Ordering of minimal energies in unicyclic signed graphs

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Abstract. Let $S = (G, \sigma)$ be a signed graph of order n and size m and let $t_1, t_2, \ldots, t_n$ be the eigenvalues of S. The energy of S is defined as $E(S) = \sum\limits_{j=1}^{n} |t_j|$. A connected signed graph is said to be unicyclic if its order and size are same. In this paper, we characterize, up to switching, the unicyclic signed graphs with first 11 minimal energies for all $n \geq 12$. For $3 \leq n \leq 7$, we provide complete ordering of unicyclic signed graphs with respect to energy. For $n = 8, 9, 10$ and 11, we determine unicyclic signed graphs with first 13 minimal energies respectively.

Keywords: Unicyclic signed graph, spectrum, energy, ordering

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1 Introduction

Let $S = (G, \sigma)$ be a signed graph of order n, where $G = (V, E)$ is its underlying graph and $\sigma : E \to \{-1, 1\}$ is its signature. Let A be the adjacency matrix of S. In a signed graph, a cycle is said to be positive if it contains an even number of negative edges, and negative, otherwise. A signed graph is said to be balanced if all its cycles are positive. For undefined terms related to signed graphs, we refer to [1]. The characteristic polynomial $P(S, x)$ of S is the characteristic polynomial of its adjacency matrix A and is given by

$$P(S, t) = \det(tI - A) = \sum_{r=0}^{n} a_r(S) t^{n-r},$$

with

$$a_r(S) = \sum_{l \in \mathscr{L}_r} (-1)^{k(l)} 2^{|c(l)|} \prod_{X \in c(l)} Z(X), \tag{1.1}$$

where $\mathscr{L}_r$ denotes the set of all linear signed subgraphs (also known as basic figures) on r vertices, $k(l)$ denotes the number of components in l, $c(l)$ denotes the set of cycles in l and $Z(X) = \prod_{e \in X} \sigma(e)$ is the sign of X. Let S be a signed graph with vertex set V. For any $X \subseteq V$, let S^X denote the signed graph obtained from S by reversing the signs of the edges between X and $V - X$. Then, we say S^X is switching equivalent to S. Here we note that switching is an equivalence relation and preserves the eigenvalues including their multiplicities. We use a single signed graph as representative of a switching class. Germina et al. [2] defined the energy of a signed graph S with eigenvalues $t_1, t_2, \ldots, t_n$ as $E(S) = \sum_{j=1}^{n} |t_j|$. Note that the definition of the energy of a signed graph is transferred from the domain of unsigned graph. Signed graphs significantly enrich algebraic and geometric properties compared to unsigned graphs [7].

It is well known that even and odd coefficients of the characteristic polynomial of a unicyclic signed graph respectively alternate in sign [[1], Lemma 2.7]. Putting $c_j(S) = |a_j(S)|$, we have the following integral representation for the energy of a unicyclic signed graph S.

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} \log \left[\left(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{2j}(S) t^{2j} \right)^2 + \left(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{2j+1}(S) t^{2j+1} \right)^2 \right] dt. \tag{1.2}$$

From the above integral formula, we see that the energy of a unicyclic signed graph is a monotonic increasing function of the coefficients c_j, where $j = 0, 1, \ldots, n$. For signed graphs S_1 and S_2 of the same order, say n, if $c_j(S_1) \leq c_j(S_2)$ for all j, then we write $S_1 \preceq S_2$. Moreover, if $S_1 \preceq S_2$ and there is a strict inequality in $c_j(S_1) \leq c_j(S_2)$ for some $j = 1, 2, \ldots, n$, then we write $S_1 \prec S_2$. Hence, if $S_1 \preceq S_2$, then $E(S_1) \leq E(S_2)$ and if $S_1 \prec S_2$, then $E(S_1) < E(S_2)$. Also if $c_j(S_1) = c_j(S_2)$ for all j, then we write $S_1 \sim S_2$. Hence, if $S_1 \sim S_2$, then $E(S_1) = E(S_2)$.

Let $S_{n,l}$ denote the set of unicyclic signed graphs with n vertices and a cycle of length $l \leq n$. Let $e = uv$ be a pendant edge of a signed graph $S \in S_{n,l}$ with v as the pendant vertex. Then the following relation holds [[1], Lemma 3.2] for c_j's of a signed graph S and its vertex deleted signed subgraphs.

$$c_j(S) = c_j(S - v) + c_{j-2}(S - u - v). \tag{1.3}$$

2 The unicyclic signed graphs of order n with the first eleven minimal energies

Let $C_r^\sigma (r = 3, 4)$ be signed cycles on 3 and 4 vertices respectively, and k be a nonnegative integer. Let $S_{n,n}^k$ be a unicyclic signed graph obtained from C_3^σ by connecting k pendent vertices to any vertex and remaining $(n - k - 3)$ pendent vertices to any other vertex of C_3^σ. Also, let $B_{n,n}^k$ be a unicyclic signed graph obtained from C_4^σ by connecting k pendent vertices to any vertex and

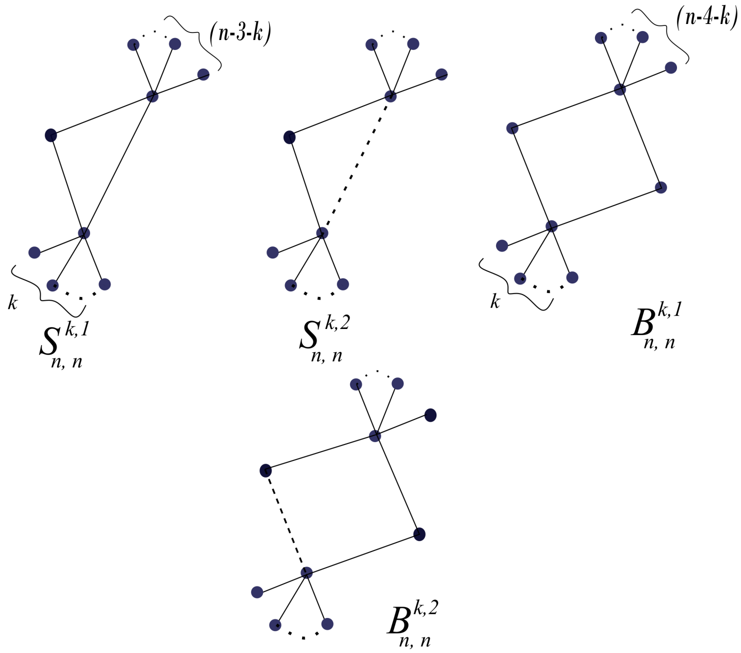

Figure 1: Switching classes corresponding to unicyclic signed graphs $S^k_{n,n}$ and $B^k_{n,n}$ respectively.

remaining $(n-k-4)$ pendent vertices to other vertex of C_4^σ which is at a distance of 2 from this vertex. There are two switching classes in $S^k_{n,n}$ and $B^k_{n,n}$ respectively. We use $S^{k,1}_{n,n}$, $S^{k,2}_{n,n}$ and $B^{k,1}_{n,n}$, $B^{k,2}_{n,n}$ respectively as the representative for these two switching classes as shown in Figure 1 (here positive edge is denoted by a plain line and negative edge by a dotted line). Note that $S^{k,1}_{n,n}$ and $B^{k,1}_{n,n}$ are balanced while as $S^{k,2}_{n,n}$ and $B^{k,2}_{n,n}$ are unbalanced. With these notations, we have the following result.

Lemma 2.1 *(i) For all $n \geq 11$ and $0 \leq k < n-5$,* $E(S^{k,1}_{n,n}) = E(S^{k,2}_{n,n}) < E(B^{k,1}_{n,n})$.
(ii) For all $n \geq 11$, for $k = 0, 1$, $E(B^{k,1}_{n,n}) < E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n})$ *and*
for all $2 \leq k \leq n-4$, $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,1}_{n,n})$.
(iii) For all $n \geq 11$ and $0 \leq k \leq n-4$, $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,2}_{n,n})$.
(iv) For all $n > 2k+9$ and $k \geq 0$, $E(B^{k,2}_{n,n}) < E(B^{k+1,1}_{n,n})$.
(v) For all $n \geq 11$ and $k \geq 0$, $E(B^{k,1}_{n,n}) < E(B^{k,2}_{n,n})$.

Proof. **(i)** By (1.1), we have

$$p(B^{k,1}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4) + 2k]\},$$

$$p(S^{k,1}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 - 2t + [(k+1)(n-k-3) + k]\}$$

and

$$p(S^{k,2}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + 2t + [(k+1)(n-k-3) + k]\}.$$

It is clear that $S_{n,n}^{k,1} \sim S_{n,n}^{k,2}$ and so $E(S_{n,n}^{k,1}) = E(S_{n,n}^{k,2})$. Therefore, to compare the energy of $S_{n,n}^{k,r}$ for $r = 1, 2$ and $B_{n,n}^{k,1}$, it is enough to compare the energy of $S_{n,n}^{k,1}$ and $B_{n,n}^{k,1}$. Clearly, $S_{n,n}^{k,1}$ and $B_{n,n}^{k,1}$ are not quasi-order comparable. We use integral formula (1.2) in this case, and we have

$$E(B_{n,n}^{k,1}) - E(S_{n,n}^{k,1}) = \frac{1}{\pi}\int_0^\infty \ln \frac{\{1 + nt^2 + [(k+2)(n-k-4) + 2k]t^4\}^2}{\{1 + nt^2 + [(k+1)(n-k-3) + k]t^4\}^2 + 4t^6} dt.$$

Put

$$f_1(t) = \{1 + nt^2 + [(k+2)(n-k-4) + 2k]t^4\}^2$$

and

$$g_1(t) = \{1 + nt^2 + [(k+1)(n-k-3) + k]t^4\}^2 + 4t^6.$$

Since $n > k + 5$, we get
$f_1(t)-g_1(t) = 2(n-k-5)t^4+2[n(n-k-5)-2]t^6+(n-k-5)[n-k-5+2(k+1)(n-k-3)+2k]t^8 > 0$ for $n \geq 11$ and $t > 0$, and thus $E(B_{n,n}^{k,1}) > E(S_{n,n}^{k,1})$.

(ii) The characteristic polynomials of $B_{n,n}^{k,1}$ and $S_{n,n}^{k+1,r}$ for $r = 1, 2$ are given by

$$p(B_{n,n}^{k,1}, t) = t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4) + 2k]\},$$

$$p(S_{n,n}^{k+1,1}, t) = t^{n-4}\{t^4 - nt^2 - 2t + [(k+2)(n-k-4) + k + 1]\}$$

and

$$p(S_{n,n}^{k+1,2}, t) = t^{n-4}\{t^4 - nt^2 + 2t + [(k+2)(n-k-4) + k + 1]\}.$$

To prove the result, it is enough to show that $E(B_{n,n}^{k,1}) < E(S_{n,n}^{k+1,1})$ for $k = 0, 1$ and $E(B_{n,n}^{k,1}) > E(S_{n,n}^{k+1,1})$ for all $2 \leq k \leq n-4$. Clearly, $B_{n,n}^{k,1} \prec S_{n,n}^{k+1,1}$, for $k = 0, 1$ and therefore $E(B_{n,n}^{k,1}) < E(S_{n,n}^{k+1,1})$ for $k = 0, 1$ and for all $n \geq 11$. To compare the energy of $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$, for all $2 \leq k \leq n-4$, it is enough to compare the energy of $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$. Clearly, $S_{n,n}^{k+1,1}$ and $B_{n,n}^{k,1}$ are not quasi-order comparable for $2 \leq k \leq n-4$. By (1.2), we have

$$E(B_{n,n}^{k,1}) - E(S_{n,n}^{k+1,1}) = \frac{1}{\pi}\int_0^\infty \ln \frac{\{1 + nt^2 + [(k+2)(n-k-4) + 2k]t^4\}^2}{\{1 + nt^2 + [(k+2)(n-k-4) + k + 1]t^4\}^2 + 4t^6} dt.$$

Put

$$f_2(t) = \{1 + nt^2 + [(k+2)(n-k-4) + 2k]t^4\}^2$$

and

$$g_2(t) = \{1 + nt^2 + [(k+2)(n-k-4) + k + 1]t^4\}^2 + 4t^6.$$

Since $n \geq k + 4$, we get
$f_2(t) - g_2(t) = 2(k-1)t^4 + 2[n(k-1) - 2]t^6 + [2(k+2)(n-k-4)(k-1) + 3k^2 - 2k - 1]t^8 > 0$

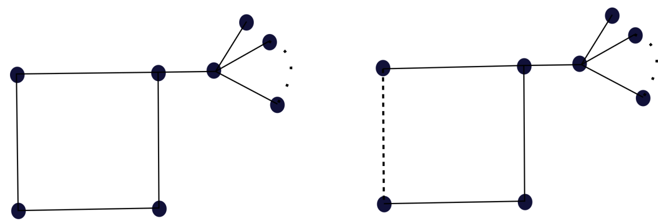

Figure 2: Unicyclic signed graphs $F^1_{n,n}$ and $F^2_{n,n}$ respectively.

for all $2 \le k \le n-4$, $n \ge 11$ and $t > 0$ and therefore $E(B^{k,1}_{n,n}) > E(S^{k+1,1}_{n,n})$ for all $2 \le k \le n-4$ and $n \ge 11$.
(iii) The characteristic polynomial of $B^{k,2}_{n,n}$ is given by

$$p(B^{k,2}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4) + 2k + 4]\},$$

We show that $E(B^{k,2}_{n,n}) > E(S^{k+1,1}_{n,n})$, for all $0 \le k \le n-4$. Clearly, $S^{k+1,1}_{n,n}$ and $B^{k,2}_{n,n}$ are not quasi-order comparable. Proceeding similarly as in part (ii), we can prove that $E(S^{k+1,1}_{n,n}) = E(S^{k+1,2}_{n,n}) < E(B^{k,2}_{n,n})$, for all $0 \le k \le n-4$ and $n \ge 11$.
(iv) We have

$$p(B^{k+1,1}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + [(k+3)(n-k-5) + 2k + 2]\}.$$

Clearly, $B^{k+1,1}_{n,n} \succ B^{k,2}_{n,n}$, for all $n > 2k+7$ and therefore $E(B^{k,2}_{n,n}) < E(B^{k+1,1}_{n,n})$ for all $n > 2k+9$.
(v) This follows by [[1], Theorem 2.10(i)]. ■

Let $F^1_{n,n}$ be a unicyclic graph as shown in Figure 2. There are two switching classes on the signings of $F^1_{n,n}$. Let $F^1_{n,n}$ and $F^2_{n,n}$ be the representative for these two switching classes, where $F^1_{n,n}$ contains a positive cycle of length 4 and $F^2_{n,n}$ contains a negative cycle of length 4. With these notations, we have the following lemma.

Lemma 2.2 (i) *For all* $n \ge 6$, *we have,* $E(B^{2,1}_{n,n}) < E(F^1_{n,n}) < E(B^{2,2}_{n,n}) < E(F^2_{n,n})$.
(ii) *For all* $n \ge 9$, $n = 2k+9$, $E(B^{k,2}_{n,n}) = E(B^{k+1,1}_{n,n})$.

Proof. **(i)** The characteristic polynomials of $B^{2,r}_{n,n}$, for $r = 1, 2$, $F^1_{n,n}$ and $F^2_{n,n}$ are given by
$$p(B^{2,2}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + (4n-16)\},$$
$$p(B^{2,1}_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + (4n-20)\},$$
$$p(F^1_{n,n}, t) = t^{n-4}\{t^4 - nt^2 + (4n-18)\}$$
and
$$p(F^2_{n,n}, t) = t^{n-6}\{t^6 - nt^4 + (4n-14)t^2 - 4(n-5)\}.$$
Clearly, $F^2_{n,n} \succ B^{2,2}_{n,n} \succ F^1_{n,n} \succ B^{2,1}_{n,n}$ and therefore $E(B^{2,1}_{n,n}) < E(F^1_{n,n}) < E(B^{2,2}_{n,n}) < E(F^2_{n,n})$ for

all $n \geq 6$.
(ii) We have

$$p(B_{n,n}^{k+1,1}, t) = t^{n-4}\{t^4 - nt^2 + [(k+3)(n-k-5) + 2k + 2]\}$$

and

$$p(B_{n,n}^{k+1,1}, t) = t^{n-4}\{t^4 - nt^2 + [(k+2)(n-k-4) + 2k + 4]\}.$$

Clearly, $B_{n,n}^{k+1,1} \sim B_{n,n}^{k,2}$, for $n = 2k+7$ and therefore $E(B_{n,n}^{k,2}) = E(B_{n,n}^{k+1,1})$ for all $n = 2k+9$. ■

Combining Lemma 2.1 and Lemma 2.2, we have the following result.

Theorem 2.3 *(i) For $n = 11$, we have*
$E(S_{11,11}^{0,1}) = E(S_{11,11}^{0,2}) < E(B_{11,11}^{0,1}) < E(S_{11,11}^{1,1}) = E(S_{11,11}^{1,2}) < E(B_{11,11}^{0,2}) < E(B_{11,11}^{1,1}) < E(S_{11,11}^{2,1}) = E(S_{11,11}^{2,2}) < E(S_{11,11}^{3,1}) = E(S_{11,11}^{3,2}) < E(B_{11,11}^{1,2}) = E(B_{11,11}^{2,1}) < E(F_{11,11}^{1}) < E(B_{11,11}^{2,2})$.
(ii) For all $n \geq 12$, we have
$E(S_{n,n}^{0,1}) = E(S_{n,n}^{0,2}) < E(B_{n,n}^{0,1}) < E(S_{n,n}^{1,1}) = E(S_{n,n}^{1,2}) < E(B_{n,n}^{0,2}) < E(B_{n,n}^{1,1}) < E(S_{n,n}^{2,1}) = E(S_{n,n}^{2,2}) < E(B_{n,n}^{1,2}) < E(S_{n,n}^{3,1}) = E(S_{n,n}^{3,2}) < E(B_{n,n}^{2,1}) < E(F_{n,n}^{1}) < E(B_{n,n}^{2,2})$.

Let $S_n^{l\sigma}$ denote the signed graph obtained by identifying the center of the signed star S_{n-l+1} with a vertex of C_l^σ. The following theorem shows that among all unicyclic signed graphs with cycle length greater than 5, S_n^{6-} has the minimal energy.

Theorem 2.4 *Let $S \in S_{n,l}$, where $S \neq S_n^{6-}$, $n \geq l$, $n \geq 7$ and $l \geq 6$. Then $S \succ S_n^{6-}$ and $E(S) > E(S_n^{6-})$.*

Proof. By (1.1), we have

$$p(S_n^{6-}, t) = t^{n-6}\{t^6 - nt^4 + (4n-15)t^2 - (3n-18)\}.$$

In view of integral formula (1.2), it suffices to prove that $c_i(S_n^{6-}) \leq c_i(S)$, for all $i = 4, 6$, with strict inequality holds for at least one i. Here, we need to consider two cases.
Case 1. Let $S \in S_{n,l}$ be unbalanced, where $n \geq l$, $n \geq 7$ and $l \geq 6$. Then, by [[1], Theorem 3.3], it suffices to show that $c_i(S_n^{6-}) \leq c_i(S_n^{l-})$ for all $i = 4, 6$ with strict inequality for at least one i. We use induction on $n - l$ for $n \geq l$, where $n \geq 7$ and $l \geq 6$.

If $n - l = 0$, then $S_n^{l-} = C_n^-$. We have $c_4(C_n^-) = \frac{n(n-3)}{2}$, $c_4(S_n^{6-}) = 4n - 15$, $c_6(C_n^-) = \frac{n(n-4)(n-5)}{6}$ and $c_6(S_n^{6-}) = 3n - 18$. Clearly, $c_i(S_n^{6-}) < c_i(C_n^-)$, for $i = 4, 6$ and $n \geq 7$. By (1.3), for $i = 4, 6$, we have

$$c_i(S_n^{l-}) = c_i(S_{n-1}^{l-}) + c_{i-2}(P_{l-1})$$

and

$$c_i(S_n^{6-}) = c_i(S_{n-1}^{6-}) + c_{i-2}(P_5).$$

By induction $S_{n-1}^{l-} \succ S_{n-1}^{6-}$. Since $l \geq 6$, therefore P_{l-1} has P_5 as a subgraph and hence $c_i(S_n^{6-}) \leq c_i(S_n^{l-})$, for all $i = 4, 6$, with strict inequality holds for at least one i.
Case 2. This is similar to Case 1. ■

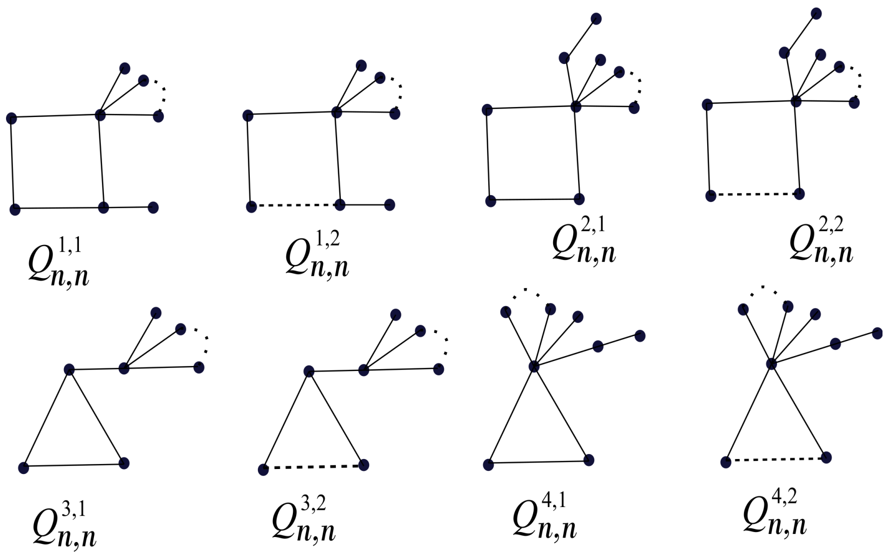

Figure 3: Unicyclic signed graphs $Q_{n,n}^{r,s}$ ($r = 1, 2, 3, 4$ and $s = 1, 2$).

Lemma 2.5 *For all $n \geq 6$, we have (i) $E(S_n^{6-}) > E(B_{n,n}^{2,2})$, (ii) $E(S_n^{5+}) > E(B_{n,n}^{2,2})$.*

Proof. (i) The characteristic polynomials of S_n^{6-} and $B_{n,n}^{2,2}$ are respectively given by

$$p(S_n^{6-}, t) = t^{n-6}\{t^6 - nt^4 + (4n - 15)t^2 - (3n - 18)\}$$

and

$$p(B_{n,n}^{2,2}, t) = t^{n-6}\{t^6 - nt^4 + (4n - 16)t^2\}.$$

Clearly, $S_n^{6-} \succ B_{n,n}^{2,2}$ for all $n \geq 6$ and therefore $E(S_n^{6-}) > E(B_{n,n}^{2,2})$ for all $n \geq 6$.
(ii) We have

$$p(S_n^{5+}, t) = t^{n-6}\{t^6 - nt^4 + (3n - 10)t^2 - 2t - (n - 5)\}.$$

The signed graphs S_n^{5+} and $B_{n,n}^{2,2}$ are not quasi-order comparable. Consider the functions $f_3(t) = t^6 - nt^4 + (3n - 10)t^2 - 2t - (n - 5)$ and $g_3(t) = t^4 - nt^2 + (4n - 16)$. It is easy to see that $f_3(\frac{3}{5}) < 0$, $f_3(1) > 0$, $f_3(\frac{7}{5}) > 0$, $f_3(2) < 0$, $f_3(\sqrt{n-3}) < 0$ and $f_3(\sqrt{n-2}) > 0$ for all $n \geq 10$. Also, $g_3(2) = 0$ and $g_3(\sqrt{n-4}) = 0$. By Descarte's rule of signs, $f_3(t)$ has three positive and three negative zeros and $g_3(t)$ has two positive and two negative zeros. As the energy of a signed graph is twice the sum of its positive eigenvalues, therefore

$$E(S_n^{5+}) > 2(2 + \sqrt{n-3}) > 2(2 + \sqrt{n-4}) = E(B_{n,n}^{2,2})$$

for all $n \geq 10$. We have verified the result directly for $n = 6, 7, 8, 9$.

Let $Q_{n,n}^{r,1}$, $r = 1, 2, 3, 4$ be the graphs as shown in Figure 3. It is easy to see that there are two switching classes on the signings of $Q_{n,n}^{r,1}$, for all $r = 1, 2, 3, 4$. Let $Q_{n,n}^{r,1}$ and $Q_{n,n}^{r,2}$ ($r = 1, 2, 3, 4$) be the representative for these two switching classes, where $Q_{n,n}^{r,1}$ ($r = 1, 2, 3, 4$) contains positive cycle and $Q^{r,2}_{n,n}$ $(r = 1, 2, 3, 4)$ contains negative cycle. We have the following lemma, the proof of which is similar to that of Lemma 2.5, and so we skip it here.

Lemma 2.6 *For all $n \geq 6$, we have*
(i) $E(B^{2,2}_{n,n}) < E(Q^{1,1}_{n,n}) < E(Q^{1,2}_{n,n})$.
(ii) $E(Q^{1,1}_{n,n}) < E(Q^{2,1}_{n,n}) < E(Q^{2,2}_{n,n})$.
(iii) $E(B^{2,2}_{n,n}) < E(Q^{3,1}_{n,n}) = E(Q^{3,2}_{n,n})$.
(vi) $E(B^{2,2}_{n,n}) < E(Q^{4,1}_{n,n}) = E(Q^{4,2}_{n,n})$.

A unicyclic signed graph can be obtained by attaching rooted signed trees to the vertices of the cycle C^{σ}_l. Thus, if $T_1, T_2, \dots, T_l$ are l rooted signed trees, then we denote by $U(T_1, T_2, \dots, T_l, \sigma)$, the signed graph obtained by attaching the rooted signed trees T_i to the vertices v_i of the cycle $C^{\sigma}_l = v_1 v_2 \dots v_l v_1$. When T_i is a rooted signed star K_{1,n_i} with the center of star as its root, then we write $U(n_1, n_2, \dots, n_l, \sigma)$ instead of $U(T_1, T_2, \dots, T_l, \sigma)$. For example, $B^{2,1}_{n,n} = U(n-6, 0, 2, 0, +)$ and $B^{2,2}_{n,n} = U(n-6, 0, 2, 0, -)$.

Also, when T_i is a rooted signed star K_{1,n_i}, with a pendent vertex of star as its root, then we simplify the notation $U(T_1, T_2, \dots, T_l, \sigma)$ by replacing T_i by the pair $(n_i - 1, 1)$. For example, $F^1_{n,n} = U((n-5, 1), 0, 0, 0, +)$ and $F^2_{n,n} = U((n-5, 1), 0, 0, 0, -)$. Let $T(m-2, 2)$ be the rooted signed tree obtained by identifying end vertex of a path of length 2 with center of the star $K_{1,m-2}$ and let vertex of degree $m-1$ be the root. Clearly, $Q^{2,1}_{n,n} = U(T(n-6, 2), 0, 0, 0, +)$ and $Q^{2,2}_{n,n} = U(T(n-6, 2), 0, 0, 0, -)$

Let $S(n)$ be the set of all unicyclic signed graphs of order n. Let $S \in S(n)$ and u be a vertex of S. Let T be a rooted signed tree and $S_u(T)$ be the signed graph obtained by attaching T to S such that the root of T is u. When T is a signed path P_{m+1} with one endpoint as the root, then we write $S_u(T)$ as $S_u(m)$. When T is a star $K_{1,m}$ with the center as its root, then we write $S_u(T)$ as $S^*_u(m)$. When T is a star $K_{1,m}$ with a pendent vertex as its root, then we write $S_u(T)$ as $S^*_u(m-1, 1)$. For example, if $S = C^-_3$, then $S^*_u(n-3) = S^{0,2}_{n,n}$ and $S^*_u(n-4, 1) = Q^{3,2}_{n,n}$. With these notations, we have following lemmas.

Lemma 2.7 *[6] Let $S \in S(n)$ be balanced and u be a vertex of S. Let T be a tree of order $m+1$ rooted at u. Then we have the following.*
(1) If $S_u(T) \neq S_u(m)$, then $S_u(T) \prec S_u(m)$.
(2) If $S_u(T) \neq S^*_u(m)$, then $S_u(T) \succ S^*_u(m)$.

Lemma 2.8 *[8] Let $S \in S(n)$ be balanced, u be a vertex of S and T be a tree of order $m+1$ $(m \geq 3)$ rooted at u. If $S_u(T) \neq S_u(T(m-2, 2)), S^*_u(m-1, 1), S^*_u(m)$, then $S_u(T) \succ S_u(T(m-2, 2))$.*

For a signed graph S, let $d_S(v)$ denote the degree of a vertex v. Recall that $U(n_1, n_2, \dots, n_l, \sigma)$ denote the unicyclic signed graph obtained by attaching the rooted signed star K_{1,n_i} with the center of star as its root, to the vertices v_i for $i = 1, 2, \dots, l$, of the cycle $C^{\sigma}_l = v_1 v_2 \dots v_l v_1$. We

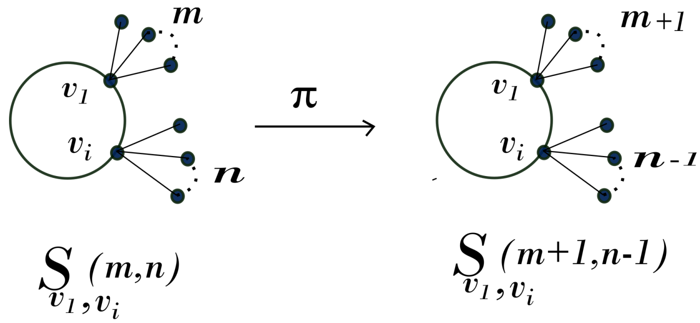

Figure 4: The edge grafting π.

denote $S_{v_1,v_i}(m,n) = U(m,0,0,0,\ldots,n,n_{i+1},n_{i+2},\ldots,n_l,\sigma)(2 \leq i \leq l)$ to be the signed graph obtained by attaching m pendent edges and n pendent edges to the vertices v_1 and v_i of signed graph $U(0,0,0,0,\ldots,0,n_{i+1},n_{i+2},\ldots,n_l,\sigma)(2 \leq i \leq l)$ respectively as shown in Figure 4. Then $S_{v_1,v_i}(m+1,n-1)$ is the signed graph obtained from the signed graph $S_{v_1,v_i}(m,n)$ by deleting a pendent edge which is adjacent to v_i and adding a pendent edge to v_1, also called edge grafting π. Proceeding exactly in a similar way in [[8], Theorem 2.2], we obtain the following lemma.

Lemma 2.9 *Let m and n be positive integers. If $m \geq n$, then $S_{v_1,v_i}(m,n) \succ S_{v_1,v_i}(m+1,n-1)$.*

Let $\mathscr{U}_n = \{S \in S(n)$ and $S \neq S^{k,r}_{n,n}$ $(k = 0,1,2,3$ and $r = 1,2)$, $B^{k,r}_{n,n}$ $(k = 0,1,2$ and $r = 1,2)$, $F^1_{n,n}\}$. Also, let $C^\sigma_l = v_1v_2\ldots v_lv_1$ be the unique cycle of the unicyclic signed graph S and $N(S) = \{v_i|d(v_i) > 2,\ v_i \in V(C^\sigma_l)\}$.

Theorem 2.10 *Let $S \in \mathscr{U}_n$. If $n \geq 11$, then $E(S) > E(B^{2,2}_{n,n})$.*

Proof. Let $C^\sigma_l = v_1v_2\ldots v_lv_1$ be the unique cycle of the unicyclic signed graph S and $N(S) = \{v_i\ :\ d(v_i) > 2,\ v_i \in V(C^\sigma_l)\}$. Then the following cases arise:

Case 1. If $l \geq 5$, then the following two subcases arise.

Subcase 1.1. If $l \geq 6$, then the result follows by Theorem 2.4 and Lemma 2.5.

Subcase 1.2. If $l = 5$, then the result follows by [[1], Theorem 2.9], [[4], Theorem 4] and Lemma 2.5.

Case 2. Let $l = 4$. If $S = F^2_{n,n}$, then the result follows by Lemma 2.2. Also, by [[1], Theorem 2.10(i)], it is enough to show that $E(S) > E(B^{2,2}_{n,n})$, where S is balanced. Therefore, the following subcases arise.

Subcase 2.1. If $|N(S)| = 1$, then by Lemma 2.8, $S \succ Q^{2,1}_{n,n}$ and therefore $E(S) > E(Q^{2,1}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 2.2. If $|N(S)| = 2$, then by Lemma 2.7, $S \succeq U(n-4-r,r,0,0,+)(r \geq 1)$ or

$S \succeq U(n-4-r,0,r,0,+)(r \geq 3)$. By Lemma 2.9, $U(n-4-r,r,0,0,+) \succeq Q^{1,1}_{n,n}(r \geq 1)$ or $U(n-4-r,0,r,0,+) \succeq B^{3,1}_{n,n} \succ B^{2,2}_{n,n}(r \geq 3)$ and therefore $E(S) \geq E(Q^{1,1}_{n,n})$ or $E(S) > E(B^{2,2}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 2.3. If $|N(S)| = 3$, then by Lemma 2.7, $S \succeq U(n-4-r-s,r,s,0,+)(r \geq 1, s \geq 1)$ or $S \succeq U(n-4-r-s,r,0,s,+)(r \geq 1, s \geq 1)$. By Lemma 2.9, $U(n-4-r-s,r,s,0,+) \succ U(n-4-s,0,s,0,+)$ or $U(n-4-r-s,r,0,s,+) \succ U(n-4-s,0,0,s,+)$. Again by Lemma 2.9, $U(n-4-s,0,s,0,+) \succeq B^{3,1}_{n,n} \succ B^{2,2}_{n,n}(s \geq 3)$ or $U(n-4-s,0,0,s,+) \succeq Q^{1,1}_{n,n}(s \geq 1)$ and therefore $E(S) > E(Q^{1,1}_{n,n})$ or $E(S) > E(B^{2,2}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 2.4. If $|N(S)| = 4$, then by Lemma 2.7, $S \succeq U(n-4-r_1-r_2-r_3,r_1,r_2,r_3,+)(r_i \geq 1, i = 1,2,3)$. Now applying Lemma 2.9 repeatedly, we have, $U(n-4-r_1-r_2-r_3,r_1,r_2,r_3,+) \succ U(n-4-r_2-r_3,0,r_2,r_3,+) \succ U(n-4-r_3,0,0,r_3,+) \succeq Q^{1,1}_{n,n}(r_3 \geq 1,)$ and therefore $E(S) > E(Q^{1,1}_{n,n})$. Hence the result follows by Lemma 2.6.

Case 3. Let $l = 3$. By [[1], Theorem 2.9], it is enough to show that $E(S) > E(B^{2,2}_{n,n})$, where S is balanced. Therefore the following subcases arise.

Subcase 3.1. If $|N(S)| = 1$, by Lemma 2.8, if $S \neq Q^{r,1}_{n,n}(r = 3,4)$, then $S \succ Q^{4,1}_{n,n}$ and therefore $E(S) > E(Q^{4,1}_{n,n})$. Hence the result follows by Lemma 2.6.

Subcase 3.2. If $|N(S)| = 2$, then by Lemma 2.7, we have $S \succeq U(n-3-r,r,0,+)(r \geq 1)$. By Lemma 2.9, $U(n-3-r,r,0,+) \succeq U(n-7,4,0,+)(r \geq 4)$. By Sach's theorem, we have

$$p(U(n-7,4,0,+),t) = t^{n-4}\{t^4 - nt^2 - 2t + (5n-31)\}.$$

Clearly, $U(n-7,4,0,+) \succ B^{2,2}_{n,n}$ and therefore $E(U(n-7,4,0,+)) > E(B^{2,2}_{n,n})$. Hence the result follows in this subcase.

Subcase 3.3. If $|N(S)| = 3$, then by Lemma 2.7, we have $S \succeq U(n-3-r_1-r_2,r_1,r_2,+)(r_i \geq 1, i = 1,2)$. Now applying Lemma 2.9 repeatedly, we have, $U(n-3-r_1-r_2,r_1,r_2,+) \succ U(n-3-r_2,0,r_2,+) \succeq U(n-7,0,4,+)(r_2 \geq 4)$. By Sach's theorem, we get

$$p(U(n-7,0,4,+),t) = t^{n-4}\{t^4 - nt^2 - 2t + (5n-31)\}.$$

Clearly, $U(n-7,0,4,+) \succ B^{2,2}_{n,n}$ and therefore $E(U(n-7,0,4,+)) > E(B^{2,2}_{n,n})$. Hence the result follows. This completes the proof.

By direct calculations via computer simulation, we have verified that the signed graphs $Q^{3,1}_{11,11}$, $Q^{3,2}_{11,11}$ have 11^{th} minimal energy, $Q^{4,1}_{11,11}$, $Q^{4,2}_{11,11}$ have 12^{th} minimal energy and $Q^{1,1}_{11,11}$ has 13^{th} minimal energy for $n = 11$. The following theorem is the main result of our paper.

Theorem 2.11 (*i*) *Among all unicyclic signed graphs with $n = 11$ vertices, $Q^{1,1}_{11,11}$ is the signed graph with 13^{th} minimal energy. Also, we have ordering of energies in ascending order as follows.* $E(S^{0,1}_{11,11}) = E(S^{0,2}_{11,11}) < E(B^{0,1}_{11,11}) < E(S^{1,1}_{11,11}) = E(S^{1,2}_{11,11}) < E(B^{0,2}_{11,11}) < E(B^{1,1}_{11,11}) < E(S^{2,1}_{11,11}) = E(S^{2,2}_{11,11}) < E(S^{3,1}_{11,11}) = E(S^{3,2}_{11,11}) < E(B^{1,2}_{11,11}) = E(B^{2,1}_{11,11}) < E(F^{1}_{11,11}) < E(B^{2,2}_{11,11}) < E(Q^{3,1}_{11,11}) = E(Q^{3,2}_{11,11}) < E(Q^{4,1}_{11,11}) = E(Q^{4,2}_{11,11}) < E(Q^{1,1}_{11,11})$.

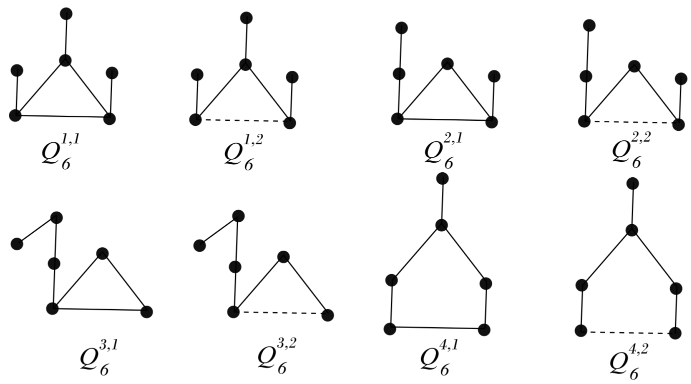

Figure 5: Unicyclic signed graphs $Q_6^{r,s}, r = 1, 2, 3, 4$ and $s = 1, 2$.

(ii) Among all unicyclic signed graphs with $n \geq 12$ vertices, $B_{n,n}^{2,2}$ is the signed graph with 11^{th} minimal energy for all $n \geq 12$. Also, we have ordering of energies in ascending order as follows. $E(S_{n,n}^{0,1}) = E(S_{n,n}^{0,2}) < E(B_{n,n}^{0,1}) < E(S_{n,n}^{1,1}) = E(S_{n,n}^{1,2}) < E(B_{n,n}^{0,2}) < E(B_{n,n}^{1,1}) < E(S_{n,n}^{2,1}) = E(S_{n,n}^{2,2}) < E(B_{n,n}^{1,2}) < E(S_{n,n}^{3,1}) = E(S_{n,n}^{3,2}) < E(B_{n,n}^{2,1}) < E(F_{n,n}^{1}) < E(B_{n,n}^{2,2})$.

Proof. The result follows by Theorems 2.3 and 2.10. ∎

Finally, we consider the partial ordering by minimal energies of unicyclic signed graphs with at most 10 vertices. There does not exist any unicyclic signed graph on one and two vertices. For $n = 3$, there is unique unicyclic unsigned graph, which gives two unicyclic signed graphs with the same number of vertices, up to switching. Let C_3^+ and C_3^- be balanced and unbalanced cycle on 3 vertices respectively. By coefficient theorem, we have $p(C_3^+, t) = t^3 - 3t - 2$ and $p(C_3^-, t) = t^3 - 3t + 2$. Therefore by integral formula (1.2), we get $E(C_3^+) = E(C_3^-)$.

For $n = 4$, there are 4 unicyclic signed graphs, up to switching. Let $S_{4,4}^{0,1}$, $S_{4,4}^{0,2}$, C_4^+ and C_4^- be 4 unicyclic signed graph, up to switching, on 4 vertices. Their characteristic polynomials is respectively given as

$p(S_{4,4}^{0,1}, t) = t^4 - 4t^2 - 2t + 1$, $p(S_{4,4}^{0,2}, t) = t^4 - 4t^2 + 2t + 1$, $p(C_4^+, t) = t^4 - 4t^2$ and $p(C_4^-, t) = t^4 - 4t^2 + 4$. By direct calculations, we have, $E(S_{4,4}^{0,1}) = 4.9622 = E(S_{4,4}^{0,2})$, $E(C_4^+) = 4$ and $E(C_4^-) = 5.657$. Therefore we obtain

$$E(C_4^+) < E(S_{4,4}^{0,1}) = E(S_{4,4}^{0,2}) < E(C_4^-).$$

For $n = 5$, there are 10 unicyclic signed graph, up to switching. Let $S_{5,5}^{0,1}$, $S_{5,5}^{0,2}$, $S_{5,5}^{1,1}$, $S_{5,5}^{1,2}$, $Q_{5,5}^{3,1}$, $Q_{5,5}^{3,2}$, $B_{5,5}^{0,1}$, $B_{5,5}^{0,2}$, C_5^+ and C_5^- be 10 unicyclic signed graph, up to switching, on 5 vertices. Their characteristic polynomials is respectively given as

$p(S_{5,5}^{0,1}, t) = t\{t^4 - 5t^2 - 2t + 2\}$, $p(S_{5,5}^{0,2}, t) = t\{t^4 - 5t^2 + 2t + 2\}$, $p(S_{5,5}^{1,1}, t) = t\{t^4 - 5t^2 - 2t + 3\}$, $p(S_{5,5}^{1,2}, t) = t\{t^4 - 5t^2 + 2t + 3\}$, $p(Q_{5,5}^{3,1}, t) = t^5 - 5t^3 - 2t^2 + 4t + 2$, $p(Q_{5,5}^{3,2}, t) = t^5 - 5t^3 + 2t^2 + 4t - 2$, $p(B_{5,5}^{0,1}, t) = t\{t^4 - 5t^2 + 2\}$, $p(B_{5,5}^{0,2}, t) = t\{t^4 - 5t^2 + 6\}$, $p(C_5^+, t) = t^5 - 5t^3 + 5t - 2$ and $p(C_5^-, t) = t^5 - 5t^3 + 5t + 2$.

By direct calculations, it is easy to see that $E(S_{5,5}^{0,1}) = 5.6272 = E(S_{5,5}^{0,2})$, $E(S_{5,5}^{1,1}) = 5.8416 =$

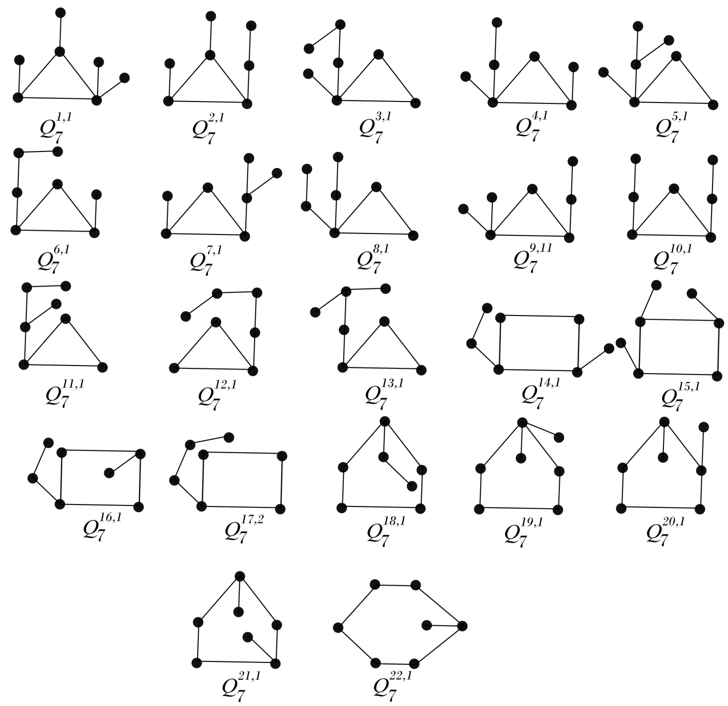

Figure 6: Unicyclic signed graphs $Q_7^{r,1}, r = 1, 2, 3, ..., 22$.

$E(S_{5,5}^{1,2})$, $E(Q_{5,5}^{3,1}) = 6.4286 = E(Q_{5,5}^{3,2})$, $E(B_{5,5}^{0,1}) = 5.596$, $E(B_{5,5}^{0,2}) = 6.2926$ and $E(C_5^+) = 6.472 = E(C_5^-)$. Therefore, we have
$E(B_{5,5}^{0,1}) < E(S_{5,5}^{0,1}) = E(S_{5,5}^{0,2}) < E(S_{5,5}^{1,1}) = E(S_{5,5}^{1,2}) < E(B_{5,5}^{0,2}) < E(Q_{5,5}^{3,1}) = E(Q_{5,5}^{3,2}) < E(C_5^+) = E(C_5^-)$.

For $n = 6$, there are 26 unicyclic signed graphs, up to switching. Let $S_{6,6}^{0,1}$, $S_{6,6}^{0,2}$, $S_{6,6}^{1,1}$, $S_{6,6}^{1,2}$, $Q_{6,6}^{3,1}$, $Q_{6,6}^{3,2}$, $Q_{6,6}^{4,1}$, $Q_{6,6}^{4,2}$, $B_{6,6}^{0,1}$, $B_{6,6}^{0,2}$, $B_{6,6}^{1,1}$, $B_{6,6}^{1,2}$, $Q_{6,6}^{1,1}$, $Q_{6,6}^{1,2}$, $F_{6,6}^{1}$, $F_{6,6}^{2}$, $Q_6^{1,1}$, $Q_6^{1,2}$, $Q_6^{2,1}$, $Q_6^{2,2}$, $Q_6^{3,1}$, $Q_6^{3,2}$, $Q_6^{4,1}$, $Q_6^{4,2}$, C_6^+, and C_6^- be 26 unicyclic signed graph, up to switching, on 6 vertices. Here, the signed graphs $Q_6^{1,1}$, $Q_6^{1,2}$, $Q_6^{2,1}$, $Q_6^{2,2}$, $Q_6^{3,1}$, $Q_6^{3,2}$, $Q_6^{4,1}$ and $Q_6^{4,2}$ are shown in Figure 5. By direct calculations, it is easy to see that, $E(S_{6,6}^{0,1}) = E(S_{6,6}^{0,2}) = 6.1722$, $E(S_{6,6}^{1,1}) = E(S_{6,6}^{1,2}) = 6.4852$, $E(Q_{6,6}^{3,1}) = E(Q_{6,6}^{3,2}) = 7.1916$, $E(Q_{6,6}^{4,1}) = E(Q_{6,6}^{4,2}) = 7.3426$, $E(B_{6,6}^{0,1}) = 6.3244$, $E(B_{6,6}^{0,2}) = 6.8284$, $E(B_{6,6}^{1,1}) = 6.4722$, $E(B_{6,6}^{1,2}) = 6.9284$, $E(Q_{6,6}^{1,1}) = 7.2078$, $E(Q_{6,6}^{1,2}) = 7.5176$, $E(F_{6,6}^{1}) = 6.6026$, $E(F_{6,6}^{2}) = 8.0548$, $E(Q_6^{1,1}) = E(Q_6^{1,2}) = 7.3004$, $E(Q_6^{2,1}) = E(Q_6^{2,2}) = 7.416$, $E(Q_6^{3,1}) = E(Q_6^{3,2}) = 7.5494$, $E(Q_6^{4,1}) = E(Q_6^{4,2}) = 7.4658$, $E(C_6^+) = 8$, and $E(C_6^-) = 6.9284$. Therefore, we have
$E(S_{6,6}^{0,1}) = E(S_{6,6}^{0,2}) < E(B_{6,6}^{0,1}) < E(B_{6,6}^{1,1}) < E(S_{6,6}^{1,1}) = E(S_{6,6}^{1,2}) < E(F_{6,6}^{1}) < E(B_{6,6}^{0,2}) < E(B_{6,6}^{1,2}) = E(C_6^-) < E(Q_{6,6}^{3,1}) = E(Q_{6,6}^{3,2}) < E(Q_{6,6}^{1,1}) < E(Q_6^{1,1}) = E(Q_6^{1,2}) < E(Q_{6,6}^{4,1}) = E(Q_{6,6}^{4,2}) < E(Q_6^{2,1}) = E(Q_6^{2,2}) < E(Q_6^{4,1}) = E(Q_6^{4,2}) < E(Q_{6,6}^{1,2}) < E(Q_6^{3,1}) = E(Q_6^{3,2}) < E(C_6^+) < E(F_{6,6}^{2})$.

For $n = 7$, there are 33 unicyclic unsigned graph on 7 vertices, which gives 66 unicyclic signed graphs with the same number of vertices, up to switching. Let $S_{7,7}^{r,s}(r = 0, 1, 2$ and $s = 1, 2)$, $B_{7,7}^{r,s}(r = 0, 1$ and $s = 1, 2)$, $Q_{7,7}^{r,s}(r = 1, 2, 3, 4$ and $s = 1, 2)$, $F_{7,7}^{r}(r = 1, 2)$, $Q_7^{r,s}(r = 1, 2, ..., 22$

and $s = 1, 2$), C_7^+ and C_7^-.
By direct calculations, it is easy to see that, $E(S_{7,7}^{0,1}) = E(S_{7,7}^{0,2}) = 6.6468 < E(B_{7,7}^{0,1}) = 6.899 < E(S_{7,7}^{1,1}) = E(S_{7,7}^{1,2}) = 7.0206 < E(B_{7,7}^{1,1}) = 7.1154 < E(S_{7,7}^{2,1}) = E(S_{7,7}^{2,2}) = 7.1232 < E(B_{7,7}^{0,2}) = E(F_{7,7}^{1}) = 7.3006 < E(B_{7,7}^{1,2}) = 7.4642 < E(Q_{7,7}^{3,1}) = E(Q_{7,7}^{3,2}) = 7.8102 < E(Q_{7,7}^{1,1}) = 7.9426 < E(Q_{7,7}^{4,1}) = E(Q_{7,7}^{4,2}) = 7.9688 < E(Q_{7,7}^{2,1}) = 8.004 < E(Q_7^{1,1}) = E(Q_7^{1,2}) = 8.0094 < E(Q_7^{16,1}) = 8.0628 < E(Q_7^{4,1}) = E(Q_7^{4,2}) = E(Q_7^{9,1}) = E(Q_7^{9,2}) = 8.0852 < E(Q_7^{5,1}) = E(Q_7^{5,2}) = 8.1178 < E(Q_7^{17,1}) = 8.12 < E(Q_7^{19,1}) = E(Q_7^{19,2}) = 8.1282 < E(Q_7^{15,1}) = 8.1528 < E(Q_7^{7,1}) = E(Q_7^{7,2}) = 8.171 < E(Q_{7,7}^{1,2}) = 8.175 < E(Q_7^{14,1}) = 8.2078 < E(Q_7^{13,1}) = E(Q_7^{13,2}) = 8.2618 < E(Q_7^{3,1}) = E(Q_7^{3,2}) = 8.3012 < E(Q_7^{21,1}) = E(Q_7^{21,2}) = 8.3184 < E(Q_7^{15,2}) = E(Q_7^{22,2}) = 8.3632 < E(Q_7^{2,1}) = E(Q_7^{2,2}) = 8.3898 < E(Q_7^{20,1}) = E(Q_7^{20,2}) = 8.4286 < E(Q_7^{6,1}) = E(Q_7^{6,2}) = 8.4556 < E(Q_{7,7}^{2,2}) = 8.647 < E(Q_7^{16,2}) = 8.6906 < E(Q_7^{22,1}) = 8.7266 < E(Q_7^{14,2}) = 8.7628 < E(Q_7^{8,1}) = E(Q_7^{2,2}) = E(F_{7,7}^{2}) = 8.8284 < E(Q_7^{11,1}) = E(Q_7^{11,2}) = 8.8696 < E(Q_7^{10,1}) = E(Q_7^{10,2}) = 8.8702 < E(Q_7^{18,1}) = E(Q_7^{18,2}) = 8.9172 < E(Q_7^{12,1}) = E(Q_7^{12,2}) = 8.9408 < E(C_7^+) = E(C_7^-) = 8.988 < E(Q_7^{17,2}) = 8.9838$. Where unicyclic signed graphs $Q_7^{r,1}, r = 1, 2, 3, ..., 22$ are shown in Figure 6. Also, $Q_7^{r,2}$ is unbalanced unicyclic signed graph corresponding to balanced signed graph $Q_7^{r,1}$. Similarly, by direct calculations with the help of MATLAB software, we obtain the following result.

Theorem 2.12 *(i) Among all 178 unicyclic signed graphs on 8 vertices, $Q_8^{1,1}$ and $Q_8^{1,2}$ are the signed graphs with 13^{th} minimal energy. Also, we have ordering of energies in ascending order as follows.*
$E(S_{8,8}^{0,1}) = E(S_{8,8}^{0,2}) < E(B_{8,8}^{0,1}) < E(S_{8,8}^{1,1}) = E(S_{8,8}^{1,2}) < E(B_{8,8}^{1,1}) < E(S_{8,8}^{2,1}) = E(S_{8,8}^{2,2}) < E(B_{8,8}^{0,2}) = E(B_{8,8}^{2,1}) < E(F_{8,8}^{1}) < E(B_{8,8}^{1,2}) < E(B_{8,8}^{2,2}) < E(Q_{8,8}^{3,1}) = E(Q_{8,8}^{3,2}) < E(Q_{8,8}^{2,1}) < E(Q_{8,8}^{1,1}) < E(Q_8^{1,1}) = E(Q_8^{1,2})$. *Where $Q_8^{1,r}, r = 1, 2$ are shown in Figure 7.*
(ii) Among all 480 unicyclic signed graphs on 9 vertices, $Q_{9,9}^{2,1}$ is the signed graph with 13^{th} minimal energy. Also, we have ordering of energies in ascending order as follows.
$E(S_{9,9}^{0,1}) = E(S_{9,9}^{0,2}) < E(B_{9,9}^{0,1}) < E(S_{9,9}^{1,1}) = E(S_{9,9}^{1,2}) < E(B_{9,9}^{0,2}) = E(B_{9,9}^{1,1}) < E(S_{9,9}^{2,1}) = E(S_{9,9}^{2,2}) < E(B_{9,9}^{2,1}) < E(F_{9,9}^{1}) = E(B_{9,9}^{1,2}) < E(B_{9,9}^{2,2}) < E(Q_{9,9}^{3,1}) = E(Q_{9,9}^{3,2}) < E(Q_{9,9}^{4,1}) = E(Q_{9,9}^{4,2}) < E(Q_{9,9}^{1,1}) < E(Q_9^{1,1}) = E(Q_9^{1,2}) < E(Q_{9,9}^{2,1})$. *Where $Q_9^{1,r}, r = 1, 2$ are shown in Figure 7.*
(iii) Among all 667 unicyclic unsigned graphs on 10 vertices, which gives 1334 unicyclic signed graphs, upto switching. $Q_{10,10}^{4,1}$ and $Q_{10,10}^{4,2}$ are the signed graph with 13^{th} minimal energy. Also, we have ordering of energies in ascending order as follows.
$E(S_{10,10}^{0,1}) = E(S_{10,10}^{0,2}) < E(B_{10,10}^{0,1}) < E(S_{10,10}^{1,1}) = E(S_{10,10}^{1,2}) < E(B_{10,10}^{0,2}) < E(B_{10,10}^{1,1}) < E(S_{10,10}^{2,1}) = E(S_{10,10}^{2,2}) < E(S_{10,10}^{3,1}) = E(S_{10,10}^{3,2}) < E(B_{10,10}^{2,1}) < E(B_{10,10}^{1,2}) < E(F_{10,10}^{1}) < E(B_{10,10}^{2,2}) < E(Q_{10,10}^{3,1}) = E(Q_{10,10}^{3,2}) < E(Q_{10,10}^{4,1}) = E(Q_{10,10}^{4,2})$.

Conclusion. In this paper we are able to provide unicyclic signed graphs with first eleven minimal energies. After that the problem becomes difficult. For example, It is easy to see that the unicyclic signed graphs $Q_{n,n}^{3,1}$ and $Q_{n,n}^{4,1}$ have 12^{th} and 13^{th} minimal energy respectively for $n = 12$. But for $n = 1000$, $Q_{n,n}^{4,1}$ has 12^{th} minimal energy and $Q_{n,n}^{3,1}$ has 13^{th} minimal energy. It

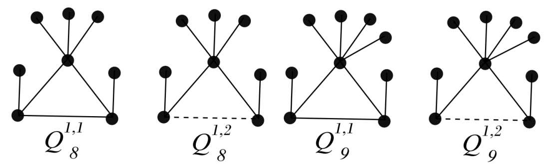

Figure 7: Unicyclic signed graphs $Q_8^{1,1}$, $Q_8^{1,2}$, $Q_9^{1,1}$ and $Q_9^{1,2}$ respectively.

will be interesting to provide further ordering with respect to minimal energy. Energy ordering in other families of signed graphs like bipartite, k-cyclic ($k \geq 2$), complete signed graphs of fixed order etc remains a problem for future study. It will be useful to see the work on weighted graphs [3] and directed graphs [5].

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