mathematics

Article

Fault-Tolerant Metric Dimension of Circulant Graphs

Laxman Saha [1], Rupen Lama [1], Kalishankar Tiwary [2], Kinkar Chandra Das [3,*] and Yilun Shang [4,*]

1 Department of Mathematics, Balurghat College, Balurghat 733101, India; laxman.math@balurghatcollege.ac.in (L.S.); rupen.maths@balurghatcollege.ac.in (R.L.)
2 Department of Mathematics, Raiganj University, Raiganj 733134, India; kstiwary@raiganjuniversity.ac.in
3 Department of Mathematics, Sungkyunkwan University, Suwon 16419, Korea
4 Department of Computer and Information Sciences, Northumbria University, Newcastle NE1 8ST, UK
* Correspondence: kinkardas2003@gmail.com or kinkar@skku.edu (K.C.D.); yilun.shang@northumbria.ac.uk (Y.S.)

Abstract: Let G be a connected graph with vertex set $V(G)$ and $d(u,v)$ be the distance between the vertices u and v. A set of vertices $S = \{s_1, s_2, \dots, s_k\} \subset V(G)$ is called a resolving set for G if, for any two distinct vertices $u, v \in V(G)$, there is a vertex $s_i \in S$ such that $d(u,s_i) \neq d(v,s_i)$. A resolving set S for G is fault-tolerant if $S \setminus \{x\}$ is also a resolving set, for each x in S, and the fault-tolerant metric dimension of G, denoted by $\beta'(G)$, is the minimum cardinality of such a set. The paper of Basak et al. on fault-tolerant metric dimension of circulant graphs $C_n(1,2,3)$ has determined the exact value of $\beta'(C_n(1,2,3))$. In this article, we extend the results of Basak et al. to the graph $C_n(1,2,3,4)$ and obtain the exact value of $\beta'(C_n(1,2,3,4))$ for all $n \geq 22$.

Keywords: circulant graphs; resolving set; fault-tolerant resolving set; fault-tolerant metric dimension

check for updates

Citation: Saha, L.; Lama, R.; Tiwary, K.; Das, K.C.; Shang, Y. Fault-Tolerant Metric Dimension of Circulant Graphs . *Mathematics* **2022**, *10*, 124. https://doi.org/10.3390/math10010124

Academic Editor: Elena Guardo

Received: 29 November 2021
Accepted: 28 December 2021
Published: 1 January 2022

1. Introduction

The distance between two vertices u and v, denoted by $d_G(u,v)$, is the length of the shortest $u-v$ path in a simple, undirected, connected graph G with the vertex set $V(G)$ and the edge set $E(G)$. Whenever there is no possibility of confusion, we will simply write $d(u,v)$ instead of $d_G(u,v)$. A vertex z resolves two vertices x and y if $d(x,z) \neq d(y,z)$. Let $S \subset V(G)$ be a set with m elements. The code of a vertex w with respect to S, denoted by $c(w|S)$, is the m-tuple $c(w|S) = (d(w,s){:}s \in S)$. A set S is a resolving set if distinct vertices have distinct codes, i.e., if $c(x|S) = c(y|S)$ for all distinct $x, y \in V(G)$. Equivalently, S is said to be a resolving set for G if for every pair of distinct vertices x and y, there is a $s \in S$ such that $c(x|S) \neq c(y|S)$. The metric dimension of G is the number $\min_S\{|S|{:}S \text{ is a resolving set of } G\}$ and it is denoted by $\beta(G)$.

Slater [1] and Harry et al. [2] have introduced the metric dimension of graphs. A metric basis is a resolving set with the cardinality $\beta(G)$. Some times metric bases elements may be considered as censors, see [3]. We will not have enough knowledge to deal with the attacker (fire, thief etc) if one of the censors malfunctions. In order to overcome this kind of problems, Hernando et al. have proposed the concept of fault-tolerant metric dimension in [4].

A resolving set S of a graph G is *fault-tolerant* if for each $u \in S$, $S \setminus \{u\}$ is also a resolving set for G. The *fault-tolerant metric dimension* of G, denoted by $\beta'(G)$, is the minimum cardinality of a fault-tolerant resolving set. A *fault-tolerant metric basis* is a fault-tolerant resolving set of order $\beta'(G)$.

Determining a graph's fault-tolerant metric dimension is a challenging combinatorial problem with potential applications in sensor networks. It has only been tested for a few simple graph families thus far. Hernendo et al. characterized the fault tolerant resolving sets in a tree T in their introductory paper [4]. They have also furnished an upper bound for the fault-tolerant metric dimension of an arbitrary graph G as $\beta'(G) \leqslant \beta(G)(1 + 2 \cdot 5^{\beta(G)-1})$. Saha [5] determined the fault-tolerant metric dimension of cube of paths, and Javaid et al. [6] obtained $\beta'(C_n)$, where C_n is a cycle of order n.

Let n and t be positive integers with $2 \leq t \leq \lfloor \frac{n}{2} \rfloor$. An undirected graph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{(v_i, v_j) : |i - j| = s \pmod{n}\}$, $s \in \{1, 2, \ldots, t\}$ is called a *circulant graph* and is denoted by $C_n(1, 2, \ldots, t)$. Note that $C_n(1, 2, \ldots, \lfloor \frac{n}{2} \rfloor)$ is isomorphic to a complete graph on n vertices. Javaid et al. [7] found $\beta'(C_n(1, 2))$, in [8], Imran et al. only bounded the metric dimension of $C_n(1, 2)$ and $C_n(1, 2, 3)$, and then Borchert and Gosselin [9] extended their results and determined the exact metric dimension of these two families of circulants for all n.

In this article, we extend the results of Basak et al. [10] to the graph $C_n(1, 2, 3, 4)$ and obtain the exact value of $\beta'(C_n(1, 2, 3, 4))$ for all $n \geq 22$. It is worth noting that the fault-tolerant problem for circulant graphs has also been studied in the context of network robustness [11], which is different from the current setting.

2. Preliminaries and Notations

The distance between two vertices v_i and v_j in $C_n(1, 2, 3, 4)$ is given by

$$d(v_i, v_j) = \begin{cases} \left\lceil \frac{|i-j|}{4} \right\rceil & \text{if } |i - j| \leqslant \lfloor \frac{n}{2} \rfloor, \\ \left\lceil \frac{n-|i-j|}{4} \right\rceil & \text{if } |i - j| > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

The diameter of $C_n(1, 2, 3, 4)$ is $\left\lceil \frac{n-1}{8} \right\rceil$. If we take n as the form $8k + r$ with $r \in \{0, 1, 2, 3, 4, 5, 6, 7\}$, then diameter is k or $k + 1$ according as $r \in \{0, 1\}$ or $r \in \{2, 3, \ldots, 7\}$. To fix this variability of diameter for different values of r, we take n is of the form $8k + r$ with $r \in \{2, 3, 4, 5, 6, 7, 8, 9\}$. Throughout this paper, we denote that $k + 1$ is the diameter of $C_n(1, 2, 3, 4)$ and $\varnothing$ is the empty set.

The following lemma gives a basic property of a fault-tolerant resolving set for an arbitrary graph.

Lemma 1 ([6])**.** *A set $F \subset V(G)$ is a fault-tolerant resolving set of G if, and only if, every pair of vertices in G is resolved by at least two vertices of F.*

Definition 1. *A vertex u is called an antipodal vertex of v if $d(u, v) = k + 1$, where $k + 1$ is the diameter of $C_n(1, 2, 3, 4)$. For each $j \in \{0, 1, 2, \ldots, n - 1\}$,, we denote the set of all antipodal vertices of $v_j \in V(C_n(1, 2, 3, 4))$ by $A(v_j)$.*

The lemma below gives the set of all antipodal vertices for each vertex $v \in C_n(1, 2, 3, 4)$, which can be verified easily.

Lemma 2. *Let $n = 8k + r$, where $r \in \{2, 3, \ldots, 7, 8, 9\}$. Then, for any vertex $v_j \in C_n(1, 2, 3, 4)$,*

$$A(v_j) = \{v_{4k+1+j}, v_{4k+2+j}, \ldots, v_{4k+r-1+j}\}$$

and hence $|A(v_j)| = r - 1$. Note that, for all $v \in C_n(1, 2, 3, 4)$, $|A(v)| = 1$ or 8, according to $n = 8k + 2$ or $n = 8k + 9$.

Definition 2. *For $m \in \{2, 3, 4, 5\}$ and $j \in \{0, 1, \ldots, n - m - 1\}$, define K_m^j as the complete subgraph of $C_n(1, 2, 3, 4)$ induced by $\{v_j, v_{j+1}, \ldots, v_{j+m-1}\}$. For the clique K_m^j, we call the vertices v_j, v_{j+m-1} end vertices and the others intermediate vertices of K_m^j. We shall denote the set of all intermediate vertices of K_m^j by $I(K_m^j)$.*

Example 1. *The clique K_5^7 in $C_{30}(1, 2, 3, 4)$ is a complete subgraph induced by $\{v_7, v_8, v_9, v_{10}, v_{11}\}$. The vertices v_7 and v_{11} are the end vertices of K_5^7, whereas v_8, v_9, v_{10} are the intermediate vertices for the same.*

Notation 1. *A vertex v_j in $C_n(1, 2, 3, 4)$ is called a right or a left side vertex of v_0 according to $j \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ or $j \in \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 1, \ldots, n-1\}$. We denote $R(v_0)$ as the set of all vertices of $C_n(1, 2, 3, 4)$ which are at right side of v_0, i.e,*

$$R(v_0) = \left\{v_i \in V(C_n(1, 2, 3, 4)) \colon 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Similarly, we define

$$L(v_0) = \left\{v_i \in V(C_n(1, 2, 3, 4)) \colon \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1 \right\}$$

and call it as the set of all left side vertices of v_0.

3. Lower Bound for Fault-tolerant Metric Dimension of $C_n(1, 2, 3, 4)$

In this section, we show that any fault-tolerant resolving set F of $C_n(1, 2, 3, 4)$ contains at least eight elements. Moreover, for $n \equiv 1 \pmod 8$, we show that one more element should be added in F for it to be a fault-tolerant resolving set .

Lemma 3. *For two positive integers a and j, $\left\lceil \frac{|j-a|}{4} \right\rceil \neq \left\lceil \frac{|j-a-1|}{4} \right\rceil$ implies $j \equiv a \pmod 4$ or $j \equiv a+1 \pmod 4$ according as $j \leq a$ or $j \geq a+1$.*

Proof. First we assume that $j \leq a$. Then there exists positive integers q and r such that $a - j = 4q + s$ with $s \in \{0, 1, 2, 3\}$. Then

$$\begin{aligned} \left\lceil \frac{|j-a|}{4} \right\rceil &= \left\lceil \frac{4q+s}{4} \right\rceil = \begin{cases} q & \text{if } s = 0, \\ q+1 & \text{if } s = 1, 2, 3. \end{cases} \\ \left\lceil \frac{|j-a-1|}{4} \right\rceil &= \left\lceil \frac{4q+s+1}{4} \right\rceil = q+1 \ \text{ for all } r = 0, 1, 2, 3. \end{aligned}$$

From the above two results, we conclude that $\left\lceil \frac{|j-a|}{4} \right\rceil \neq \left\lceil \frac{|j-a-1|}{4} \right\rceil$ if $s = 0$, that is, if $j = a - 4q \equiv a \pmod 4$ provided $j \leq a$.

Next we assume that $a + 1 \leq j$ and let $j - a - 1 = 4q + s$ for some positive integer q and $s \in \{0, 1, 2, 3\}$. Then

$$\begin{aligned} \left\lceil \frac{|j-a-1|}{4} \right\rceil &= \left\lceil \frac{4q+s}{4} \right\rceil = \begin{cases} q & \text{if } s = 0, \\ q+1 & \text{if } s = 1, 2, 3. \end{cases} \\ \left\lceil \frac{|j-a|}{4} \right\rceil &= \left\lceil \frac{4q+s+1}{4} \right\rceil = q+1 \ \text{ for all } r = 0, 1, 2, 3. \end{aligned}$$

From the above two results, we conclude that $\left\lceil \frac{|j-a|}{4} \right\rceil \neq \left\lceil \frac{|j-a-1|}{4} \right\rceil$ if $s = 0$, that is, if $j = 4q + a + 1 \equiv a + 1 \pmod 4$ provided $j \geq a + 1$. □

Using Lemma 3, we have the following result.

Lemma 4. *Let $n = 8k + r$ be a positive integer. Let $a, j \in \{0, 1, \ldots, n-1\}$ be two distinct integers. Then $\left\lceil \frac{n-|j-a|}{4} \right\rceil \neq \left\lceil \frac{n-|j-a-1|}{4} \right\rceil$ implies $j \equiv a + n \pmod 4$ or $j \equiv a + 1 - n \pmod 4$ according as $j > a$ or $j < a$.*

Notation 2. *Recall that a vertex u resolve two vertices v and w if $d(u, v) \neq d(u, w)$. We denote the set of all vertices which resolve two consecutive vertices v_a and v_{a+1} by $R_{a,a+1}$.*

The lemma below gives an explicit form of $R_{a,a+1}$ for each $a \in \{0, 1, \ldots, n-1\}$. From here to onward, a non-negative integer $j \in [a]$, we mean $j - a \equiv 0 \pmod 4$.

Lemma 5. *Let $n = 8k + r$ for some positive integer k and $r \in \{2, 3, \ldots, 9\}$. For any two consecutive vertices v_a and v_{a+1} of $C_n(1, 2, 3, 4)$, the following are hold:*

(a) If $a \leq \lfloor \frac{n}{2} \rfloor - 1$, then $R_{a,a+1} = \{v_j \colon j \in [a],\ 0 \leq j \leq a\} \cup \{v_j \colon j \in [a+1],\ a+1 \leq j \leq a+1+4k\} \cup \{v_j \colon j \in [r+a],\ a+r+4k \leq j \leq n-1\}$.

(b) If $a \geq \lfloor \frac{n}{2} \rfloor$, then $R_{a,a+1} = \{v_j \colon j \in [a],\ a-4k \leq j \leq a\} \cup \{v_j \colon j \in [a+1],\ a+1 \leq j \leq n-1\} \cup \{v_j \colon j \in [a+1-r],\ 0 \leq j \leq a+1-r-4k\}$.

Proof. It is clear that $A(v_a) \cap A(v_{a+1}) = \phi$ when $r = 2$ and for $r \neq 2$,

$$A(v_a) \cap A(v_{a+1}) = \{v_{a+4k+2}, \ldots, v_{a+4k+r-1}\}.$$

(a) Let v_j resolve the vertices v_a and v_{a+1}. Then $v_j \notin A(v_a) \cap A(v_{a+1})$. Now the distances of v_a and v_{a+1} from v_j are given by

$$d(v_j, v_a) = \begin{cases} \left\lceil \frac{|j-a|}{4} \right\rceil & \text{if } |j-a| \leq \frac{n}{2}, \\ \left\lceil \frac{n-|j-a|}{4} \right\rceil & \text{if } |j-a| > \frac{n}{2}, \end{cases}$$

$$d(v_j, v_{a+1}) = \begin{cases} \left\lceil \frac{|j-a-1|}{4} \right\rceil & \text{if } |j-a-1| \leq \frac{n}{2}, \\ \left\lceil \frac{n-|j-a-1|}{4} \right\rceil & \text{if } |j-a-1| > \frac{n}{2}. \end{cases}$$

Since v_j resolve v_a and v_{a+1}, $d(v_j, v_a) \neq d(v_j, v_{a+1})$, and when $|j-a| \leq \lfloor \frac{n}{2} \rfloor$, applying Lemma 3, we have $j \equiv a \pmod 4$ or $j \equiv a+1 \pmod 4$ according as $j \leq a$ or $j \geq a+1$. Again if $|j-a| > \lfloor \frac{n}{2} \rfloor$, then $j > a$ as $a \leq \lfloor \frac{n}{2} \rfloor - 1$ and hence applying Lemma 4, we have $j \equiv a + r \pmod 4$. Hence, proof of part (a) is complete. For part (b), proof will be similar. □

Corollary 1. *Let $F = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for some fixed $i \in \{0, 1, \ldots, n-4\}$. Then for each ℓ there exists an element $v_j \in F$ such that v_ℓ and $v_{\ell+1}$ are resolved by v_j, provided both v_ℓ and $v_{\ell+1}$ are not in $A(v_j)$.*

Corollary 2. *For $n \equiv 1 \pmod 8$, $R_{a,a+1} = \{v_j \colon j \in [a],\ 0 \leq j \leq a\} \cup \{v_j \colon j \in [a+1],\ a+1 \leq j \leq n-1\} \setminus (A(v_a) \cap A(v_{a+1}))$.*

Proof. Let $n = 8k + 9$ for some positive integer k. Note that

$$A(v_a) \cap A(v_{a+1}) = \{v_{a+4k+2}, v_{a+4k+3}, \ldots, v_{a+4k+8}\},$$

where the indices of vertices are taken to be modulo n. First we take $a \leq \lfloor \frac{n}{2} \rfloor - 1$. Then from Lemma 5$(a)$, we have $R_{a,a+1} = \{v_j \colon j \in [a],\ 0 \leq j \leq a\} \cup \{v_j \colon j \in [a+1],\ a+1 \leq j \leq a+1+4k\} \cup \{v_j \colon j \in [a+1],\ a+9+4k \leq j \leq n-1\}$. Therefore, the result is true if $a \leq \lfloor \frac{n}{2} \rfloor - 1$. Again if $a \geq \lfloor \frac{n}{2} \rfloor$, then $A(v_a) \cap A(v_{a+1}) = \{v_{a-4k-7}, v_{a-4k-6}, \ldots, v_{a-4k-1}\}$ and hence from Lemma 5(b), we have $R_{a,a+1} = \{v_j \colon j \in [a],\ a-4k-2 \leq j \leq a\} \cup \{v_j \colon j \in [a+1],\ a+1 \leq j \leq n-1\} \cup \{v_j \colon j \in [a],\ 0 \leq j \leq a-8-4k\}$. Therefore the result is true. □

Lemma 6. *Let $n = 8k + r$, where k being a positive integer and $r \in \{2, 3, \ldots, 8, 9\}$. Let K_5 be a clique in $C_n(1, 2, 3, 4)$. Then for every pair of vertices v_a, v_b in $V(K_5)$ with $a < b < a + 4$, we have the following.*

(a) *When $r = 2$, then*

$$R_{a,\,a+1} \cap R_{b,\,b+1} = \begin{cases} \{v_{a+1}, v_{a+4k+2}\} & \text{if } b = a+1, \\ \emptyset & \text{otherwise.} \end{cases}$$

(b) *When $r = 3$, then*

$$R_{a,\,a+1} \cap R_{b,\,b+1} = \begin{cases} \{v_{a+1}\} & \text{if } b = a+1, \\ \{v_{a+4k+3}\} & \text{if } b = a+2, \\ \emptyset & b = a+3. \end{cases}$$

(c) *When $r = 4$, then*

$$R_{a,\,a+1} \cap R_{b,\,b+1} = \begin{cases} \{v_{a+1}\} & \text{if } b = a+1, \\ \emptyset & \text{if } b = a+2, \\ \{v_{a+4k+4}\}, & \text{if } b = a+3. \end{cases}$$

(d) *When $r \in \{5, 6, 7, 8, 9\}$, then*

$$R_{a,\,a+1} \cap R_{b,\,b+1} = \begin{cases} \{v_{a+1}\} & \text{if } b = a+1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. For symmetry of $C_n(1, 2, 3, 4)$, we prove the result for a K_5 with $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$. Then from Lemma 5, we obtain

$$\begin{aligned} R_{0,1} &= \{v_0\} \cup \{v_j : j \in [1],\ 1 \le j \le 4k+1\} \cup \{v_j : j \in [r],\ 4k+r \le j \le n-1\}, \\ R_{1,2} &= \{v_1\} \cup \{v_j : j \in [2],\ 2 \le j \le 4k+2\} \cup \{v_j : j \in [r+1],\ 4k+r+1 \le j \le n-1\}, \\ R_{2,3} &= \{v_2\} \cup \{v_j : j \in [3],\ 3 \le j \le 4k+3\} \cup \{v_j : j \in [r+2],\ 4k+r+2 \le j \le n-1\}, \\ R_{3,4} &= \{v_3\} \cup \{v_j : j \in [0],\ 4 \le j \le 4k+4\} \cup \{v_j : j \in [r+3],\ 4k+r+3 \le j \le n-1\}. \end{aligned}$$

By putting the different values of r and on simple calculations, we get the required result. □

Example 2. *Let $n = 8 \cdot 6 + 5$ and let us take $V(K_5) = \{v_7, v_8, v_9, v_{10}, v_{11}\}$ in the circulant graph $C_{53}(1, 2, 3, 4)$. Then we have the following*

$$\begin{aligned} R_{7,8} &= \{v_3, v_7\} \cup \{v_8, v_{12}, \ldots, v_{32}\} \cup \{v_{36}, v_{40}, \ldots, v_{52}\}, \\ R_{8,9} &= \{v_0, v_4, v_8\} \cup \{v_9, v_{13}, \ldots, v_{33}\} \cup \{v_{37}, v_{41}, \ldots, v_{49}\}, \\ R_{9,10} &= \{v_1, v_5, v_9\} \cup \{v_{10}, v_{14}, \ldots, v_{34}\} \cup \{v_{38}, v_{42}, \ldots, v_{50}\}, \\ R_{10,11} &= \{v_2, v_6, v_{10}\} \cup \{v_{11}, v_{15}, \ldots, v_{35}\} \cup \{v_{39}, v_{43}, \ldots, v_{51}\}. \end{aligned}$$

Here we see that $R_{a,a+1} \cap R_{b,b+1} = \emptyset$ for $|b - a| \ge 2$, whereas $R_{a,a+1} \cap R_{a+1,a+2} = \{v_{a+1}\}$ for $a \in \{7, 8, 9\}$.

Definition 3. *Let U and S be two subsets of vertices of $C_n(1, 2, 3, 4)$. We call the set U an S-block, if all vertices of U are at equal distance from every vertex of S, or equivalently, $\mathrm{C}(u|S) = \mathrm{C}(v|S)$ for u, $v \in U$.*

In the lemma below, we give the least number of elements that should be included in a fault-tolerant resolving set F to resolve a clique K_m for $m \in \{2, 3, 4, 5\}$.

Lemma 7. *Let F be a fault-tolerant resolving set of $C_{8k+r}(1, 2, 3, 4)$, where $5 \le r \le 9$. Let S be a subset of F and $I(K_t)$ denotes the set of intermediate vertices of K_t. If there exists a clique K_m $(2 \le m \le 5)$ in $C_{8k+r}(1, 2, 3, 4)$ such that $F \cap I(K_m) = \emptyset$ and $V(K_m)$ is an S-block, then $|F| \ge 2m - 2 + |S|$.*

Proof. For symmetricity of $C_n(1, 2, 3, 4)$, it is sufficient to show that the result is true for a clique K_m with $V(K_m) \subset R(v_0)$. Let $V(K_m) = \{v_i, \ldots, v_{i+m-1}\}$. Since $V(K_m)$ is an S $(\subset F)$-block, $\mathrm{C}(u|S) = \mathrm{C}(v|S)$ for every pair of vertices $u, v \in V(K_m)$. Again, as F is a fault-tolerant resolving set of $C_n(1, 2, 3, 4)$, applying Lemma 1, we have $|R_{a,\,a+1} \cap (F \setminus S)| \ge 2$ for each $a \in \{i, \ldots, i+m-2\}$. Again, since $F \cap I(K_m) = \emptyset$, so from Lemma 6, we have $(F \cap R_{a,\,a+1}) \cap (F \cap R_{b,\,b+1}) = \emptyset$ for distinct $a, b \in \{i, \ldots, i+m-2\}$. Therefore, $|F \setminus S| \ge 2(m-1)$, that is, $|F| \ge 2(m-1) + |S|$. □

Lemma 8. *Let F be a fault-tolerant resolving set of $C_{8k+r}(1, 2, 3, 4)$, where $2 \le r \le 4$. If there exists a clique K_5^i in $C_{8k+r}(1, 2, 3, 4)$ such that $F \cap I(K_5^i) = \emptyset$ and $F \cap I(K_{7-r}^{i+4k+r-1}) = \emptyset$, then $|F| \ge 8$.*

Proof. Since F is a fault-tolerant resolving, applying Lemma 1, $|F \cap R_{a,a+1}| \ge 2$ for every a with $0 \le a \le n-1$. If F is a fault-tolerant resolving set of $C_{8k+r}(1, 2, 3, 4)$ such that $F \cap I(K_5^i) = \emptyset$ and $F \cap I(K_{7-r}^{i+4k+r-1}) = \emptyset$, then applying Lemma 6, we have $(F \cap R_{a,a+1}) \cap (F \cap R_{b,b+1}) = \emptyset$ for distinct a and b in $\{i, i+1, i+2, i+3\}$. Thus $|F| \ge \sum_{a=i}^{i+3} |F \cap R_{a,a+1}| \ge 8$. □

Lemma 9. *Let $n \equiv 5, 6, 7, 8, 9 \pmod 8$ and F be a fault-tolerant resolving set of $C_n(1, 2, 3, 4)$. Then for every clique K_5 in $C_n(1, 2, 3, 4)$, $|F| \ge 8 - |F \cap I(K_5)|$, where $I(K_5)$ denotes the set of intermediate vertices of K_5.*

Proof. From the symmetries of $C_n(1, 2, 3, 4)$, we assume $V(K_5) \subset R(v_0)$ and let $V(K_5) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$. For a fault-tolerant resolving set F with $F \cap V(K_5) = \emptyset$, $|F| \ge 8$ due to Lemma 8 with $S = \emptyset$. Since F is a fault-tolerant resolving set of $C_n(1, 2, 3, 4)$, $|F \cap R_{a,a+1}| \ge 2$ for every a with $0 \le a \le n-1$ and in particular, $a \in \{i, i+1, i+2, i+3\}$. Let $|F \cap I(K_5)| = \ell$ $(1 \le \ell \le 3)$. In view of the values of ℓ, we consider the following three cases.

Case 1: $|F \cap I(K_5)| = 1$. Suppose $F \cap I(K_5) = \{v_p\}$ for some $p \in \{i+1, i+2, i+3\}$. Let $F_1 = F \setminus \{v_p\}$. First we assume that $p = i+1$. Since $|F \cap R_{a,a+1}| \ge 2$ for all $a \in \{i, i+1, i+2, i+3\}$ and $v_{i+1} \notin R_{a,a+1}$ for $a \in \{i+2, i+3\}$, we have $|R_{i,i+1} \cap F_1| \ge 1$, $|R_{i+1,i+2} \cap F_1| \ge 1$ and $|R_{a,a+1} \cap F_1| \ge 2$ for $a \in \{i+2, i+3\}$. As $F_1 \cap \{v_{i+1}, v_{i+2}, v_{i+3}\} = \emptyset$, so applying Lemma 6, we have $(F_1 \cap R_{a,a+1}) \cap (F_1 \cap R_{b,b+1}) = \emptyset$ for distinct $a, b \in \{i, i+1, i+2, i+3\}$. Therefore, $|F_1| \ge \sum_{a=i}^{i+3} |R_{a,a+1} \cap F_1| \ge 6$ and hence $|F| \ge 7$ as $F_1 = F \setminus \{v_p\}$. Similarly, we obtain $|F| \ge 7$ when $p = i+3$. Again, if we take $p = i+2$, then by a similar argument, one can easily prove that $|F_1| \ge 6$ as in this case $|R_{i,i+1} \cap F_1| \ge 2$, $|R_{i+1,i+2} \cap F_1| \ge 1$, $|R_{i+2,i+3} \cap F_1| \ge 1$ and $|R_{i+3,i+4} \cap F_1| \ge 2$. Therefore, the result holds when $|F \cap I(K_5)| = 1$.

Case 2: $|F \cap I(K_5)| = 2$. First we assume that $F \cap I(K_5) = \{v_{i+1}, v_{i+2}\}$. Let $F_1 = F \setminus \{v_{i+1}, v_{i+2}\}$. Then by a similar argument as in Case 1, we have $|R_{i,i+1} \cap F_1| \ge 1$, $|R_{i+2,i+3} \cap F_1| \ge 1$ and $|R_{i+3,i+4} \cap F_1| \ge 2$ as in this case none of v_{i+1} and v_{i+2} are in $R_{i+3,i+4}$. Therefore $|F_1| \ge 4$ and hence $|F| \ge 6$. By a similar argument, we can prove the result when $F \cap I(K_5) = \{v_{i+2}, v_{i+3}\}$. Next, we assume that $F \cap I(K_5) = \{v_{i+1}, v_{i+3}\}$. Let $F_2 = F \setminus \{v_{i+1}, v_{i+3}\}$. Then, by a similar argument as in Case 1, we have $|R_{t,t+1} \cap F_2| \ge 1$

for all $t \in \{i, i+1, i+2, i+3\}$. Thus, we have $|F_2| \geq 4$ and consequently, $|F| \geq 6$. So, in this case, the result is true.

Case 3: $|F \cap I(K_5)| = 3$. Here $F \cap I(K_5) = \{v_{i+1}, v_{i+2}, v_{i+3}\}$. Let $F_1 = F \setminus \{v_{i+1}, v_{i+2}, v_{i+3}\}$. Then, $|R_{i,i+1} \cap F_1| \geq 1$ and $|R_{i+3,i+4} \cap F_1| \geq 1$ and hence $|F_1| \geq 2$, and consequently, $|F| \geq 5$.

On account of the above three cases, we have $|F| \geq 8 - |F \cap I(K_5)|$. □

Using a similar argument of Lemma 9, we have the following results.

Lemma 10. *Let $n \equiv r \pmod 8$ and F be a fault-tolerant resolving set of $C_n(1, 2, 3, 4)$, where $2 \leq r \leq 4$. Then, for every clique K_5^i in $C_n(1, 2, 3, 4)$, $|F| \geq 8 - \left|F \cap \left(I(K_5^i) \cup I(K_{7-r}^{i+4k+r-1})\right)\right|$, where $I(K_t)$ denotes the set of intermediate vertices of K_t.*

Lemma 11. *Let $n \equiv 5, 6, 7, 8, 9 \pmod 8$ and F be a fault-tolerant resolving set of $C_n(1, 2, 3, 4)$. Let $S \subset F$. Then, for every clique K_5 in $C_n(1, 2, 3, 4)$ with $V(K_5)$ as an S-block, $|F| \geq 8 - |F \cap I(K_5)| + |S|$, where $I(K_5)$ denotes the set of intermediate vertices of K_5.*

Theorem 1. *For $n \geq 22$ and $n \notin \{26, 27, 34, 35, 42\}$,*

$$\beta'(C_n(1, 2, 3, 4)) \geq \begin{cases} 8 & \text{if } n \not\equiv 1 \pmod 8, \\ 9 & \text{if } n \equiv 1 \pmod 8. \end{cases}$$

Proof. Let F be an arbitrary fault-tolerant resolving set of $C_n(1, 2, 3, 4)$. Let $n = 8k + r$ for some positive integers k and r, where $2 \leq r \leq 9$. We consider the following three cases.

Case 1: $r \in \{2, 3\}$. Since $n \geq 22$ and $n \notin \{26, 27, 34, 35, 42\}$, so in this case, we have $n \geq 43$. If there exists a clique K_5^i such that $F \cap I(K_5^i) = \emptyset$ and $F \cap I(K_{7-r}^{i+4k+r-1}) = \emptyset$, then applying Lemma 8, we get $|F| \geq 8$. So, we assume $F \cap I(K_5^i) \neq \emptyset$ or $F \cap I(K_{7-r}^{i+4k+r-1}) \neq \emptyset$ for every i satisfying $0 \leq i \leq n-1$, that is, $\left|F \cap \left(I(K_5^i) \cup I\left(K_{7-r}^{i+4k+r-1}\right)\right)\right| \geq 1$ for all $i \in \{0, 1, \ldots, n-1\}$. Without loss of generality, we can assume that $v_0 \in F$. Recall that $I(K_5^i) = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $I(K_{7-r}^{i+4k+r-1}) = \{v_{i+4k+r}, \ldots, v_{i+4k+4}\}$. Now, from $\left|F \cap \left(I(K_5^i) \cup I\left(K_{7-r}^{i+4k+r-1}\right)\right)\right| \geq 1$, we obtain

$$|F| \geq 1 + \sum_{\ell=0,\ \ell \equiv 0 \pmod 3}^{\lfloor \frac{n}{2} \rfloor - 4} |F \cap \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4k+r}, \ldots, v_{i+4k+4}\}|$$

(extra one is here as $v_0 \in F$). Since $n \geq 43$, the sets

$$S_p = \{v_{p+1}, v_{p+2}, v_{p+3}, v_{p+4k+r}, \ldots, v_{p+4k+4}\}$$
$$\text{and } S_q = \{v_{q+1}, v_{q+2}, v_{q+3}, v_{q+4k+r}, \ldots, v_{q+4k+4}\}$$

are disjoint for $|p - q| \geq 3$ and $0 \leq p, q \leq \lfloor \frac{n}{2} \rfloor - 3$. Thus

$$\sum_{\ell=0,\ \ell \equiv 0 \pmod 3}^{\lfloor \frac{n}{2} \rfloor - 4} |F \cap \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4k+r}, \ldots, v_{i+4k+4}\}| \geq 7$$

and hence we obtain the result.

Case 2: $r = 4$. In this case, n is of the form $8k + 4$ for some positive integer k. Since $n \geq 22$ and $n \notin \{26, 27, 34, 35, 42\}$, so in this case $n \geq 28$. We prove the result for $n \geq 36$. The proof for $n = 28$ will be similar. Note that $I(K_5^i) = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $I(K_3^{i+4k+3}) = \{v_{i+4k+4}\}$ for all i. Thus, if there exists an i such that $F \cap \{v_{i+1}, v_{i+2}, v_{i+3}\} = \emptyset$ and

$F \cap \{v_{i+4k+4}\} = \emptyset$, then applying Lemma 8, we get $|F| \geq 8$. So, we assume that at least one of $F \cap \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $F \cap \{v_{i+4k+4}\}$ is non-empty, that is, for all $i \in \{0, 1, \ldots, n-1\}$,

$$|F \cap \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4k+4}\}| \geq 1. \quad (1)$$

Let $S_i = \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4k+4}\}$ for $i \in \{0, 1, \ldots, n-1\}$. Then $S_i \cap S_{i+3} = \emptyset$. Our claim is

$$|F \cap (S_i \cup S_{i+3} \cup S_{i+4k+4})| \geq 3 \quad (2)$$

for every i. Since (1) holds for every i, we have the following

$$|F \cap S_\ell| \quad \geq \quad 1 \text{ for } \ell \in \{i, i+3, i+4k+3, i+4k+4\}.$$

Note that all $S_i \cap S_{i+3}$, $S_{i+3} \cap S_{i+4k+3}$ and $S_{i+3} \cap S_{i+4k+4}$ are empty set. Also we have $S_i \cap S_{i+4k+3} = \{v_{i+2}, v_{i+4k+4}\}$, $S_i \cap S_{i+4k+4} = \{v_{i+3}\}$. Thus if $v_{i+3} \notin F$, then the sets $F \cap S_i$, $F \cap S_{i+3}$ and $F \cap S_{i+4k+4}$ are mutually disjoint and hence (2) holds. Again if $v_{i+3} \in F$, then (2) is also true because $|F \cap S_{i+3}| \geq 1$ and $|F \cap S_{i+4k+4}| \geq 1$ with $v_{i+3} \notin S_{i+3} \cup S_{i+4k+4}$. Thus our claim (2) is true for every i. Without loss of generality, we can assume that $v_0 \in F$. Since $n \geq 36$ and hence $k \geq 4$, so by virtue of inequality (2) with $i = 0, 6$ and (1) with $i = 12$, we have the following

$$\begin{aligned} |F \cap \{v_1, v_2, \ldots, v_6, v_{4k+4}, v_{4k+5}, \ldots, v_{4k+8}| &\geq 3, \\ |F \cap \{v_7, v_8, \ldots, v_{12}, v_{4k+10}, v_{4k+11}, \ldots, v_{4k+14}| &\geq 3, \\ |F \cap \{v_{13}, v_{14}, v_{15}, v_{4k+16}| &\geq 1. \end{aligned}$$

Since $v_0 \in F$, above inequalities imply that $|F| \geq 8$.

Case 3: $r \in \{5, 6, 7, 8, 9\}$. If there exists a clique K_5^i such that $F \cap I(K_5^i) = \emptyset$, then applying Lemma 9, we get $|F| \geq 8$. So we can assume that $F \cap I(K_5^i) \neq \emptyset$ for all i, $0 \leq i \leq n-1$. Without loss of generality, we can assume that $v_0 \in F$. Note that $I(K_5^p) \cap I(K_5^q) = \emptyset$, provided $|p-q| \geq 3$. Thus $|F| \geq 1 + \sum_{\ell=0,\ \ell \equiv 0 \pmod 3}^{n-4} \left|F \cap I(K_5^\ell)\right|$ (extra one is added as $v_0 \in F$). Since $n \geq 22$ and $\left|F \cap I(K_5^\ell)\right| \geq 1$, we have $\sum_{\ell=0,\ \ell \equiv 0 \pmod 3}^{n-4} \left|F \cap I(K_5^\ell)\right| \geq 7$. Therefore $|F| \geq 8$. Now we prove the theorem for $n = 8k+9$. Assume to the contrary that there is a fault-tolerant resolving set F with $|F| = 8$. Without loss of generality, we can assume that $v_0 \in F$. Note that $A(v_0) = \{v_\ell : \ell \in \{4k+1, 4k+2, \ldots, 4k+8\}\}$. Let $S_0 = \{v_{4k+2}, v_{4k+3}, v_{4k+4}\} \subset A(v_0)$ and $S_0^{'} = \{v_{4k+5}, v_{4k+6}\, v_{4k+7}\} \subset A(v_0)$. Since $|F| = 8$, so applying Lemma 11 to the clique K_5^{4k+1} and K_5^{4k+4} with $S = \{v_0\}$, we get $|F \cap S_0| \geq 1$ and $|F \cap S_0'| \geq 1$, respectively.

It is clear that $A(v_{4k+2}) \cap A(v_{4k+3}) \cap A(v_{4k+4}) = \{v_{8k+5}, v_{8k+6}, v_{8k+7}, v_{8k+8}, v_0, v_1\} = U_1$ and $A(v_{4k+5}) \cap A(v_{4k+6}) \cap A(v_{4k+7}) = \{v_0, v_1, v_2, v_3, v_4, v_{8k+8}\} = U_2$. Note that for every $u \in S_0$, $d(u, x) = k+1$ for all $x \in U_1$ because the elements of U_1 are the common antipodal vertices of three vertices v_{4k+2}, v_{4k+3} and v_{4k+3}. Similarly, for each $w \in S_0^{'}$, $d(w, y) = k+1$ for all $x \in U_2$. Now our aim is to show $|F \cap S_1| \geq 1$ and $|F \cap S_1^{'}| \geq 1$, where S_1 and S_2 are defined as $S_1 = \{v_{8k+6}, v_{8k+7}, v_{8k+8}\} \subset U_1$ and $S_1^{'} = \{v_1, v_2, v_3\} \subset U_2$. As $|F| = 8$, applying Lemma 11 to the clique K_5^{8k+5} with $S = F \cap S_0$, we have $|F \cap S_1| \geq 1$ (as

$|F \cap S_0| \geq 1$). Again applying the same lemma to the clique K_5^0 with $S = F \cap S_0^{'}$, we get $|F \cap S_1^{'}| \geq 1$.

Claim 1. $|F \cap S_\ell| \geq 2$ and $|F \cap S_\ell^{'}| \geq 2$ for $\ell \in \{0, 1\}$.

Proof of Claim 1. From the above, we have $|F \cap S_\ell| \geq 1$ and $|F \cap S_\ell^{'}| \geq 1$ for each $\ell \in \{0, 1\}$. First we show that the claim is true for $\ell = 0$. Here $A(v_{8k+6}) \cap A(v_{8k+7}) \cap A(v_{8k+8}) = \{v_{4k}, v_{4k+1}, v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}$ and $A(v_1) \cap A(v_2) \cap A(v_3) = \{v_{4k+4}, v_{4k+5}, v_{4k+6}, v_{4k+7}, v_{4k+8}, v_{4k+9}\}$. Assume to the contrary that $|F \cap S_0| = 1$. Since $|F \cap S_1| \geq 1$ and $v_0 \notin F \cap S_1$, we have $|(F \cap S_1) \cup \{v_0\}| \geq 2$. By applying Lemma 11 to K_5^{4k+1} with $S = (F \cap S_1) \cup \{v_0\}$, we obtain $|F| \geq 8 + |(F \cap S_1) \cup \{v_0\}| - 1 \geq 9$, a contradiction. Hence $|F \cap S_0| \geq 2$. Similarly, if $|F \cap S_0^{'}| = 1$, then applying the same lemma to K_5^{4k+4} with $S = (F \cap S_1^{'}) \cup \{v_0\}$, we have $|F| \geq 9$, a contradiction. Therefore, $|F \cap S_0^{'}| \geq 2$.

Now we prove the claim for $\ell = 1$. If $|F \cap S_1| = 1$, then applying Lemma 11 to K_5^{8k+5} with $S = F \cap S_0$, we obtain $|F| \geq 8 + |F \cap S_0| - 1 \geq 9$, a contradiction (as $|F \cap S_0| \geq 2$). Hence $|F \cap S_1| \geq 2$. Again if $|F \cap S_1^{'}| = 1$, then we apply Lemma 11 to K_5^0 with $S = F \cap S_0^{'}$ and we get $|F| \geq 9$, a contradiction. Hence $|F \cap S_1^{'}| \geq 2$. This finishes the proof of the Claim 1.

Since $v_0 \in F \setminus \left(S_0 \cup S_1 \cup S_0' \cup S_1'\right)$ and the sets S_0, S_1, S_0', S_1' are mutually disjoint, we obtain

$$8 = |F| \geq 1 + |F \cap S_0| + |F \cap S_0^{'}| + |F \cap S_1| + |F \cap S_1^{'}|,$$
$$\text{that is,} \qquad |F \cap S_0| + |F \cap S_0^{'}| + |F \cap S_1| + |F \cap S_1^{'}| \leq 7.$$

By Claim 1, we obtain

$$|F \cap S_0| + |F \cap S_0^{'}| + |F \cap S_1| + |F \cap S_1^{'}| \geq 8, \text{ a contradiction.}$$

Hence $|F| \geq 9$. This completes the proof of the theorem. □

4. Upper Bound for $\beta^{'}(C_n(1, 2, 3, 4))$

In this section, we determine optimal fault-tolerant resolving set for $C_n(1, 2, 3, 4)$.

Lemma 12. *Let ℓ and m be two integers in $\{4, 5, \ldots, \lfloor \frac{n}{2} \rfloor\}$. If $|\ell - m| \geq 2$, then v_ℓ and v_m are resolved by at least two elements of $\{v_0, v_1, v_2, v_3\}$. Moreover, if $|\ell - m| = 1$, then v_ℓ and v_m are resolved by at least one element of $\{v_0, v_1, v_2, v_3\}$.*

Proof. Let $F_R = \{v_0, v_1, v_2, v_3\}$. Suppose that $|\ell - m| \geq 2$. Without loss of generality, we can assume that $m \geq \ell + 2$. Let $\ell \equiv a \pmod 4$, where $a \in \{0, 1, 2, 3\}$. First we suppose that $a \in \{0, 1, 2\}$. Then $v_a, v_{a+1} \in F_R$. Now $d(v_a, v_\ell) = \frac{\ell - a}{4} = d(v_{a+1}, v_\ell)$, $d(v_a, v_m) = \lceil \frac{m-a}{4} \rceil \geq \lceil \frac{\ell+2-a}{4} \rceil \geq \frac{\ell-a}{4} + 1$ and $d(v_{a+1}, v_m) = \lceil \frac{m-a-1}{4} \rceil \geq \lceil \frac{\ell+1-a}{4} \rceil \geq \frac{\ell-a}{4} + 1$. Therefore, $d(v_x, v_\ell) \neq d(v_x, v_m)$ for $x \in \{a, a+1\}$. Next we suppose that $a = 3$, that is, $\ell \equiv 3 \pmod 4$. We now calculate the distances of v_ℓ and v_m from v_0 and v_3:

$$d(v_0, v_\ell) = \left\lceil \frac{\ell}{4} \right\rceil = \frac{\ell+1}{4}, \ d(v_0, v_m) = \left\lceil \frac{m}{4} \right\rceil \geq \left\lceil \frac{\ell+2}{4} \right\rceil = \frac{\ell+5}{4},$$
$$d(v_3, v_\ell) = \frac{\ell-3}{4}, \ d(v_3, v_m) = \left\lceil \frac{m-3}{4} \right\rceil \geq \left\lceil \frac{\ell-1}{4} \right\rceil = \frac{\ell+1}{4}.$$

Therefore, v_ℓ and v_m are resolved by both vertices v_0 and v_3. Hence v_l and v_m are resolved by at least two elements of $\{v_0, v_1, v_2, v_3\}$ provided $|\ell - m| \geq 2$.

Now we suppose that $|\ell - m| = 1$. Without loss of generality, we can assume that $m = \ell + 1$. Let $\ell = a \pmod 4$. Then $d(v_a, v_\ell) = \frac{\ell-a}{4}$ and $d(v_a, v_m) = \lceil \frac{m-a}{4} \rceil =$

$\lceil \frac{\ell+1-a}{4} \rceil = \frac{\ell+4-a}{4}$. Hence v_ℓ and v_m are resolved by $v_a \in F_R$ when $\ell = a \pmod 4$, where $a \in \{0, 1, 2, 3\}$. □

Lemma 13. *Let $F_R = \{v_0, v_1, v_2, v_3\}$ be an ordered set and ℓ be an integer with $4 \le \ell \le \lfloor \frac{n}{2} \rfloor$. Then $\mathrm{C}(v_\ell|F_R)$ and $\mathrm{C}(v_{n+3-\ell}|F_R)$ are reverse to each other.*

Proof. The distances of v_ℓ and $v_{n+3-\ell}$ from v_a, where $a \in \{0,1,2,3\}$, are $d(v_a, v_\ell) = \left\lceil \frac{\ell-a}{4} \right\rceil$ and $d(v_a, v_{n+3-\ell}) = \left\lceil \frac{\ell+a-3}{4} \right\rceil$, respectively. Now the j-th coordinate in $\mathrm{C}(v_\ell|F_R)$ and $\mathrm{C}(v_{n+3-\ell}|F_R)$ are $\left\lceil \frac{\ell+1-j}{4} \right\rceil$ and $\left\lceil \frac{\ell+j-4}{4} \right\rceil$, respectively, where $j \in \{1,2,3,4\}$. Now $\mathrm{C}(v_\ell|F_R)$ and $\mathrm{C}(v_{n+3-\ell}|F_R)$ are in reverse order only if i-th element in $\mathrm{C}(v_\ell|F_R)$ is equal to $(5-i)$-th element in $\mathrm{C}(v_{n+3-\ell}|F_R)$ for each $i \in \{1,2,3,4\}$. The $(5-i)$-th element in $\mathrm{C}(v_{n+3-\ell}|F_R)$ is $\left\lceil \frac{\ell+1-i}{4} \right\rceil$, which is equal to the i-th element in $\mathrm{C}(v_\ell|F_R)$ for each $i \in \{1,2,3,4\}$. Hence the result is proved. □

Corollary 3. $\mathrm{C}(v_\ell|F_R) = \mathrm{C}(v_{n+3-\ell}|F_R)$ *only if* $\ell \equiv 0 \pmod 4$.

From Lemmas 12 and 13, we have the following result.

Lemma 14. *Let $\ell, m \in \left\{ \lceil \frac{n}{2} \rceil + 2, \lceil \frac{n}{2} \rceil + 3, \ldots, n-1 \right\}$ be two integers. If $|\ell - m| \ge 2$, then v_ℓ and v_m are resolved by at least two elements of $\{v_0, v_1, v_2, v_3\}$. Moreover, if $|\ell - m| = 1$, then v_ℓ and v_m are resolved by at least one element of $\{v_0, v_1, v_2, v_3\}$.*

Lemma 15. *Let $n \equiv 4, 5, 6, 7, 8 \pmod 8$ and $F_L = \left\{ v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}, v_{\lfloor \frac{n}{2} \rfloor+2}, v_{\lfloor \frac{n}{2} \rfloor+3} \right\}$. If $\ell \in \{0, 1, 2, 3\}$ and $m \in \{4, 5, \ldots \lfloor \frac{n}{2} \rfloor - 1\} \cup \left\{ \lfloor \frac{n}{2} \rfloor + 4, \lfloor \frac{n}{2} \rfloor + 5, \ldots, n-1 \right\}$, then v_ℓ and v_m are resolved by at least one element of F_L. Moreover, the result is also true for $n \equiv 9 \pmod 8$ if we add an extra vertex $\left\{ v_{\lfloor \frac{n}{2} \rfloor + 4} \right\}$ to F_L.*

Proof. Let us assume $n = 8k + r$, where $r \in \{4, 5, 6, 7, 8, 9\}$. Let $S_1 = \{4, 5, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ and $S_2 = \left\{ \lfloor \frac{n}{2} \rfloor + 4, \lfloor \frac{n}{2} \rfloor + 5, \ldots, n-1 \right\}$. To prove the result, we show that there exist $u_1, u_2 \in F_L$ such that $A(u_i) \cap S_i = \emptyset$ and $\{v_0, v_1, v_2, v_3\} \subset A(u_i)$ for $i \in \{1, 2\}$, where $A(u_i)$ denotes the set of all antipodal vertices of u_i. Recall that $|A(u_i)| = r - 1$. First, we take $r \in \{5, 6, 7, 8, 9\}$ so that $|A(u_i)|$ is at least four. Now $A(v_{4k+r-1}) = \{v_0, v_1, \ldots, v_{r-2}\}$ and $A(v_{4k+4}) = \{v_{8k+5}, v_{8k+6}, \ldots, v_{8k+r+3}\}$, where the indices of vertices in $A(v_{4k+4})$ are to be taken modulo n. Here the set $\{v_{4k+r-1}, v_{4k+4}\}$ is contained in F_L or $F_L \cup \left\{ v_{\lfloor \frac{n}{2} \rfloor + 4} \right\}$ according as $r \in \{5, 6, 7, 8\}$ or $r = 9$; and the set $\{v_0, v_1, v_2, v_3\}$ is contained in both $A(v_{4k+r-1})$ and $A(v_{4k+4})$. Moreover, we have $S_1 \cap A(v_{4k+r-1}) = \emptyset$ and $S_2 \cap A(v_{4k+4}) = \emptyset$. Therefore, $d(v_\ell, v_{4k+r-1}) = k+1 > d(v_m, v_{4k+r-1})$ and $d(v_\ell, v_{4k+4}) = k+1 > d(v_{m'}, v_{4k+r-1})$ for each $\ell \in \{0, 1, 2, 3\}$, where $m \in S_1$ and $m' \in S_2$.

Now the remaining case is $r = 4$. As $n = 8k + 4$, the set $\left\{ v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}, v_{\lfloor \frac{n}{2} \rfloor+2}, v_{\lfloor \frac{n}{2} \rfloor+3} \right\}$ transfer to $\{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}$. Since $A(v_{4k+3}) = \{v_0, v_1, v_2\}$ and $A(4k+4) = \{v_1, v_2, v_3\}$, $d(v_\ell, v_x) = k+1 > d(v_m, v_x)$ for all $m \in S_1$, where $x = 4k+3$ or $x = 4k+4$ according as $\ell \in \{0, 1, 2\}$ or $\ell \in \{1, 2, 3\}$. □

Lemma 16. *Let $n \equiv 3 \pmod 8$ and $F_L = \left\{ v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2} \right\}$. If $\ell \in \{0, 1, 2, 3\}$ and $m \in \left\{ 4, 5, \ldots, \frac{n-1}{2} - 2 \right\} \cup \left\{ \frac{n-1}{2} + 3, \frac{n-1}{2} + 4, \ldots, n-1 \right\}$, then v_ℓ and v_m are resolved by at least one element of $F_L \cup \{v_2\}$.*

Proof. Let $n = 8k + 3$, where k being a positive integer. Then $F_L = \{v_{4k}, v_{4k+1}, v_{4k+2}, v_{4k+3}\}$. Note that every vertex u has two antipodal vertices. It is easy to see that $A(v_{4k+2}) = \{v_0, v_1\}$ and $A(v_{4k+3}) = \{v_1, v_2\}$. Thus the result is true if $\ell \in \{0, 1, 2\}$ and $m \in$

$\left\{4, 5, \ldots, \frac{n-1}{2} - 2\right\} \cup \left\{\frac{n-1}{2} + 3, \frac{n-1}{2} + 4, \ldots, n-1\right\}$. Now if $\ell = 3$ and $m \in \left\{4, 5, \ldots, \frac{n-1}{2} - 2\right\}$, then $d(v_{4k}, v_\ell) = k$ and $d(v_{4k}, v_m) = \left\lceil \frac{4k-m}{4} \right\rceil \leq k-1$. So v_ℓ and v_m are resolved by $v_{4k} \in F_L$ when $\ell = 3$ and $m \in \left\{4, 5, \ldots, \frac{n-1}{2} - 2\right\}$. Similarly, if $\ell = 3$ and $m \in \left\{\frac{n-1}{2} + 3, \frac{n-1}{2} + 4, \ldots, n-4\right\}$, then we can prove that v_ℓ and v_m are resolved by $v_{4k+3} \in F_L$. Now we search for an element $u \in F_L \cup \{v_2\}$ that resolve v_3 and v_m when $m \in \{n-3, n-2, n-1\} = \{8k, 8k+1, 8k+2\}$. Note that $d(v_{4k}, v_\ell) = k$ and $d(v_{4k}, v_m) = k+1$ for $m \in \{v_{8k+1}, v_{8k+2}\}$. Moreover, v_3 & v_2 are adjacent, and v_{n-3} & v_2 are non-adjacent. Therefore v_3 and v_m are resolved by an element of $u \in F_L \cup \{v_2\}$ when $m \in \{n-3, n-2, n-1\}$. On accounts of all cases considered here the lemma is proved. □

Lemma 17. *Let $n \equiv 2 \pmod 8$ and $F_L = \left\{v_{\frac{n}{2}-3}, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}\right\}$. If $\ell \in \{0, 1, 2, 3\}$ and $m \in \left\{4, 5, \ldots, \frac{n}{2} - 4\right\} \cup \left\{\frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1\right\}$, then v_ℓ and v_m are resolved by at least one element of F_L.*

Proof. Let $n = 8k + 2$, where k being a positive integer. Then $F_L = \{v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$ and $m \in \{4, 5, \ldots, 4k-3\} \cup \{4k+1, 4k+2, \ldots, 8k+1\}$. If $\ell \in \{0, 1, 2, 3\}$ and $m \in \{4, 5, \ldots, 4k-3\} \cup \{4k+1, 4k+2, \ldots, 8k-4\}$, then v_ℓ and v_m are resolved by v_{4k} as $d(v_\ell, v_{4k}) = k$, $d(v_m, v_{4k}) \leq k-1$. So we consider $\ell \in \{0, 1, 2, 3\}$ and $m \in \{8k-3, 8k-2, 8k-1, 8k, 8k+1\}$. Note that the antipodal vertex of an element $v_{4k-2+a} \in F_L$ is v_{8k-1+a}, where $a \in \{0, 1, 2\}$. Therefore, if $m = 8k-1+a$, then $d(v_\ell, v_{4k-2+a}) = k$ and $d(v_m, v_{4k-2+a}) = k+1$ for $a \in \{0, 1, 2\}$. Thus the lemma is also true for $\ell \in \{0, 1, 2, 3\}$ and $m \in \{8k-1, 8k, 8k+1\}$. Now take $m \in \{8k-3, 8k-2\}$. For $\ell \in \{2, 3\}$ and $m \in \{8k-3, 8k-2\}$, we have $d(v_\ell, v_{4k-2}) = k-1$ and $d(v_m, v_{4k-2}) = k$. Again if $\ell = 0$ and $m \in \{8k-3, 8k-2\}$, $d(v_\ell, v_{4k+1}) = k+1$ and $d(v_m, v_{4k+1}) = k$. Moreover, $d(v_1, v_{4k+1}) = k$ and $d(v_{8k-3}, v_{4k+1}) = k-1$. Therefore, the only remaining case is $\ell = 1$ and $m = 8k-2$. In this case v_ℓ and v_m can be resolved by v_2. □

Lemma 18. *Let $\ell = 4q + r$, where $r \in \{-1, 0, 1, 2\}$ with $4 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$ and m be an integer from the set $\{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n+1-4q\} \cup \{n-4q+5, \ldots, n-1\}$. Then there are at least two elements in $\{v_0, v_1, v_2, v_3\}$ that resolve the vertices v_ℓ and v_m, provided both v_ℓ and v_m are not in $A(v_2) \cap A(v_3)$. Moreover, if $\ell = 4q+2$ and $m \in \{n+2-4q, n+3-4q, n+4-4q\}$, then v_ℓ and v_m are also resolved by at least two elements from $\{v_0, v_1, v_2, v_3\}$.*

Proof. Now we calculate the distances of v_ℓ from the vertices $\{v_0, v_1, v_2, v_3\}$:

$$\begin{aligned}
d(v_0, v_\ell) &= \begin{cases} q & \text{if } r \in \{-1, 0\}, \\ q+1 & \text{if } r \in \{1, 2\}, \end{cases} \\
d(v_1, v_\ell) &= \begin{cases} q & \text{if } r \in \{-1, 0, 1\}, \\ q+1 & \text{if } r = 2, \end{cases} \\
d(v_2, v_\ell) &= q \quad \text{for all } r \in \{-1, 0, 1, 2\}, \\
d(v_3, v_\ell) &= \begin{cases} q-1 & \text{if } r = -1, \\ q & \text{if } r \in \{0, 1, 2\}. \end{cases}
\end{aligned}$$

Now the distances of v_m from the vertices $\{v_0, v_1, v_2, v_3\}$ are given by

$$d(v_a, v_m) = \left\lceil \frac{n-m+a}{4} \right\rceil \quad (a = 0, 1, 2, 3).$$

For $m \in \{\lfloor \frac{n}{2} \rfloor + 4, \ldots, n-4q, n+1-4q\}$ with $a \in \{2, 3\}$, we obtain

$$d(v_a, v_m) = \left\lceil \frac{n-m+a}{4} \right\rceil \geq \left\lceil \frac{4q-1+a}{4} \right\rceil \geq q+1.$$

Thus if $m \in \{\lfloor \frac{n}{2} \rfloor + 4, \ldots, n-4q, \; n+1-4q\}$, then v_ℓ and v_m are resolved by v_2 and v_3. For $m \in \{n-4q+5, \ldots, n-1\}$ with $a \in \{0, 1\}$, we obtain

$$d(v_a, v_m) \quad = \quad \left\lceil \frac{n-m+a}{4} \right\rceil \leq \left\lceil \frac{4q-5+a}{4} \right\rceil \leq q-1.$$

Therefore, if $m \in \{n-4q+5, \ldots, n-1\}$, then v_ℓ and v_m are resolved by v_0 and v_1. So the lemma is true when $\ell = 4q+r$ with $r \in \{-1, 0, 1, 2\}$ and $m \in \{\lfloor \frac{n}{2} \rfloor + 4, \ldots, n-4q, \; n+1-4q\} \cup \{n-4q+5, \ldots, n-1\}$. □

Theorem 2. *For $n \equiv 4, 5, 6, 7, 8 \pmod 8$, the set $F = \left\{v_0, v_1, v_2, v_3, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}, v_{\lfloor \frac{n}{2} \rfloor + 3}\right\}$ is a fault tolerant resolving set of $C_n(1,2,3,4)$. Moreover, $F \cup \left\{v_{\lfloor \frac{n}{2} \rfloor + 4}\right\}$ is a fault tolerant resolving set of $C_n(1,2,3,4)$, when $n \equiv 9 \pmod 8$.*

Proof. First we take $n = 8k + t$, where k is a positive integer and $t \in \{4, 5, 6, 7, 8\}$. Let $F_R = \{v_0, v_1, v_2, v_3\}$ and $F_L = \left\{v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}, v_{\lfloor \frac{n}{2} \rfloor + 3}\right\}$. Then $F_R \cup F_L$ is a disjoint union of F. Here we show that any two distinct vertices x and y of $C_n(1, 2, 3, 4)$ are resolved by at least two elements of F. As $V(C_n(1, 2, 3, 4)) = \{v_i : 0 \leq i \leq n-1\}$, we assume $x = v_\ell$ and $y = v_m$ for some ℓ and m with $\ell, m \in \{0, 1, \ldots, n-1\}$. If both $x, y \in F$, there is nothing to prove. Otherwise, we consider the following cases.

Case 1: *Exactly one of v_ℓ and v_m belongs to F.* Suppose $v_\ell \in F$. Without loss of generality, we can assume that $v_\ell \in F_R$. Since $v_m \notin F$, then v_ℓ and v_m are resolved by v_ℓ. Again from Lemma 15, v_ℓ and v_m are resolved by at least one element of F_L. Therefore, v_ℓ and v_m are resolved by at least two element of F.

Case 2: *Neither v_ℓ nor v_m is in F.* Let $S = \{4, 5, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ and $T = \{\lfloor \frac{n}{2} \rfloor + 4, \lfloor \frac{n}{2} \rfloor + 5, \ldots, n-1\}$. Since $v_\ell, v_m \notin F$, then $\ell, m \in S \cup T$.

Case 2.1: *Both ℓ and m are from S or T.* If v_ℓ and v_m are two consecutive vertices, then from Corollary 1, v_ℓ and v_m are resolved by two elements of F, one from $\{v_0, v_1, v_2, v_3\}$ and another from $\{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3\}$. Otherwise, v_ℓ and v_m are not consecutive. Then applying Lemma 12 accordingly $\ell, m \in S$ or $\ell, m \in T$, we have v_ℓ and v_m are resolved by at least two vertices of $\{v_0, v_1, v_2, v_3\}$.

Case 2.2: *One of ℓ and m is in S and another is in T.* Here we take $\ell \in S$ and $m \in T$. We may write $\ell = 4q + r$ for some integers q and r, where $-1 \leq r \leq 2$. If $m \in \{\lfloor \frac{n}{2} \rfloor + 4, \ldots, n-4q\} \cup \{n-4q+5, \ldots, n-1\}$, then by Lemma 18, v_ℓ and v_m are resolved by at least two elements from $\{v_0, v_1, v_2, v_3\}$. Now we determine the codes of remaining vertices with respect to F, that is, for v_ℓ and v_m, where $\ell \in \{4q-1, 4q, 4q+1, 4q+2\}$ and $m \in \{n-4q+1, n-4q+2, n-4q+3, n-4q+4\}$. The codes of v_ℓ and v_m with respect to F_R are given by

$$c(v_{4q+r}|F_R) = \begin{cases} (q, q, q, q-1) & \text{for } r = -1, \\ (q, q, q, q) & \text{for } r = 0, \\ (q+1, q, q, q) & \text{for } r = 1, \\ (q+1, q+1, q, q) & \text{for } r = 2 \end{cases}$$

and

$$c(v_{n-4q+s}|F_R) = \begin{cases} (q, q, q+1, q+1) & \text{for } s = 1, \\ (q, q, q, q+1) & \text{for } s = 2, \\ (q, q, q, q) & \text{for } s = 3, \\ (q-1, q, q, q) & \text{for } s = 4. \end{cases}$$

Let k be the diameter of $C_n(1, 2, 3, 4)$ and denote $k-q$ by b. With this notation of $k-q$, the codes of v_ℓ and v_m with respect to F_L are listed in below for different values of n.

(a) When $n \equiv 4 \pmod 8$,

$$c(v_{4q+r}|F_L) = \begin{cases} (b+1, b+1, b+2, b+2) & \text{for } r=-1, \\ (b+1, b+1, b+1, b+2) & \text{for } r=0, \\ (b+1, b+1, b+1, b+1) & \text{for } r=1, \\ (b, b+1, b+1, b+1) & \text{for } r=2, \end{cases}$$

$$c(v_{n-4q+s}|F_L) = \begin{cases} (b+1, b+1, b+1, b) & \text{for } s=1, \\ (b+1, b+1, b+1, b+1) & \text{for } s=2, \\ (b+2, b+1, b+1, b+1) & \text{for } s=3, \\ (b+2, b+2, b+1, b+1) & \text{for } s=4. \end{cases}$$

(b) When $n \equiv 5 \pmod 8$,

$$c(v_{4q+r}|F_L) = \begin{cases} (b+1, b+1, b+2, b+2) & \text{for } r=-1, \\ (b+1, b+1, b+1, b+2) & \text{for } r=0, \\ (b+1, b+1, b+1, b+1) & \text{for } r=1, \\ (b, b+1, b+1, b+1) & \text{for } r=2, \end{cases}$$

$$c(v_{n-4q+s}|F_L) = \begin{cases} (b+1, b+1, b+1, b+1) & \text{for } s=1, \\ (b+2, b+1, b+1, b+1) & \text{for } s=2, \\ (b+2, b+2, b+1, b+1) & \text{for } s=3, \\ (b+2, b+2, b+2, b+1) & \text{for } s=4. \end{cases}$$

(c) When $n \equiv 6 \pmod 8$,

$$c(v_{4q+r}|F_L) = \begin{cases} (b+1, b+2, b+2, b+2) & \text{for } r=-1, \\ (b+2, b+2, b+3, b+3) & \text{for } r=0, \\ (b+2, b+2, b+2, b+3) & \text{for } r=1, \\ (b+2, b+2, b+2, b+2) & \text{for } r=2, \end{cases}$$

$$c(v_{n-4q+s}|F_L) = \begin{cases} (b+1, b+1, b+1, b+1) & \text{for } s=1, \\ (b+2, b+1, b+1, b+1) & \text{for } s=2, \\ (b+2, b+2, b+1, b+1) & \text{for } s=3, \\ (b+2, b+2, b+2, b+1) & \text{for } s=4. \end{cases}$$

(d) When $n \equiv 7 \pmod 8$,

$$c(v_{4q+r}|F_L) = \begin{cases} (b+1, b+2, b+2, b+2) & \text{for } r=-1, \\ (b+2, b+2, b+3, b+3) & \text{for } r=0, \\ (b+2, b+2, b+2, b+3) & \text{for } r=1, \\ (b+2, b+2, b+2, b+2) & \text{for } r=2, \end{cases}$$

$$c(v_{n-4q+s}|F_L) = \begin{cases} (b+2, b+1, b+1, b+1) & \text{for } s=1, \\ (b+2, b+2, b+1, b+1) & \text{for } s=2, \\ (b+2, b+2, b+2, b+1) & \text{for } s=3, \\ (b+2, b+2, b+2, b+2) & \text{for } s=4. \end{cases}$$

(e) When $n \equiv 8 \pmod 8$

$$c(v_{4q+r}|F_L) = \begin{cases} (b+2, b+2, b+2, b+2) & \text{for } r = -1, \\ (b+1, b+2, b+2, b+2) & \text{for } r = 0, \\ (b+1, b+1, b+2, b+2) & \text{for } r = 1, \\ (b+1, b+1, b+1, b+2) & \text{for } r = 2, \end{cases}$$

$$c(v_{n-4q+s}|F_L) = \begin{cases} (b+2, b+1, b+1, b+1) & \text{for } s = 1, \\ (b+2, b+2, b+1, b+1) & \text{for } s = 2, \\ (b+2, b+2, b+2, b+1) & \text{for } s = 3, \\ (b+2, b+2, b+2, b+2) & \text{for } s = 4. \end{cases}$$

Thus $c(v_{4q+r}|F_L)$ and $c(v_{n-4q+s}|F_L)$ for respective values of r and s, are different by at least two places.

Finally, we take $n \equiv 9 \pmod 8$, that is, $n = 8k + 9$ for some positive integer k. Here it is sufficient to show that codes of v_ℓ and v_m with respect to $F_1 = F \cup \left\{ v_{\lfloor \frac{n}{2} \rfloor + 4} \right\}$ are differ by at least two positions, where $\ell \in \{4q-1,\ 4q,\ 4q+1,\ 4q+2\}$ and $m \in \{n-4q+1,\ n-4q+2,\ n-4q+3,\ n-4q+4\}$. For these values of ℓ and m, codes are listed in below. In these codes b stands for $k - q$.

$$c(v_{4q+r}|F) = \begin{cases} (q, q, q, q-1, b+2, b+2, b+2, b+2, b+3), & \text{for } r = -1, \\ (q, q, q, q, b+1, b+2, b+2, b+2, b+2), & \text{for } r = 0, \\ (q+1, q, q, q, b+1, b+1, b+2, b+2, b+2), & \text{for } r = 1, \\ (q+1, q+1, q, q, b+1, b+1, b+1, b+2, b+2), & \text{for } r = 2, \end{cases}$$

$$c(v_{n-4q+s}|F) = \begin{cases} (q, q, q+1, q+1, b+2, b+2, b+1, b+1, b+1), & \text{for } s = 1, \\ (q, q, q, q+1, b+2, b+2, b+2, b+1, b+1), & \text{for } s = 2, \\ (q, q, q, q, b+2, b+2, b+2, b+2, b+1), & \text{for } s = 3, \\ (q-1, q, q, q, b+3, b+2, b+2, b+2, b+2), & \text{for } s = 4. \end{cases}$$

Thus from the above it is easy to verify that $c(v_{4q+r}|F)$ and $c(v_{n-4q+s}|F)$ are differ by at least two positions. This completes the proof of the theorem. □

Theorem 3. *For $n \equiv 3 \pmod 8$, the set $F = \{v_0, v_1, v_2, v_3, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}\}$ forms a fault-tolerant resolving set of $C_n(1,2,3,4)$.*

Proof. Let $n = 8k + 3$ for some positive integer k. Suppose v_ℓ and v_m be arbitrary two vertices of $C_n(1,2,3,4)$. For $v_\ell,\ v_m \in F$, we are done. So we consider the following cases. If exactly one of v_ℓ and v_m is in F, then using Lemma 15 and by a similar argument as in Case 1 of Theorem 2, we get that v_ℓ and v_m are resolved by at least two elements of F. Therefore we assume that none of v_ℓ and v_m are in F. Then $\ell,\ m \in S \cup T$, where $S = \{4, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$ and $T = \{\lfloor \frac{n}{2} \rfloor + 3, \ldots, n-1\}$. If both v_ℓ and v_m are in S or in T, then by a similar argument as in Case 2.1 of Theorem 2, we obtained that v_ℓ and v_m are resolved by at least two elements of F. Otherwise, we assume that $v_\ell \in S$ and $v_m \in T$. Let $\ell = 4q + r$, where $r \in \{-1,\ 0,\ 1,\ 2\}$. If $m \in \left\{ \lfloor \frac{n}{2} \rfloor + 4, \ldots, n-4q \right\} \cup \{n-4q+5, \ldots, n-1\}$, then we obtain the result due to Lemma 18. Now we calculate the codes of the remaining vertices with respect to F, that is, for v_ℓ and v_m, where $\ell \in \{4q-1,\ 4q,\ 4q+1,\ 4q+2\}$ and $m \in \{n-4q+1,\ n-4q+2,\ n-4q+3,\ n-4q+4\} \cup \left\{ \lfloor \frac{n}{2} \rfloor + 3 \right\}$. For $m = \lfloor \frac{n}{2} \rfloor + 3$ and $\ell \in S$, it is easy to see that v_ℓ and v_m are resolved by both v_2 and v_3. Now we calculate codes of $v_\ell,\ v_m$, where $m \in \{n-4q+1,\ n-4q+2,\ n-4q+3,\ n-4q+4\}$. In the following codes b stands for $k + 1 - q$.

$$c(v_{4q+r}|F) = \begin{cases} (q,q,q,q-1,b,b,b,b) & \text{for } r=-1, \\ (q,q,q,q,b-1,b,b,b) & \text{for } r=0, \\ (q+1,q,q,q,b-1,b-1,b,b) & \text{for } r=1, \\ (q+1,q+1,q,q,b-1,b-1,b-1,b) & \text{for } r=2 \end{cases}$$

and

$$c(v_{n-4q+s}|F) = \begin{cases} (q,q,q+1,q+1,b,b,b,b) & \text{for } s=1, \\ (q,q,q,q+1,b+1,b,b,b) & \text{for } s=2, \\ (q,q,q,q,b+1,b+1,b,b) & \text{for } s=3, \\ (q-1,q,q,q,b+1,b+1,b+1,b) & \text{for } s=4. \end{cases}$$

Thus $c(v_{4q+r}|F)$ and $c(v_{n-4q+s}|F)$ are differ by at least two positions. This completes the proof of the theorem. □

Theorem 4. *For $n \equiv 2 \pmod 8$, $F = \{v_0, v_1, v_2, v_3, v_{\lfloor \frac{n}{2} \rfloor -3}, v_{\lfloor \frac{n}{2} \rfloor -2}, v_{\lfloor \frac{n}{2} \rfloor -1}, v_{\lfloor \frac{n}{2} \rfloor}\}$ is a fault-tolerant resolving set of $C_n(1,2,3,4)$.*

Proof. Let $n = 8k+2$. Suppose v_ℓ and v_m be arbitrary two vertices of $C_n(1,2,3,4)$. Let $S = \{4, \dots \lfloor \frac{n}{2} \rfloor - 4\}$ and $T = \{\lfloor \frac{n}{2} \rfloor + 1 \dots, n-1\}$. Also let $\ell = 4q+r$, where $r \in \{-1, 0, 1, 2\}$. We prove this theorem only for $\ell \in \{4q-1, 4q, 4q+1, 4q+2\}$ and $m \in \{n-4q+1, n-4q+2, n-4q+3, n-4q+4\} \cup \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3\}$; because we can prove the theorem for other values of ℓ and m using similar arguments of Theorems 2 and 3. The codes of v_ℓ and v_m are listed as below for $\ell \in \{4q-1, 4q, 4q+1, 4q+2\}$ and $m \in \{n-4q+1, n-4q+2, n-4q+3, n-4q+4\}$. In these codes $b = k+1-q$.

$$c(v_{4q+r}|F) = \begin{cases} (q,q,q-1,q-1,b-1,b-1,b,b) & \text{for } r=-1, \\ (q,q,q,q,b-1,b-1,b-1,b) & \text{for } r=0, \\ (q+1,q,q,q,b-1,b-1,b-1,b-1) & \text{for } r=1, \\ (q+1,q+1,q,q,b-2,b-1,b-1,b-1) & \text{for } r=2 \end{cases}$$

and

$$c(v_{n-4q+s}|F) = \begin{cases} (q,q,q+1,q+1,b+1,b,b,b) & \text{for } s=1, \\ (q,q,q,q+1,b+1,b+1,b,b) & \text{for } s=2, \\ (q,q,q,q,b+1,b+1,b+1,b) & \text{for } s=3, \\ (q-1,q,q,q,b+1,b+1,b+1,b+1) & \text{for } s=4. \end{cases}$$

Thus $c(v_{4q+r}|F)$ and $c(v_{n-4q+s}|F)$ are differ by at least two positions. Now we take $m \in \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3\}$ and $\ell \in S$. Then it is easy to see that v_ℓ and v_m are resolved by both v_1 and v_2. Hence the theorem. □

Theorem 5. *For the circulant graph $C_n(1,\, 2,\, 3,\, 4)$ with $n \geq 22$ and $n \notin \{26,\, 27,\, 34,\, 35,\, 42\}$,*

$$\beta'(C_n(1,\, 2,\, 3,\, 4)) = \begin{cases} 8 & \text{if } n \not\equiv 1 \pmod 8, \\ 9 & \text{if } n \equiv 1 \pmod 8. \end{cases}$$

Moreover, if $n \in \{26,\, 27,\, 34,\, 35,\, 42\}$, then $\beta'(C_n(1,\, 2,\, 3,\, 4)) = 7$.

Proof. The first part follows immediately from Theorems 1–4. Now we have to prove that if $n \in \{26,\, 27,\, 34,\, 35,\, 42\}$, then $\beta'(C_n(1,\, 2,\, 3,\, 4)) \geq 7$. Here n is of the form $8k+r$ with $r \in \{2,3\}$ and $k \geq 3$. We prove the result for $r = 2$. The proof will be similar for $r = 3$. Let F be an arbitrary fault-tolerant resolving set of $C_n(1,\, 2,\, 3,\, 4)$. Note that $I(K_5^i) = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $I(K_5^{i+4k+1}) = \{v_{i+4k+2}, \dots, v_{i+4k+4}\}$. If $\left|F \cap \left(I(K_5^i) \cup I(K_5^{i+4k+1})\right)\right| \leq 1$ for some clique K_5^i, then applying Lemma 10, we get $|F| \geq 7$. Thus we assume

$\left|F \cap \left(I(K_5^i) \cup I(K_5^{i+4k+1})\right)\right| \geq 2$ for every i. Without loss of generality, we can assume that $v_0 \in F$. Then we have

$$\begin{aligned} |F \cap \{v_1, v_2, v_3, v_{4k+2}, v_{4k+3}, v_{4k+4}\}| &\geq 2, \\ |F \cap \{v_4, v_5, v_6, v_{4k+5}, v_{4k+6}, v_{4k+7}\}| &\geq 2, \\ |F \cap \{v_7, v_8, v_9, v_{4k+8}, v_{4k+9}, v_{4k+10}\}| &\geq 2. \end{aligned}$$

Since $v_0 \in F$ and $k \geq 3$, so from the above inequalities, we have $|F| \geq 7$ for $n \in \{26, 34, 42\}$. Reader can verify that the sets $\{v_0, v_1, v_2, v_3, v_4, v_7, v_{10}\}$, $\{v_0, v_1, v_3, v_6, v_9, v_{12}, v_{15}\}$ and $\{v_0, v_5, v_8, v_{11}, v_{14}, v_{17}, v_{20}\}$ are fault-tolerant resolving sets of $C_{26}(1, 2, 3, 4)$, $C_{34}(1, 2, 3, 4)$ and $C_{42}(1, 2, 3, 4)$, respectively. By a similar argument as described in above, it can be shown that $\beta'(C_n(1, 2, 3, 4)) \geq 7$ when $n \in \{27, 35\}$. Also it is easy to verify that the sets $\{v_0, v_1, v_6, v_{11}, v_{12}, v_{17}, v_{22}\}$ and $\{v_0, v_5, v_{10}, v_{15}, v_{20}, v_{25}, v_{30}\}$ are resolving sets of $C_{27}(1, 2, 3, 4)$ and $C_{35}(1, 2, 3, 4)$, respectively. □

Author Contributions: Conceptualization—L.S., R.L., K.T., K.C.D. and Y.S.; investigation—L.S., R.L., K.T., K.C.D. and Y.S.; writing—original draft preparation—L.S., R.L., K.T., K.C.D. and Y.S.; writing—review and editing—L.S., R.L., K.T., K.C.D. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: L. Saha is supported by Science and Research Board (SERB), DST, India (Grant No. CRG/2019/006909). K. C. Das is supported by National Research Foundation funded by the Korean government (Grant No. 2021R1F1A1050646).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Slater, P.J. Leaves of trees. *Congr. Numer.* **1975**, *14*, 549–559.
2. Harary, F.; Melter, R.A. On the metric dimension of a graph. *Ars Comb.* **1976**, *2*, 191–195.
3. Chartrand, G.; Zhang, P. The theory and applicatons of resolvability in graphs, A survey. *Congr. Numer.* **2003**, *160*, 47–68.
4. Hernando, C.; Mora, M.; Slater, P.J.; Wood, D.R. Fault-Tolerant metric dimension of graphs. In *Ramanujan Math. Society Lecture Notes, Proceedings of the International Instructional Workshop on Convexity in Discrete Structures, Thiruvananthapuram, India, 22 March–2 April 2006*; Ramanujan Mathematical Society: Tiruchirappalli, India, 2008; Volume 5, pp. 81–85.
5. Saha, L. Fault-Tolerant metric dimension of cube of paths. *J. Phys. Conf. Ser.* **2021**, *1714*, 012–019.
6. Javaid, I.; Salman, M.; Chaudhry, M.A.; Shokat, S. Fault-Tolerance in Resolvability. *Util. Math.* **2009**, *80*, 263–275.
7. Javaid, I.; Rahim, M.T.; Ali, K. Families of regular graphs with constant metric dimension. *Util. Math.* **2008**, *75*, 21–33.
8. Imran, M.; Baig, A.; Bokhary, S.A.U.H.; Javaid, I. On the metric dimension of circulant graphs. *Appl. Math. Lett.* **2012**, *25*, 320–325.
9. Borchert, A.; Gosselin, S. The metric dimension of circulant graphs and Cayley hypergraphs. *Util. Math.* **2014**, *106*, 125–147.
10. Basak, M.; Saha, L.; Das, G.; Tiwary, K. Fault-tolerant metric dimension of circulant graphs $C_n(1, 2, 3)$. *Theor. Comput. Sci.* **2020**, *817*, 66–79.
11. Shang, Y. On the Estrada index of Ramanujan graphs. *J. Comb. Inf. Syst. Sci.* **2012**, *37*, 69–74.