(H)||K| - 1)(|Y_i||K| - |Z(H)||K| - 2),$$

where $\mathrm{Cent}(H) = \{H, Y_1, \dots, Y_n\}$.

Proof. Clearly $Z(H \times K) = Z(H) \times K$ and $\mathrm{Cent}(H \times K) = \{H \times K, Y_1 \times K, Y_2 \times K, \dots, Y_n \times K\}$. Hence, $H \times K$ is an AC-group and so, by Theorem 6, the result follows. □

We shall conclude this section by obtaining some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic.

Proposition 1. *The commuting graphs of* D_{4k} *and* Q_{4k} *for* $k \geq 2$ *are CN-equienergetic.*

Proof. The result follows from parts (3) and (4) of Corollary 2. □

Using Corollary 2 (parts (3) and (4)) and Theorem 4 (2) we also have the following result.

Proposition 2. *The commuting graphs of* D_{2^k}*,* Q_{2^k} *and* QD_{2^k} *for* $k \geq 4$ *are pairwise CN-equienergetic.*

4. Some Consequences

In this section we derive some consequences of the results obtained in Section 3.

Theorem 7. *Let G be a finite group with center* $Z(G)$*. If* $\frac{G}{Z(G)}$ *is isomorphic to* $Sz(2), \mathbb{Z}_p \times \mathbb{Z}_p$ *or* D_{2m} *(where p is any prime and* $m \geq 2$*) then* $\Gamma_c(G)$ *is not CN-hyperenergetic.*

Proof. If $\frac{G}{Z(G)} \cong Sz(2)$ then, by Theorem 3 (1), we have

$$E_{cn}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12).$$

Since $|V(\Gamma_c(G))| = 19|Z(G)|$, by Lemma 1 we have

$$E_{cn}(K_{19|Z(G)|}) = 2(19|Z(G)| - 1)(19|Z(G)| - 2) = 2(361|Z(G)|^2 - 57|Z(G)| + 2).$$

Clearly, $361|Z(G)|^2 + 2 > 61|Z(G)|^2 + 12$ which gives $E_{cn}(K_{19|Z(G)|}) > E_{cn}(\Gamma_c(G))$.
If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then, by Theorem 3 (2), we have

$$E_{cn}(\Gamma_c(G)) = 2(p+1)((p-1)|Z(G)| - 1)((p-1)|Z(G)| - 2).$$

Since $|V(\Gamma_c(G))| = (p^2 - 1)|Z(G)|$, by Lemma 1 we have

$$E_{cn}(K_{(p^2-1)|Z(G)|}) = 2((p^2-1)|Z(G)| - 1)((p^2-1)|Z(G)| - 2).$$

Clearly

$$\begin{aligned} &((p^2-1)|Z(G)| - 1)((p^2-1)|Z(G)| - 2) \\ &> ((p^2-1)|Z(G)| - (p+1))((p^2-1)|Z(G)| - 2(p+1)) \\ &> (p+1)((p-1)|Z(G)| - 1)((p-1)|Z(G)| - 2. \end{aligned}$$

Thus $E_{cn}(K_{(p^2-1)|Z(G)|}) > E_{cn}(\Gamma_c(G))$.
If $\frac{G}{Z(G)} \cong D_{2m}$ then we have

$$E_{cn}(\Gamma_c(G)) = 2((m^2 - m + 1)|Z(G)|^2 - (6m-3)|Z(G)| + 2m + 2),$$

by Theorem 3 (3). Since $|V(\Gamma_c(G))| = (2m-1)|Z(G)|$, by Lemma 1 we have

$$\begin{aligned} E_{cn}(K_{(2m-1)|Z(G)|}) &= 2(2m|Z(G)| - |Z(G)| - 1)(2m|Z(G)| - |Z(G)| - 2) \\ &= 2((4m^2 - 4m + 1)|Z(G)|^2 - (6m-3)|Z(G)| + 2). \end{aligned}$$

Clearly $(4m^2 - 4m + 1)|Z(G)|^2 > (m^2 - m + 1)|Z(G)|^2 + 2m$. Therefore, $E_{cn}(K_{(p^2-1)|Z(G)|}) > E_{cn}(\Gamma_c(G))$. This completes the proof. □

We have the following two corollaries.

Corollary 4. *If G is isomorphic to one of the groups listed in Corollary 1, then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. Since $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows from Theorem 7 considering $p = 2$. □

Corollary 5. *Let G be a non-abelian group. If G is isomorphic to M_{2mn}, D_{2m}, Q_{4n} or a group of order p^3 then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If G is isomorphic to M_{2mn}, D_{2m} or Q_{4n} then $\frac{G}{Z(G)}$ is isomorphic to some dihedral groups. If G is isomorphic to a group of order p^3 then $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, by Theorem 7, the result follows. □

We have the following results regarding commuting graphs of finite n-centralizer groups.

Theorem 8. *If G is a finite 4-centralizer group then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by [25] (Theorem 2). Hence, using Theorem 7 for $p = 2$, the result follows. □

Theorem 9. *Let G be a finite $(p+2)$-centralizer p-group. Then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by [33] (Lemma 2.7). Hence, by Theorem 7, the result follows. □

Theorem 10. *If G is a finite* 5*-centralizer group then* $\Gamma_c(G)$ *is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6, by [25] (Theorem 4). Hence, by Theorem 7, the result follows. □

As a corollary to Theorems 8 and 10 we have the following result.

Corollary 6. *Let G be a finite non-abelian group and* $\{x_1, x_2, \ldots, x_r\}$ *be a set of pairwise non-commuting elements of G having maximal size. Then* $\Gamma_c(G)$ *is not CN-hyperenergetic if* $r = 3, 4$.

Proof. By [34] (Lemma 2.4), we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Theorems 8 and 10. □

Theorem 11. *Let G be a non-abelian group. If G is isomorphic to* QD_{2^n}, $PSL(2, 2^k)$, $A(n, \sigma)$, $GL(2, q)$, $A(n, p)$ *or a group of order pr, where p and r are primes with* $p \mid (r-1)$ *and* $q = p^m > 2$, *then* $\Gamma_c(G)$ *is not CN-hyperenergetic.*

Proof. If G is isomorphic to QD_{2^n} then, by Theorem 4, we have $E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4)$. Since $|V(\Gamma_c(G))| = 2^n - 2$, by Lemma 1 we have

$$E_{cn}(K_{2^n-2}) = 2(2^n - 3)(2^n - 4).$$

Clearly, $(2^n - 3)(2^n - 4) > (2^{n-1} - 3)(2^{n-1} - 4)$. Hence, $E_{cn}(K_{2^n-2}) > E_{cn}(\Gamma_c(G))$. If G is isomorphic to $PSL(2, 2^k)$ then, by Theorem 4 (3), we have

$$E_{cn}(\Gamma_c(G)) = 2^{4k+1} - 4 \cdot 2^{3k+1} + 2^{2k+1} + 6 \cdot 2^{k+1} + 12.$$

Since $|V(\Gamma_c(G))| = 2^k(2^{2k} - 1) - 1 = 2^{3k} - 2^k - 1$, by Lemma 1 we have

$$\begin{aligned} E_{cn}(K_{2^{3k}-2^k-1}) &= 2(2^{3k} - 2^k - 1)(2^{3k} - 2^k - 3) \\ &= 2^{6k+1} - 2 \cdot 2^{4k+1} - 3 \cdot 2^{3k+1} + 2^{2k+1} + 5 \cdot 2^{k+1} + 12. \end{aligned}$$

Therefore,

$$\begin{aligned} E_{cn}(K_{2^{3k}-2^k-1}) - E_{cn}(\Gamma_c(G)) &= 2^{6k+1} - 3 \cdot 2^{4k+1} + 2^{3k+1} - 2^{k+1} \\ &= 2^{4k+1}(2^{2k} - 3) + 2^{k+1}(2^{2k} - 1). \end{aligned}$$

Since $2^{2k} - 3 > 0$ and $2^{2k} - 1 > 0$ we have $E_{cn}(K_{2^{3k}-2^k-1}) - E_{cn}(\Gamma_c(G))$ is positive. Hence, the result follows.

If G is isomorphic to $GL(2, q)$ then, by Theorem 4 (4), we have

$$E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q.$$

Since $|V(\Gamma_c(G))| = (q^2 - 1)(q^2 - q) - (q - 1) = q^4 - q^3 - q^2 + 1$, by Lemma 1 we have

$$E_{cn}(K_{q^4-q^3-q^2+1}) = 2(q^4 - q^3 - q^2)(q^4 - q^3 - q^2 - 1) = 2q^8 - 4q^7 - 2q^6 + 4q^5 + 2q^3 + 2q^2.$$

Therefore,

$$\begin{aligned} E_{cn}(K_{q^4-q^3-q^2+1}) - E_{cn}(\Gamma_c(G)) &= 2q^8 - 4q^7 - 4q^6 + 10q^5 + 2q^4 - 8q^3 - 4q^2 - 2q \\ &= 2q^6(q^2 - 2q - 2) + 2q^2(5q^3 - 4q - 2) + 2q(q^3 - 2). \end{aligned}$$

We have $q^2 - 2q - 2 = q(q-2) - 2 > 0$, $5q^3 - 4q - 2 = q(5q^2 - 4) - 2 > 0$ and $q^3 - 2 > 0$ since $q = p^m > 2$ for some prime p. Therefore, $E_{cn}(K_{q^4-q^3-q^2+1}) - E_{cn}(\Gamma_c(G))$ is positive and hence the result follows.

If G is isomorphic to $A(n, \sigma)$ then, by Theorem 5 (1), we have $E_{cn}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2)$. Since $|V(\Gamma_c(G))| = 2^n(2^n - 1) = 2^{2n} - 2^n$, by Lemma 1 we have

$$E_{cn}(K_{2^{2n}-2^n}) = 2(2^{2n} - 2^n - 1)(2^{2n} - 2^n - 2).$$

Clearly, $2^{2n} - 2^n - 1 > 2^{2n} - 2 \cdot 2^n - 1 = (2^n - 1)^2$ and $2^{2n} - 2^n - 2 > 2^n - 2$. Therefore, $E_{cn}(K_{2^{2n}-2^n}) > E_{cn}(\Gamma_c(G))$.

If $G \cong A(n, p)$ then, by Theorem 5 (2), we have $E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$. Since $|V(\Gamma_c(G))| = (p^n + 1)(p^{2n} - p^n)$, by Lemma 1 we have

$$E_{cn}(K_{(p^n+1)(p^{2n}-p^n)}) = 2((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2).$$

We have

$$\begin{aligned}
&(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2) \\
&< (p^n + 1)(p^{2n} - p^n - 1)(p^n + 1)(p^{2n} - p^n - 2) \\
&= ((p^n + 1)(p^{2n} - p^n) - (p^n + 1))((p^n + 1)(p^{2n} - p^n) - 2(p^n + 1)) \\
&< ((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2).
\end{aligned}$$

Hence, $E_{cn}(\Gamma_c(G)) < E_{cn}(K_{(p^n+1)(p^{2n}-p^n)})$.

If G is isomorphic to a non-abelian group of order pr then, by Theorem 4 (1), we have

$$E_{cn}(\Gamma_c(G)) = 2(r^2 + p^2 r - 5pr + r + 6).$$

Since $|V(\Gamma_c(G))| = pr - 1$, by Lemma 1 we have

$$E_{cn}(K_{pr-1}) = 2(pr - 2)(pr - 3) = 2(p^2 r^2 - 5pr + 6).$$

Since $r + 1 \leq 2(r - 1) < p^2(r - 1)$ we have $r^2 + p^2 r + r < p^2 r^2$. Hence, $E_{cn}(K_{pr-1}) > E_{cn}(\Gamma_c(G))$. This completes the proof. □

It is already mentioned that $\Pr(G)$, the commutativity degree of a group G, is the probability that a randomly chosen pair of elements of G commute. Therefore, it measures the abelianness of a group. For any finite group G, its commutativity degree can be computed using the formula

$$\Pr(G) = \frac{1}{|G|^2} \sum_{w \in G} |C_G(w)| \text{ or } \Pr(G) = \frac{k(G)}{|G|},$$

where $k(G)$ is the number of conjugacy classes in G. In finite group theory, it is an interesting problem to find all the rational numbers $r \in (0, 1]$ such that $\Pr(G) = r$ for some finite group G. Over the decades, many values of such r have obtained and characterized finite groups such that $\Pr(G) = r$. In the following theorem we list some values of r such that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Pr(G) = r$.

Theorem 12. *If* $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}\}$ *then* $\Gamma_c(G)$ *is not CN-hyperenergetic.*

Proof. If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}\}$ then $\frac{G}{Z(G)}$ is isomorphic to the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$ (by [35] (p. 246) and [36] (p. 451)). Hence, the result follows from Theorem 7. □

Theorem 13. *Let G be a finite group and $\Pr(G) = \frac{p^2+p-1}{p^3}$, where p is the smallest prime divisor of $|G|$. Then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by [37] (Theorem 3). Hence the result follows from Theorem 7. □

Theorem 14. *If G is a finite non-solvable group and $\Pr(G) = \frac{1}{12}$ then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $G \cong A_5 \times K$ for some abelian group K, by [27] (Proposition 3.3.7). It can be seen that $\Gamma_c(G) = 5K_{3|K|} \sqcup 10K_{2|K|} \sqcup 6K_{4|K|}$. Therefore, by Theorem 2, we have

$$\begin{aligned} E_{cn}(\Gamma_c(G)) &= 2(5(3|K|-1)(3|K|-2) + 10(2|K|-1)(2|K|-2)) + 6(4|K|-1)(4|K|-2)) \\ &= 2(181|K|^2 - 177|K| + 42). \end{aligned}$$

Additionally, by Lemma 1, we have $E_{cn}(K_{59|K|}) = 2(3481|K|^2 - 177|K| + 2)$. Therefore

$$E_{cn}(K_{59|K|}) - E_{cn}(\Gamma_c(G)) = 2(3300|K|^2 - 40) > 0.$$

This completes the proof. □

The following three theorems show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is planar/toroidal or the complement of $\Gamma_c(G)$ is planar.

Theorem 15. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is planar then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If $G \cong D_{12}, D_{10}, D_8, D_6, Q_8$ or Q_{12} then, by Corollary 5, we have that $\Gamma_c(G)$ is not CN-hyperenergetic.

If G is isomorphic to one of the groups listed in Corollary 1 then, by Corollary 4, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong A_4$ then it can be seen that $\Gamma_c(G) = K_3 \sqcup 4K_2$. Using Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 4$. Also, by Lemma 1, we have $E_{cn}(K_{11}) = 180$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong Sz(2)$ then $\frac{G}{Z(G)} \cong Sz(2)$. Therefore, by Theorem 7, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong SL(2,3)$ then it can be seen that $\Gamma_c(G) = 3K_2 \sqcup 4K_4$. Therefore, by Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 48$. Also, by Lemma 1, we have $E_{cn}(K_{22}) = 840$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic.

We have $PSL(2,4) \cong A_5$. Therefore, if $G \cong A_5$ then it follows that $\Gamma_c(G)$ is not CN-hyperenergetic (follows from Theorem 11).

If $G \cong S_4$ then the characteristic polynomial of $CN(\Gamma_c(G))$ is given by $x^8(x-3)^2(x+1)^{11}(x^2-5x-30)$ and so

$$\text{CN-spec}(\Gamma_c(G)) = \left\{0^8, 3^2, (-1)^{11}, \left(\frac{5+\sqrt{145}}{2}\right)^1, \left(\frac{5-\sqrt{145}}{2}\right)^1\right\}.$$

Therefore, $E_{cn}(\Gamma_c(G)) = 17 + \sqrt{145}$. Additionally, by Lemma 1, we have $E_{cn}(K_{23}) = 924$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. Hence, the result follows from [38] (Theorem 2.2). □

Theorem 16. *Let G be a finite non-abelian group. If $\Gamma_c(G)$ is toroidal then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If $G \cong D_{14}, D_{16}$ or Q_{16} then by Corollary 5 it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong QD_{16}$ then, by Theorem 11, we have that $\Gamma_c(G)$ is not CN-hyperenergetic. If G is isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then $\Gamma_c(G)$ is not CN-hyperenergetic, follows from Theorem 11 considering $p = 3$ and $r = 7$. If $G \cong D_6 \times \mathbb{Z}_3$ then $\frac{G}{Z(G)} \cong D_6$. Therefore, by Theorem 7, $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong A_4 \times \mathbb{Z}_2$ then it can be seen that $\Gamma_c(G) = K_6 \sqcup 4K_4$. Therefore, by Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 2(5 \cdot 4 + 4 \cdot 3 \cdot 2) = 88$. Also, by Lemma 1, we have $E_{cn}(K_{22}) = 2 \cdot 21 \cdot 20 = 840$. Hence, $\Gamma_c(G)$ is not CN-hyperenergetic. Hence, the result follows from [39] (Theorem 6.6). □

We also have the following result.

Theorem 17. *Let G be a finite non-abelian group. If the complement of $\Gamma_c(G)$ is planar then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. The result follows from [40] (Proposition 2.3) and Corollary 5. □

In view of the above results we conclude this paper with a few conjectures.

Conjecture 1. *A planar or toroidal graph is not CN-hyperenergetic.*

Conjecture 2. *$\Gamma_c(G)$ is not CN-hyperenergetic.*

Conjecture 3. *If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $1 \leq i \leq k$, then it is not CN-hyperenergetic.*

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