Scaled consensus and reference tracking in multiagent networks with constraints

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Abstract—**This paper considers the scaled consensus problem for networks of groups of agents having state constraints. We first address the problem by using a gradient projection approach and design distributed consensus protocols that implement state restriction and steer agents asymptotically to proportions in terms of pre-assigned scales. We then extend the method to multiagent systems with static reference values. We propose and analyze sufficient conditions on agent dynamics such that scaled reference tracking problem is solved. The results are applied to a ship steering system and numerical examples to verify the effectiveness of the proposed cooperative coordination system scheme.**

Index Terms—**scaled consensus; complex network; tracking; constraint; cooperative system.**

I. Introduction

In the last two decades there has been an ongoing intensive attention of multiagent networks and their cooperative control. Consensus-based control approaches are critically entailed in modern autonomous systems. Two classical problems have been considered in the literature of cooperative control of multiagent systems, namely, consensus and tracking. The consensus problem involves designing distributed algorithms that would allow agents to converge to an agreement on a scalar or vector state using only local information exchange in a leaderless scenario [1]–[3], whereas the objective of tracking is for all follower agents to track the leader's behavior [4]–[6].

The problem of reaching standard consensus requires the convergence of states of all agents in the network. However, in many cooperative coordination problems in real-world complex systems, agents operate in multiple scales and different consensus values among them can emerge. Examples include compartmental mass-action systems, water distribution systems [7], social network opinion cluster formation [8], [9], transcale coordination of multi-vehicle systems with space vehicles and simulating robots on ground [10]. Continuous-time scaled consensus problems are firstly characterized by using M-matrices in [7], where agents are driven to different initial-state-dependent values in terms of proportions of assumed scales. Besides standard consensus, many other related consensus problems such as bipartite or signed consensus [2], [11] as well as cluster or group consensus [12] can be investigated under the framework of scaled consensus. The scaled consensus approach developed in [7] has been extended to investigate a wide range of practical coordinated control issues including communication delays [13], convergence speed including finite-time and fixed-time convergence [10], [14], resilient consensus against attacks [15], [16], and leader-follower behavior [17], [18]. In [19], time-dependent scale values have been examined for a linear time-varying system. Moreover, scaled consensus problem has been studied in [20] for continuous-time systems with output saturation.

A significant and realistic issue encountered in the study of leader tracking and leaderless consensus problems is the possible constraints on agents' states, inputs, or communication. These constraints include, for instance, formation of autonomous robots or vehicles with limited speed and restricted safe operation area, smart building control of humidity and temperature in an appropriate range, and public and private opinion expression conforming to social norms. In [21], a discarded consensus algorithm has been introduced to achieve state constrained consensus for both continuous-time and discrete-time multiagent systems. Velocity constraint has been considered in [22] by transforming the system matrix into a state-dependent stochastic matrix. Projected consensus algorithms have been adopted to confine states to a convex set for multiagent networks [23]–[25]. An integral barrier Lyapunov function has been used in [26] to implement fuzzy tracking control under full state constraints. A survey of cooperative coordination with constraints can be found in [27]. However, in the existing literature of constrained consensus, general scaled consensus is still an open problem.

Based on the above-mentioned issues, this paper aims to address scaled consensus problems for networks under state constraints. A continuous-time multiagent network is considered, where each agent proposes a convex constraining set which is only known to itself. We first present a scaled consensus algorithm by adopting gradient projection operators and nearest neighbor rules. We provide sufficient conditions for reaching scaled consensus and ensuring the scaled states of agents remain inside the intersection of constraints if nonempty. Different from the previous works on constrained control [21]–[25], the agents here are allowed to move out of the individual constraining sets in the evolution and only the scaled values are restricted. The proposed idea is conceptually similar to the recent framework of soft interval consensus [28], [29], where the final consensus value is in a given interval but transient trajectories are trespassable. However, the dynamics therein is tailored for real-valued agent states and only standard consensus has been considered. The approach used there is also fundamentally different from the current

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work. On the other hand, the methods developed in dealing with unconstrained scaled consensus problems such as [7], [10], [14], [18] are not applicable here due to the intrinsic nonlinearity induced by the constraints.

As a further contribution, we study scaled consensus tracking problem with a static reference signal. It is shown that if the reference value is inside the intersection of all constraining sets, our proposed tracking protocol enables all other agents to converge to this reference value in the sense of scaled consensus. This framework offers a flexibility to restrict the trajectories of agents within their respective constraining sets while prescribed proportions are still reached asymptotically. It is worth noting that although in the previous works on scaled tracking (e.g. [17], [18]) dynamical leaders have been considered, the agents with state constraints may not track the leader due to constraint nonlinearities. Our proposed tracking protocol accommodates both convex constraints and scaled states in tracking the reference signal.

The rest of the paper is organized as follows. Section 2 provides background information for network topology and system dynamics, setting the stage for Section 3, which introduces the scaled consensus and tracking protocols and presents the main results. Section 4 is devoted to detailed proofs and some numerical examples including a ship steering system are illustrated in Section 5. Conclusion in Section 6 ends the paper.

II. Preliminaries

A. Graph theory

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a pair of sets $\mathcal{V}$ and $\mathcal{E}$, with $\mathcal{V} = \{1, 2, \cdots, n\}$ the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. The communication topology between agents is characterized by the graph $\mathcal{G}$, where $(i, j) \in \mathcal{E}$ represents that agent i can receive information from agent j. We assume $\mathcal{G}$ is undirected, namely, $(i, j) \in \mathcal{E}$ is equivalent to $(j, i) \in \mathcal{E}$. We denote by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ the set of neighbors of node i. The elements of the adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ are defined as $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We assume $a_{ij} = a_{ji}$ and $a_{ii} = 0$ for all $i, j \in \mathcal{V}$. The matrix $L = (l_{ij}) \in \mathbb{R}^{n\times n}$ denotes the Laplacian matrix of $\mathcal{G}$, where $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = \sum_{j\in\mathcal{V},j\neq i} a_{ij}$. It is well known that L is a symmetric and positive semidefinite matrix. $\mathcal{G}$ is said to be connected if there is a path linking any pair of nodes i and j. L has a single zero eigenvalue if and only if $\mathcal{G}$ is connected.

We further introduce an extended graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$, with $\overline{\mathcal{V}} = \mathcal{V} \cup \{n + 1\}$. The added node $n + 1$ is viewed as the leader agent, which emits directed edges towards some of the nodes in $\mathcal{V}$ whereas $\mathcal{N}_{n+1} = \emptyset$. $\overline{\mathcal{G}}$ is said to have a directed spanning tree with root $n + 1$ if for any node $i \in \mathcal{V}$ there is a directed path from $n + 1$ to i. This is equivalent to the existence of some $i \in \mathcal{V}$ satisfying $(i, n + 1) \in \overline{\mathcal{E}}$ when $\mathcal{G}$ is connected.

B. System dynamics

Consider a multiagent network $\mathcal{G}$ with dynamical agents in $\mathcal{V} = \{1, 2, \cdots, n\}$, where the i-th agent is described as follows:

$$\dot{x}_i(t) = u_i(t), \qquad i \in \mathcal{V}, \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ is the state, $u_i(t) \in \mathbb{R}^m$ is the control input, and $t \geq 0$ is the time variable. Let $(\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}} \in \mathbb{R}^n$ be a list of scalars, where T means matrix transpose and $\alpha_i \neq 0$ for $1 \leq i \leq n$. For $i \in \mathcal{V}$, define

$$w_i(t) = \mathrm{sgn}(\alpha_i) \sum_{j\in\mathcal{N}_i} a_{ij}(\alpha_j x_j(t) - \alpha_i x_i(t)), \tag{2}$$

where $\mathrm{sgn}(\cdot)$ is the signum function, which returns 1 for a positive argument and -1 for a negative argument. The control input $u_i(t)$ will be designed in Section 3 as a function of $w_i(t)$ to accommodate the constraints defined below.

Each node $i \in \mathcal{V}$ holds a state constraining set $\Omega_i \in \mathbb{R}^m$, which is a closed convex set. The intersection is denote by $\Omega = \cap_{i\in\mathcal{V}}\Omega_i$. In particular, we assume:

Assumption 1. The set Ω_i can be expressed as $\Omega_i = \{y_i \in \mathbb{R}^m : \psi_i(y_i) \leq 1\}$, where $\psi_i : \mathbb{R}^m \to \mathbb{R}$ is a twice differentiable convex function. Moreover, $\Omega_i \backslash \partial\Omega_i \neq \emptyset$, where $\partial\Omega_i = \{y_i \in \mathbb{R}^m : \psi_i(y_i) = 1\}$ is its boundary.

Remark 1. The state constraining set Ω_i is expressed as a level set of a convex function. In general, if ψ_i is not twice differentiable, we can find a twice differentiable convex function ψ_i' to approximate ψ_i in the sense of Hausdorff metric [30]. The assumption $\Omega_i \backslash \partial\Omega_i \neq \emptyset$ indicates that the interior of Ω_i is non-empty. It is also worth mentioning that the state constraints here are expressed in terms of convex functions, which is a geometric characterization compared to other works using constraint boundary functions; e.g. [35], [36]. This approach is more amenable to our analysis of projection operator below while not too restrictive.

Assumption 2. Assume that $\alpha_i x_i(0) \in \Omega_i$ for each $i \in \mathcal{V}$.

In the reference tracking scenario, we assume that the leader agent $n + 1$ follows

$$\dot{x}_{n+1}(t) = 0_m, \tag{3}$$

where $0_m \in \mathbb{R}^m$ is the all-zero vector. In other words, the agent $n + 1$ offers a static signal. We denote $\alpha_{n+1} = 1$. With a slight abuse of notation, the neighborhood set $\mathcal{N}_i$ in the expression for $w_i(t)$ in (2) will be interpreted with respect to $\overline{\mathcal{G}}$ in the reference tracking case.

Assumption 3. Assume that $x_{n+1}(0) \in \Omega$.

The objective is to design distributed control protocols $u_i(t)$ such that scaled consensus for the multiagent system (1) is achieved, namely, $\lim_{t\to\infty} \|\alpha_i x_i(t) - \alpha_j x_j(t)\| = 0$ for all $i, j \in \mathcal{V}$ [7], and $x_i(t)$ is constrained relevant to the set Ω_i in both leaderless and leader tracking senses, where $\|\cdot\|$ is the Euclidean norm.

III. Problem formulation and main results

In this section, we propose projection based strategies involving a gradient operator to achieve scaled consensus and reference tracking for the multiagent system (1) under state constraints. The design is to partition each convex set Ω_i into a core area Ω_i^c and a peripheral area Ω_i^p, where different control input $u_i(t)$ will be applied.

For $i \in \mathcal{V}$ and $y_i \in \mathbb{R}^m$, define $\phi_i(y_i) = (\psi_i(y_i) - \xi_i)(1 - \xi_i)^{-1} \in \mathbb{R}$, where the number ξ_i can be chosen by each node i individually such that $\xi_i \in (\min_{y_i \in \Omega_i} \psi_i(y_i), 1)$. Note that this interval is non-empty by Assumption 1. Moreover, the boundary $\partial\Omega_i = \{y_i \in \mathbb{R}^m : \phi_i(y_i) = 1\}$. Define a positive semidefinite matrix

$$\Gamma_i(y_i) = \frac{\nabla\phi_i(y_i)\nabla\phi_i(y_i)^{\mathrm{T}}}{\|\nabla\phi_i(y_i)\|^2}, \tag{4}$$

where $\nabla : \mathbb{R} \to \mathbb{R}^m$ is the gradient operator. Write $y_i = (y_{i1}, y_{i2}, \cdots, y_{im})^{\mathrm{T}} \in \mathbb{R}^m$ for $i \in \mathcal{V}$. We have

$$\begin{aligned} &\nabla\phi_i(y_i) \\ =&\Big(\frac{\partial\phi_i(y_i)}{\partial y_{i1}}, \frac{\partial\phi_i(y_i)}{\partial y_{i2}}, \cdots, \frac{\partial\phi_i(y_i)}{\partial y_{im}}\Big)^{\mathrm{T}} \\ =&\frac{1}{1-\xi_i} \cdot \Big(\frac{\partial\psi_i(y_i)}{\partial y_{i1}}, \frac{\partial\psi_i(y_i)}{\partial y_{i2}}, \cdots, \frac{\partial\psi_i(y_i)}{\partial y_{im}}\Big)^{\mathrm{T}}. \end{aligned} \tag{5}$$

For each $i \in \mathcal{V}$, let

$$u_i(t) = \begin{cases} w_i(t), & \text{if } \phi_i(\alpha_i x_i(t)) \le 0, \\ w_i(t), & \text{if } \phi_i(\alpha_i x_i(t)) > 0, \\ & \alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t) \le 0, \\ [I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))]\, w_i(t), & \\ & \text{if } \phi_i(\alpha_i x_i(t)) > 0, \\ & \alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t) > 0, \end{cases} \tag{6}$$

where $t \ge 0$, $w_i(t) \in \mathbb{R}^m$ is given in (2) and $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix.

Remark 2. For any $x_i \in \Omega_i$, by definition $\phi_i(\alpha_i x_i) \le 1$. At time $t \ge 0$, the constraining set Ω_i is divided into a convex core area $\Omega_i^c(t) = \{y_i \in \mathbb{R}^m : \phi_i(y_i) \le 0\}$ and a peripheral area $\Omega_i^p(t) = \{y_i \in \mathbb{R}^m : 0 < \phi_i(y_i) \le 1\}$. Note that the time derivative $\dot{\phi}_i(\alpha_i x_i(t)) = \alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} \dot{x}_i(t)$. Hence, the control input in (6) is taken as $w_i(t)$ if $\alpha_i x_i$ is inside $\Omega_i^c(t)$ or $\alpha_i x_i$ is in $\Omega_i^p(t)$ but it is moving towards the core area. For example, at the critical state $\alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} \dot{x}_i(t) = 0$ for some t, $\phi_i(\alpha_i x_i(t))$ does not change and the scaled state $\alpha_i x_i$ is still governed by the usual diffusion term (2) driving agents towards consensus. On the other hand, the control input $u_i(t)$ is chosen as $[I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))]\, w_i(t)$ if $\alpha_i x_i$ is within $\Omega_i^p(t)$ but it is moving towards the boundary. Recall that $\partial\Omega_i = \{y_i \in \mathbb{R}^m : \phi_i(y_i) = 1\}$. We will see in Lemma 1 below that the operator $[I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))]$ forms a projection over Ω_i, which will turn the trajectory of x_i towards the boundary to avoid potential trespassing. In other words, when the scaled state of an agent has a tendency to move out of its respective constraint, the projector takes action to divert its trajectory back into the set. Otherwise, only the usual diffusion term (2) applies. Similar projection based design has been extensively used in adaptive control and signal processing; see e.g. [23]–[25], [31], [32].

Theorem 1. *Consider the multiagent system (1) with (6) over the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose that $\Omega = \cap_{i\in\mathcal{V}}\Omega_i \neq \emptyset$ and $\mathcal{G}$ is connected. Under Assumptions 1 and 2, for any $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \|\alpha_i x_i(t) - \alpha_j x_j(t)\| = 0$. Furthermore, for each $i \in \mathcal{V}$, $\alpha_i x_i(t) \in \Omega_i$ for any $t \ge 0$.*

Remark 3. Let $x_i := (x_{i1}, x_{i2}, \cdots, x_{im})^{\mathrm{T}}$ for $i \in \mathcal{V}$. We conclude from Theorem 1 that the ratio of the states $x_{il}(t)/x_{jl}(t) \to \alpha_j/\alpha_i$ as $t \to \infty$ for any $1 \le l \le m$, namely, the scaled consensus is achieved. Moreover, $\alpha_i x_i(t) \in \Omega$ for sufficiently large t.

Remark 4. Since the interior of Ω is nonempty by Assumption 1, Ω is not a single point. Thus, the convergence of scaled state $\alpha_i x_i(t)$ is not guaranteed in general. This is different from the standard unconstrained scaled consensus problem, where the existence of equilibrium and even the value may be analytically identified through finite-time stability analysis [14]. Moreover, although $x_i(t)$ remains in the convex set $\alpha_i^{-1}\Omega_i$, it is not known whether or when the state $x_i(t)$ will be inside the original constraining set Ω_i.

To address these issues, we next consider the reference tracking scenario where the leader x_{n+1} offers a reference signal.

Theorem 2. *Consider the multiagent system (1) and (3) with (6) over the network $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$. Suppose that $\overline{\mathcal{G}}$ has a directed spanning tree with root $n+1$ and $\mathcal{G}$ is connected. Under Assumptions 1, 2 and 3, for any $i \in \mathcal{V}$, $\lim_{t\to\infty} \|\alpha_i x_i(t) - x_{n+1}(0)\| = 0$ and $\alpha_i x_i(t) \in \Omega_i$ for any $t \ge 0$.*

Remark 5. All followers in $\mathcal{G}$ will track the leader signal in the sense $x_{il} \to x_{n+1,l}(0)/\alpha_i$ for any $1 \le l \le m$ and $i \in \mathcal{V}$. It is worth noting that the absolute value $|\alpha_i|$ is not essential but the ratio α_i/α_j is for $i, j \in \mathcal{V}$. Hence, without loss of generality, we can always assume $\min_{i\in\mathcal{V}} |\alpha_i| = 1$. In the reference tracking case, we can often choose $x_{n+1}(0) \in \Omega$ such that $x_{n+1}(0)/\alpha_i \in \Omega$ for all $i \in \mathcal{V}$, and hence $x_i(t) \in \Omega$ for sufficiently large t. For example, if there exists a ball $B \subseteq \Omega$ containing the origin, this is always true. In fact, if Ω itself is a ball centered at the origin, then by taking $x_{n+1}(0)$ as any point on the sphere, we obtain $x_{n+1}(0)/\alpha_i \in \Omega$ for all $i \in \mathcal{V}$. The general case can be seen similarly.

Remark 6. We mention that for both leaderless and leader tracking scenarios we rely on a high level LaSalle's invariance principle argument, which does not offer insights on the rate of consensus. It would be interesting to incorporate the techniques of finite-time convergence [10], [14] to address fast consensus. However, the protocols therein cannot be directly applied due to the convex constraints and substantial effort may be needed.

IV. Technical proofs

A. Leaderless consensus

We start with the scaled consensus case over $\mathcal{G}$ without a leader agent. The first result shows that the constraining sets $\{\Omega_i\}_{i=1}^n$ are invariant relevant to the scaled trajectories.

Lemma 1. *Consider the multiagent system (1) with (6). Under Assumptions 1 and 2, for any $i \in \mathcal{V}$, $\alpha_i x_i(t) \in \Omega_i$ for $t \ge 0$.*

Proof. Fix $i \in \mathcal{V}$. Initially, $\alpha_i x_i(0) \in \Omega_i$ by Assumption 2. In order for the scaled state to move outside Ω_i, it will be in the peripheral area Ω_i^p and point outward. Hence, by (4) and (6) we have

$$\begin{aligned}&\dot{\phi}_i(\alpha_i x_i(t))\\=&\alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} \dot{x}_i(t)\\=&\alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} \left[I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))\right] w_i(t)\\=&\alpha_i \nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t) - \alpha_i\phi_i(\alpha_i x_i(t))\nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}}\\&\cdot \frac{\nabla\phi_i(\alpha_i x_i)\nabla\phi_i(\alpha_i x_i)^{\mathrm{T}}}{\|\nabla\phi_i(\alpha_i x_i)\|^2} w_i(t)\\=&(1-\phi_i(\alpha_i x_i(t)))\alpha_i\nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t), \end{aligned} \tag{7}$$

where the first term $1-\phi_i(\alpha_i x_i(t))$ is positive but decreasingly approaches to zero as the scaled state move towards $\partial\Omega_i$, and $\alpha_i\nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t) > 0$. Note that $\nabla\phi_i(\alpha_i x_i(t))$ is the gradient direction pointing toward outside of $\partial\Omega_i$. It is also perpendicular to the tangent space of boundary points. Hence, the scaled state $\alpha_i x_i(t)$ will be driven along the boundary instead of going outside of Ω_i as it approaches $\partial\Omega_i$. □

Let $\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n) \in \mathbb{R}^{n\times n}$ be the diagonal matrix with diagonal elements $\alpha_1,\alpha_2,\cdots,\alpha_n$ and $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^{\mathrm{T}} \in \mathbb{R}^{mn}$. Define a set

$$\begin{aligned}\mathcal{Z} = \Big\{&(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\otimes I_m)x^* \in \mathbb{R}^{mn} : \sum_{i\in\Phi}|\alpha_i| w_i^{*\mathrm{T}}\\&\cdot\left[I_m - \phi_i(\alpha_i x_i^*)\Gamma_i(\alpha_i x_i^*)\right] w_i^*\\&+\sum_{i\in\mathcal{V}\setminus\Phi}|\alpha_i| w_i^{*\mathrm{T}} w_i^* = 0\Big\},\end{aligned} \tag{8}$$

where $\Phi := \{i\in\mathcal{V} : \phi_i(\alpha_i x_i^*) \in (0,1], \alpha_i\nabla\phi_i(\alpha_i x_i^*)^{\mathrm{T}} w_i^* > 0\}$ and $w_i^* = \mathrm{sgn}(\alpha_i)\sum_{j\in\mathcal{N}_i} a_{ij}(\alpha_j x_j^* - \alpha_i x_i^*)$ for $i\in\mathcal{V}$. Clearly, the above expressions determine those points x^*. We will keep this notation throughout the paper.

Lemma 2. *Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^{mn}$ be a solution of the multiagent system (1) with (6). Under Assumptions 1 and 2, the set of limit points (as $t\to\infty$) of $(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\otimes I_m)x(t)$ is contained in $\mathcal{Z}$.*

Proof. The strategy is to make use of LaSalle's invariance principle; see e.g. [33]. To this end, we introduce the Lyapunov function

$$\begin{aligned}V(x) =&\frac{1}{2}x^{\mathrm{T}}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)(L\otimes I_m)\\&\cdot\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)x,\end{aligned} \tag{9}$$

which is continuous and locally Lipschitz. By Assumptions 1, 2 and Lemma 1, we obtain

$$\begin{aligned}\dot{V}(x(t)) =&x(t)^{\mathrm{T}}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)(L\otimes I_m)\\&\cdot\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\dot{x}(t)\\=&-\sum_{i=1}^{n}|\alpha_i| w_i(t)^{\mathrm{T}}\dot{x}_i(t)\\=&-\sum_{i\in\Phi(t)}|\alpha_i| w_i(t)^{\mathrm{T}}\left[I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))\right]\\&\cdot w_i(t) - \sum_{i\in\mathcal{V}\setminus\Phi(t)}|\alpha_i| w_i(t)^{\mathrm{T}} w_i(t),\end{aligned} \tag{10}$$

where (2) and (6) are used, and the set $\Phi(t)$ is given by $\Phi(t) := \{i\in\mathcal{V} : \phi_i(\alpha_i x_i(t)) \in (0,1], \alpha_i\nabla\phi_i(\alpha_i x_i(t))^{\mathrm{T}} w_i(t) > 0\}$.

Note that

$$\begin{aligned}&I_m - \phi_i(\alpha_i x_i(t))\Gamma_i(\alpha_i x_i(t))\\=&I_m - \Gamma_i(\alpha_i x_i(t)) + (1-\phi_i(\alpha_i x_i(t)))\Gamma_i(\alpha_i x_i(t)).\end{aligned} \tag{11}$$

By (4), the matrix $\Gamma_i(\alpha_i x_i(t))$ is positive semidefinite. By Lemma 1, $1-\phi_i(\alpha_i x_i(t)) \geq 0$. Moreover, for any $w_i = (w_{i1}, w_{i2}, \cdots, w_{im})^{\mathrm{T}} \in \mathbb{R}^m$,

$$\begin{aligned}&w_i^{\mathrm{T}}\left[I_m - \Gamma_i(\alpha_i x_i(t))\right] w_i =\\&\frac{\sum_{1\leq l_1<l_2\leq m}\left(\nabla\phi_i(\alpha_i x_i(t))_{l_1} w_{il_2} - \nabla\phi_i(\alpha_i x_i(t))_{l_2} w_{il_1}\right)^2}{\sum_{l=1}^{m}\nabla\phi_i(\alpha_i x_i(t))_l^2},\end{aligned} \tag{12}$$

where we write $\nabla\phi_i(\alpha_i x_i(t)) := (\nabla\phi_i(\alpha_i x_i(t))_1, \nabla\phi_i(\alpha_i x_i(t))_2, \cdots, \nabla\phi_i(\alpha_i x_i(t))_m)^{\mathrm{T}} \in \mathbb{R}^m$. It follows from (12) that $I_m - \Gamma_i(\alpha_i x_i(t))$ is positive semidefinite and $w_i^{\mathrm{T}}\left[I_m - \Gamma_i(\alpha_i x_i(t))\right] w_i = 0$ if and only if $\nabla\phi_i(\alpha_i x_i(t))_{l_1} w_{il_2} = \nabla\phi_i(\alpha_i x_i(t))_{l_2} w_{il_1}$ for all $1\leq l_1 < l_2 \leq m$. Therefore, we have $\dot{V}(x(t)) \leq 0$ by (10) and (11). Lemma 2 follows by an application of LaSalle's invariance principle. □

To investigate the scaled consensus, we need to have a further look at the set $\mathcal{Z}$. The following observation will be used in the proof of Theorem 1.

Lemma 3. *Consider the multiagent system (1) with (6). Under Assumptions 1 and 2, if $\alpha_i x_i^* \in \Omega$ for all $i\in\mathcal{V}$, then $w_i^* = 0_m$ for all $i\in\mathcal{V}$.*

Proof. Suppose there exists a node $k\in\mathcal{V}$ satisfying $w_k^* \neq 0_m$. We have

$$\frac{-\alpha_k w_k^*}{|\alpha_k|\sum_{j\in\mathcal{N}_k} a_{kj}} = \alpha_k x_k^* - \frac{\sum_{j\in\mathcal{N}_k} a_{kj}\alpha_j x_j^*}{\sum_{j\in\mathcal{N}_k} a_{kj}}. \tag{13}$$

By assumption $\alpha_k x_k^* \in \Omega_k$. If $\alpha_k x_k^* \in \Omega_k\setminus\partial\Omega_k$, we obtain $\phi_k(\alpha_k x_k^*) < 1$. From the analysis in Lemma 2, it is easy to see if there is some vector $y_k\in\mathbb{R}^m$ satisfying $y_k^{\mathrm{T}}\Gamma_k(\alpha_k x_k^*)y_k = 0$ then $y_k^{\mathrm{T}}[I_m - \Gamma_k(\alpha_k x_k^*)]y_k > 0$. Hence, by a similar decomposition as in (11), we know that $I_m - \phi_k(\alpha_k x_k^*)\Gamma_k(\alpha_k x_k^*)$ is positive definite. Since $(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\otimes I_m)x^* \in \mathcal{Z}$, we conclude $k\in\mathcal{V}\setminus\Phi$ via (8). But this again forces $w_k^* = 0_m$, which contradicts our assumption in the beginning.

If, on the other hand, $\alpha_k x_k^* \in \partial\Omega_k$, then $\alpha_k w_k^*$ is a direction pointing inside Ω_k since the last term on the right-hand side of (13) is within $\Omega \subseteq \Omega_k$ by virtue of the convex combination. Hence, $\alpha_k\nabla\phi_k(\alpha_k x_k^*)^{\mathrm{T}} w_k^* < 0$ as the angle between the two vectors is obtuse. This means $k\notin\Phi$ by the definition of Φ. Similarly as the above case, by using (8) we know $w_k^* = 0_m$, which contradicts our assumption. This concludes the proof of Lemma 3. □

Now, we are at the stage to show the main result.

Proof of Theorem 1. If we can show $\Phi = \emptyset$, then $\mathcal{Z} = \{(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\otimes I_m)x^* \in \mathbb{R}^{mn} : w_i^* = 0_m,\ \text{for any}\ i\in\mathcal{V}\}$ by (8). By Lemma 2 and the connectivity of $\mathcal{G}$, this means scaled consensus is achieved, which concludes the proof of Theorem 1.

What remains to show is $\Phi = \emptyset$. Suppose this is not true, namely there exists a node $k_1\in\Phi$. Clearly, $w_{k_1}^* \neq 0_m$. Denote by

$$\rho := \max_{i\in\mathcal{V}} \mathrm{d}(\alpha_i x_i^*, \Omega), \tag{14}$$

where $\mathrm{d}(\cdot,\Omega)$ means the distance between a point and the set Ω. By Lemma 3, there must be some node $k_2\in\mathcal{V}$ such that $\alpha_{k_2}x^*_{k_2}\notin\Omega$, and hence $\rho>0$. We investigate two complementary situations: (i) there exists $k\in\mathcal{V}$ satisfying $\rho=\mathrm{d}(\alpha_k x^*_k,\Omega)$ and $w^*_k=0_m$; and (ii) for all i with $\rho=\mathrm{d}(\alpha_i x^*_i,\Omega)$ we have $w^*_i\neq 0_m$. Using vector analysis [24], [33], we aim to derive some contradiction in both cases.

(i). Let $\hat{x}\in\partial\Omega$ be the projection of $\alpha_k x^*_k$ over Ω and denote $\theta:=\alpha_k x^*_k-\hat{x}$. Hence, $\|\theta\|=\rho$. Define $\Theta:=\{y\in\mathbb{R}^m:\theta^{\mathrm{T}}y=\theta^{\mathrm{T}}\alpha_k x^*_k\}$. It is easy to see Θ is a hyperplane at $\alpha_k x^*_k$ and it is parallel to the tangent plane $\{y\in\mathbb{R}^m:\theta^{\mathrm{T}}y=\theta^{\mathrm{T}}\hat{x}\}$ at $\hat{x}$. By the fact that Ω is convex and (14), we obtain

$$\theta^{\mathrm{T}}\alpha_i x^*_i\leq\theta^{\mathrm{T}}\alpha_k x^*_k \tag{15}$$

for all $i\in\mathcal{V}$. In other words, all points $\{\alpha_i x^*_i\}_{i\in\mathcal{V}}$ and Ω are at the same side of the hyperplane Θ or on Θ.

Since $\Phi\neq\emptyset$, as assumed above there is a node $k_1\in\Phi$ satisfying $w^*_{k_1}\neq 0_m$. This means there is a node $i_1\in\mathcal{V}$ such that $\alpha_{i_1}x^*_{i_1}\neq\alpha_k x^*_k$. By (14), we have $\mathrm{d}(\alpha_{i_1}x^*_{i_1},\Omega)\leq\rho$. Since Ω is convex, $\alpha_{i_1}x^*_{i_1}$ is not on Θ and hence

$$\theta^{\mathrm{T}}\alpha_{i_1}x^*_{i_1}<\theta^{\mathrm{T}}\alpha_k x^*_k. \tag{16}$$

Without loss of generality, we can assume i_1 is adjacent to k in the graph $\mathcal{G}$. [In fact, if this is not the case, for any $i\in\mathcal{N}_k$ we have $\alpha_i x^*_i=\alpha_k x^*_k$. Since $\mathcal{G}$ is connected, by repeating the above same argument we obtain $\alpha_j x^*_j=\alpha_k x^*_k$ for all $j\in\mathcal{V}$. This contradicts the choice of k_1.] Recall that $0_m=w^*_k=\mathrm{sgn}(\alpha_k)\sum_{j\in\mathcal{N}_k}a_{kj}(\alpha_j x^*_j-\alpha_k x^*_k)$. We obtain $\sum_{j\in\mathcal{N}_k}a_{kj}\theta^{\mathrm{T}}(\alpha_j x^*_j-\alpha_k x^*_k)=0_m$. In view of (16) and the positivity of coefficients a_{kj}, there must exist $i_2\in\mathcal{N}_k$ satisfying $\theta^{\mathrm{T}}\alpha_{i_2}x^*_{i_2}>\theta^{\mathrm{T}}\alpha_k x^*_k$. However, this is contradictory to (15).

(ii). In this case, fix any node $k\in\mathcal{V}$ satisfying $\rho=\mathrm{d}(\alpha_k x^*_k,\Omega)$ and $w^*_k\neq 0_m$. Since $(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\otimes I_m)x^*\in\mathcal{Z}$, we have $k\in\Phi$ in view of (8). Hence, $\phi_k(\alpha_k x^*_k)\in(0,1]$ and

$$\alpha_k\nabla\phi_k(\alpha_k x^*_k)^{\mathrm{T}}w^*_k>0 \tag{17}$$

by the definition of Φ. Moreover, we have $|\alpha_k|w^{*\mathrm{T}}_k\left[I_m-\phi_k(\alpha_k x^*_k)\Gamma_k(\alpha_k x^*_k)\right]w^*_k=0$ by (8) again. Therefore, since $\alpha_k\neq 0$, we obtain

$$\begin{aligned}0=&w^{*\mathrm{T}}_k\left[I_m-\phi_k(\alpha_k x^*_k)\Gamma_k(\alpha_k x^*_k)\right]w^*_k\\=&\phi_k(\alpha_k x^*_k)w^{*\mathrm{T}}_k\left[I_m-\Gamma_k(\alpha_k x^*_k)\right]w^*_k\\&+[1-\phi_k(\alpha_k x^*_k)]w^{*\mathrm{T}}_i w^*_i.\end{aligned} \tag{18}$$

Recall that $I_m-\Gamma_k(\alpha_k x^*_k)$ is positive semidefinite. In view of the above comments in case (ii), we have

$$w^{*\mathrm{T}}_k\left[I_m-\Gamma_k(\alpha_k x^*_k)\right]w^*_k=0 \tag{19}$$

and $\phi_k(\alpha_k x^*_k)=1$. Hence, $\alpha_k x^*_k\in\partial\Omega_k$ and

$$\nabla\phi_k(\alpha_k x^*_k)_{l_1}w_{kl_2}=\nabla\phi_k(\alpha_k x^*_k)_{l_2}w_{kl_1} \tag{20}$$

for all $1\leq l_1<l_2\leq m$ by (19) and the analysis in Lemma 2.

Let $\hat{x}\in\partial\Omega$ be the projection of $\alpha_k x^*_k$ over Ω and $\theta:=\alpha_k x^*_k-\hat{x}\in\mathbb{R}^m$. We have $\|\theta\|=\rho$. Define the hyperplane $\Theta:=\{y\in\mathbb{R}^m:\theta^{\mathrm{T}}y=\theta^{\mathrm{T}}\alpha_k x^*_k\}$ at $\alpha_k x^*_k$, which is parallel to the tangent plane $\{y\in\mathbb{R}^m:\theta^{\mathrm{T}}y=\theta^{\mathrm{T}}\hat{x}\}$ at $\hat{x}$. By the convexity of Ω and choice of k, we know

$$\theta^{\mathrm{T}}\alpha_i x^*_i\leq\theta^{\mathrm{T}}\alpha_k x^*_k \tag{21}$$

for all $i\in\mathcal{V}$. Namely, all points $\{\alpha_i x^*_i\}_{i\in\mathcal{V}}$ and Ω are at the same side of the hyperplane Θ or on Θ. Following (2) and (21), we get

$$\theta^{\mathrm{T}}|\alpha_k|\sum_{i\in\mathcal{N}_k}a_{ki}(\alpha_i x^*_i-\alpha_k x^*_k)=\theta^{\mathrm{T}}\alpha_k w^*_k\leq 0. \tag{22}$$

Recall $\alpha_k x^*_k\in\partial\Omega_k$ and $\Omega\subseteq\Omega_k$. We know $-\theta$ is a vector at $\alpha_k x^*_k$ pointing toward the inside of Ω_k. On the other hand, the gradient $\nabla\phi_k(\alpha_k x^*_k)$ is perpendicular to the tangent plane of Ω_k at $\alpha_k x^*_k$, and points toward the outside of Ω_k. Therefore, $\theta^{\mathrm{T}}\nabla\phi_k(\alpha_k x^*_k)>0$. Combining this with (17) we have $\theta^{\mathrm{T}}\nabla\phi_k(\alpha_k x^*_k)\alpha_k\nabla\phi_k(\alpha_k x^*_k)^{\mathrm{T}}w^*_k>0$. This can be further written as follows by involving (20):

$$\begin{aligned}0&<\theta^{\mathrm{T}}\nabla\phi_k(\alpha_k x^*_k)\alpha_k\nabla\phi_k(\alpha_k x^*_k)^{\mathrm{T}}w^*_k\\&=\alpha_k\sum_{l_1,l_2=1}^{m}\theta_{l_1}\nabla\phi_k(\alpha_k x^*_k)_{l_1}\nabla\phi_k(\alpha_k x^*_k)_{l_2}w^*_{kl_2}\\&=\alpha_k\sum_{l_1,l_2=1}^{m}\theta_{l_1}\nabla\phi_k(\alpha_k x^*_k)^2_{l_2}w^*_{kl_1}\\&=\alpha_k\theta^{\mathrm{T}}w^*_k\sum_{l_2=1}^{m}\nabla\phi_k(\alpha_k x^*_k)^2_{l_2},\end{aligned} \tag{23}$$

where we have written $\theta=(\theta_1,\theta_2,\cdots,\theta_m)^{\mathrm{T}}$ and $w^*_k=(w^*_{k1},w^*_{k2},\cdots,w^*_{km})^{\mathrm{T}}$. This means $\alpha_k\theta^{\mathrm{T}}w^*_k>0$. Obviously, it is at odds with (22). Since we have derived contradictions in both cases, the proof of Theorem 1 is complete. □

Remark 7. The information exchange of the networked system is assumed to take place over an undirected graph $\mathcal{G}$, which is essential for the above analysis of projection operator and the geometric presentation of vectors. However, it also limits potential applicability as directed information flow is arguably more general. Directed networks have been considered in some recent works on state constrained control for example by using equivalent system transformations [37] and a fixed-point theorem [28] in different applications.

B. Reference tracking

In this section, we consider the extended graph $\overline{\mathcal{G}}=(\overline{\mathcal{V}},\overline{\mathcal{E}})$ with a leader node $n+1$, which follows (3). The following lemma can be shown by the same argument in Lemma 1.

Lemma 4. *Consider the multiagent system (1) and (3) with (6). Under Assumptions 1, 2 and 3, for any $i\in\mathcal{V}$, $\alpha_i x_i(t)\in\Omega_i$ for $t\geq 0$.*

Let $\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\in\mathbb{R}^{(n+1)\times(n+1)}$ be the diagonal matrix with diagonal elements $\alpha_1,\alpha_2,\cdots,\alpha_{n+1}$, where we recall that $\alpha_{n+1}=1$. Let $\overline{x}^*=(x_1^*,x_2^*,\cdots,x_n^*,x_{n+1}^*)^{\rm T}\in\mathbb{R}^{m(n+1)}$, where $x_{n+1}^*=x_{n+1}(0)$. Define a set

$$\begin{aligned}\overline{\mathcal{Z}}=&\Big\{(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\otimes I_m)\overline{x}^*\in\mathbb{R}^{m(n+1)}:\\&\sum_{i\in\Phi}|\alpha_i|w_i^{*\rm T}\left[I_m-\phi_i(\alpha_ix_i^*)\Gamma_i(\alpha_ix_i^*)\right]w_i^*\\&+\sum_{i\in\mathcal{V}\backslash\Phi}|\alpha_i|w_i^{*\rm T}w_i^*=0\Big\},\end{aligned}\tag{24}$$

where $\Phi:=\{i\in\mathcal{V}:\phi_i(\alpha_ix_i^*)\in(0,1],\alpha_i\nabla\phi_i(\alpha_ix_i^*)^{\rm T}w_i^*>0\}$ and $w_i^*=\mathrm{sgn}(\alpha_i)\sum_{j\in\mathcal{N}_i}a_{ij}(\alpha_jx_j^*-\alpha_ix_i^*)$ for $i\in\mathcal{V}$. Note that as we mentioned in Section 2.2, the neighborhood $\mathcal{N}_i$ here is interpreted relevant to $\overline{\mathcal{G}}$ to avoid excessive notations.

Lemma 5. *Let $\overline{x}(t)=(x_1(t),x_2(t),\cdots,x_{n+1}(t))^{\rm T}\in\mathbb{R}^{m(n+1)}$ be a solution of the multiagent system (1) and (3) with (6). Under Assumptions 1, 2 and 3, the set of limit points (as $t\to\infty$) of $(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\otimes I_m)\overline{x}(t)$ is contained in $\overline{\mathcal{Z}}$.*

Proof. Define the matrix

$$Q=\begin{pmatrix} & & -a_{1,n+1}\\ & L & \vdots\\ & & -a_{n,n+1}\\ 0 & \cdots\ \ 0 & 0\end{pmatrix}\in\mathbb{R}^{(n+1)\times(n+1)}\tag{25}$$

and the Lyapunov candidate

$$\begin{aligned}V(\overline{x})=&\frac{1}{2}\overline{x}^{\rm T}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})(Q\otimes I_m)\\&\cdot\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\overline{x},\end{aligned}\tag{26}$$

which is a continuous and locally Lipschitz function. By Assumptions 1, 2, 3 and Lemma 4, we have

$$\begin{aligned}\dot V(\overline{x}(t))=&\overline{x}(t)^{\rm T}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})(Q\otimes I_m)\\&\cdot\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\dot{\overline{x}}(t)\\=&-\sum_{i=1}^n|\alpha_i|w_i(t)^{\rm T}\dot x_i(t)\\=&-\sum_{i\in\Phi(t)}|\alpha_i|w_i(t)^{\rm T}\left[I_m-\phi_i(\alpha_ix_i(t))\Gamma_i(\alpha_ix_i(t))\right]\\&\cdot w_i(t)-\sum_{i\in\mathcal{V}\backslash\Phi(t)}|\alpha_i|w_i(t)^{\rm T}w_i(t),\end{aligned}\tag{27}$$

where $\Phi(t):=\{i\in\mathcal{V}:\phi_i(\alpha_ix_i(t))\in(0,1],\alpha_i\nabla\phi_i(\alpha_ix_i(t))^{\rm T}w_i(t)>0\}$. Noting that (27) is equivalent to (10) in Lemma 2, we can similarly proceed as Lemma 2 and show that $\dot V(\overline{x}(t))\le 0$. An application of LaSalle's invariance principle concludes the proof. □

The next lemma follows from the same argument in Lemma 3 by considering $\overline{\mathcal{Z}}$ in (24) instead of $\mathcal{Z}$ in (8).

Lemma 6. *Consider the multiagent system (1) and (3) with (6). Under Assumptions 1, 2 and 3, if $\alpha_ix_i^*\in\Omega$ for all $i\in\mathcal{V}$, then $w_i^*=0_m$ for all $i\in\mathcal{V}$.*

Proof of Theorem 2. We can proceed along the same lines in the proof of Theorem 1. The aim is to show $\Phi=\emptyset$, which would indicate $\overline{\mathcal{Z}}=\{(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\otimes I_m)\overline{x}^*\in\mathbb{R}^{m(n+1)}:w_i^*=0_m,\text{ for any }i\in\mathcal{V}\}$ by (24). It follows from Lemma 5 and the directed spanning tree with root node $n+1$ in $\overline{\mathcal{G}}$ that the desired reference tracking is achieved, which will conclude the proof of Theorem 2.

To this end, we use the method of contradiction. Suppose that $\Phi\neq\emptyset$. Define the maximum distance $\rho:=\max_{i\in\mathcal{V}}\mathrm{d}(\alpha_ix_i^*,\Omega)$, which is positive in view of Lemma 6. We then consider two cases separately: (i) there exists $k\in\mathcal{V}$ satisfying $\rho=\mathrm{d}(\alpha_kx_k^*,\Omega)$ and $w_k^*=0_m$; and (ii) for all i satisfying $\rho=\mathrm{d}(\alpha_ix_i^*,\Omega)$ we have $w_i^*\neq 0_m$.

(i). Let $\hat x\in\partial\Omega$ be the projection of $\alpha_kx_k^*$ over Ω and denote $\theta:=\alpha_kx_k^*-\hat x$. We can analogously show as in Theorem 1 that

$$\theta^{\rm T}\alpha_ix_i^*\le\theta^{\rm T}\alpha_kx_k^*\tag{28}$$

for all $i\in\mathcal{V}$.

Since $\Phi\neq\emptyset$, we assume there is a node $k_1\in\Phi$ satisfying $w_{k_1}^*\neq 0_m$. This means there is a node $i_1\in\mathcal{V}$ such that $\alpha_{i_1}x_{i_1}^*\neq\alpha_kx_k^*$. By definition, we have $\mathrm{d}(\alpha_{i_1}x_{i_1}^*,\Omega)\le\rho$ and hence $\theta^{\rm T}\alpha_{i_1}x_{i_1}^*<\theta^{\rm T}\alpha_kx_k^*$. We can always assume $i_1\in\mathcal{N}_k$ with respect to $\overline{\mathcal{G}}$. [In fact, if this is not the case, for any $i\in\mathcal{N}_k$ we have $\alpha_ix_i^*=\alpha_kx_k^*$. Since $\mathcal{G}$ is connected and $\overline{\mathcal{G}}$ has a spanning tree with root $n+1$, by repeating the above same argument we obtain $\alpha_jx_j^*=\alpha_kx_k^*$ for all $j\in\overline{\mathcal{V}}$. This contradicts the choice of k_1.]

Since $0_m=w_k^*=\mathrm{sgn}(\alpha_k)\sum_{j\in\mathcal{N}_k}a_{kj}(\alpha_jx_j^*-\alpha_kx_k^*)$, we arrive at $\sum_{j\in\mathcal{N}_k}a_{kj}\theta^{\rm T}(\alpha_jx_j^*-\alpha_kx_k^*)=0_m$. Noting that the coefficients $a_{kj}>0$ for $j\in\mathcal{N}_k$ and $\theta^{\rm T}\alpha_{i_1}x_{i_1}^*<\theta^{\rm T}\alpha_kx_k^*$, there exists some node $i_2\in\mathcal{N}_k$ such that $\theta^{\rm T}\alpha_{i_2}x_{i_2}^*>\theta^{\rm T}\alpha_kx_k^*$. Moreover, we can assume $n+1\notin\mathcal{N}_k$. Otherwise, we can apply the above argument to a neighbor of k given the connectivity of $\mathcal{G}$ and $\mathcal{N}_{n+1}=\emptyset$. Therefor, we reach a contradiction of (28).

(ii). Consider a node $k\in\mathcal{V}$ satisfying $\rho=\mathrm{d}(\alpha_kx_k^*,\Omega)$ and $w_k^*\neq 0_m$. Since $(\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})\otimes I_m)x^*\in\overline{\mathcal{Z}}$, we have $k\in\Phi$ by (24). Therefore, we have $\phi_k(\alpha_kx_k^*)\in(0,1]$ and

$$\alpha_k\nabla\phi_k(\alpha_kx_k^*)^{\rm T}w_k^*>0\tag{29}$$

Since $|\alpha_k|w_k^{*\rm T}\left[I_m-\phi_k(\alpha_kx_k^*)\Gamma_k(\alpha_kx_k^*)\right]w_k^*=0$ and $\alpha_k\neq 0$, we arrive at

$$\begin{aligned}0=&\phi_k(\alpha_kx_k^*)w_k^{*\rm T}\left[I_m-\Gamma_k(\alpha_kx_k^*)\right]w_k^*\\&+[1-\phi_k(\alpha_kx_k^*)]w_i^{*\rm T}w_i^*.\end{aligned}\tag{30}$$

Noting that $I_m-\Gamma_k(\alpha_kx_k^*)$ is positive semidefinite, we have $\phi_k(\alpha_kx_k^*)=1$ and

$$w_k^{*\rm T}\left[I_m-\Gamma_k(\alpha_kx_k^*)\right]w_k^*=0.\tag{31}$$

By (31) and the proof of Lemma 5, we have the exchangeable property $\alpha_kx_k^*\in\partial\Omega_k$ and $\nabla\phi_k(\alpha_kx_k^*)_{l_1}w_{kl_2}=\nabla\phi_k(\alpha_kx_k^*)_{l_2}w_{kl_1}$ for $1\le l_1<l_2\le m$.

We then proceed similarly as in Theorem 1 by examining the hyperplane $\Theta:=\{y\in\mathbb{R}^m:\theta^{\rm T}y=\theta^{\rm T}\alpha_kx_k^*\}$ at $\alpha_kx_k^*$, which is parallel to the tangent plane $\{y\in\mathbb{R}^m:\theta^{\rm T}y=\theta^{\rm T}\hat x\}$ at $\hat x$. Here, $\theta:=\alpha_kx_k^*-\hat x\in\mathbb{R}^m$ with $\hat x$ being the projection of $\alpha_kx_k^*$ over Ω. We have $\theta^{\rm T}\alpha_ix_i^*\le\theta^{\rm T}\alpha_kx_k^*$ for all $i\in\overline{\mathcal{V}}$. Accordingly,

$$\theta^{\rm T}|\alpha_k|\sum_{i\in\mathcal{N}_k}a_{ki}(\alpha_ix_i^*-\alpha_kx_k^*)=\theta^{\rm T}\alpha_kw_k^*\le 0.\tag{32}$$

Using the fact that $-\theta$ is a vector at $\alpha_k x_k^*$ pointing toward the inside of Ω_k and that the gradient $\nabla\phi_k(\alpha_k x_k^*)$ is perpendicular to the tangent plane of Ω_k at $\alpha_k x_k^*$ pointing toward the outside of Ω_k, we derive analogously as above that $\theta^{\mathrm{T}}\nabla\phi_k(\alpha_k x_k^*) > 0$. Combining this with (29) we obtain $\theta^{\mathrm{T}}\nabla\phi_k(\alpha_k x_k^*)\alpha_k\nabla\phi_k(\alpha_k x_k^*)^{\mathrm{T}} w_k^* > 0$. Invoking the above exchangeable property, we derive as in Theorem 1 that $0 < \alpha_k\theta^{\mathrm{T}} w_k^*$. This contradicts (32) and concludes the proof of Theorem 2. □

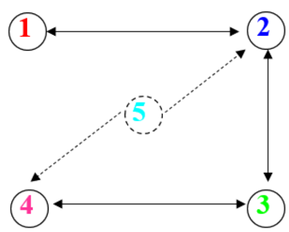

Fig. 1. Communication network $\mathcal{G}$ and extended network $\overline{\mathcal{G}}$ with $n = 4$ for Examples 1 and 2.

V. Numerical simulations

In the following we present some examples to illustrate the theoretical results of scaled consensus and scaled reference tracking.

Example 1. We start by considering scaled consensus over a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, 3, 4\}$; see Fig. 1. Write $x = (x_1, \cdots, x_4)^{\mathrm{T}}$ and $x_i = (x_{i1}, x_{i2})^{\mathrm{T}}$ for $i \in \mathcal{V}$. The constraining sets $\{\Omega_i\}_{i\in\mathcal{V}}$ are characterized in $\mathbb{R}^2$ by

$$\begin{aligned}\psi_1(x_1) =& 2(x_{11}+4)^4 + 2x_{12}^2 + (x_{11}+4)x_{12} - 19,\\ \psi_2(x_2) =& 4(x_{21}+4)^2 + (x_{22}+3)^2\\ &- 2(x_{21}+4)(x_{22}+3) - 16,\\ \psi_3(x_3) =& (x_{31}+1)^2 + 3(x_{32}-1)^2 - 9,\\ \psi_4(x_4) =& 0.\end{aligned} \tag{33}$$

The adjacency matrix A is taken as a binary 0-1 matrix and we choose $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, -2, -2)$. Define $\xi_1 = \xi_2 = -5$, $\xi_3 = -2$, and $\xi_4 = 0.5$. Fig. 2 shows the trajectories for the system (1) with (6) under initial conditions $x_1(0) = (-5, 2)^{\mathrm{T}}$, $x_2(0) = (-6, -6)^{\mathrm{T}}$, $x_3(0) = (1, 0)^{\mathrm{T}}$, and $x_4(0) = (-1, 5)^{\mathrm{T}}$. The result is consistent with our prediction in Theorem 1. The first 3 agents have convex constraints and agent 4 is unconstrained. The scaled states tend to a point inside the intersection $\Omega = \cap_{i\in\mathcal{V}}\Omega_i$ as shown by a magnified view in the inset of Fig. 2(b). In Fig. 2(c), the first components of agents 1 and 2 tend to $x_{11}(\infty) \approx -3.1$ and the counterpart for agents 3 and 4 is around 1.55, which is $\alpha_1 x_{11}(\infty)/\alpha_3 = -x_{11}(\infty)/2$. Similarly, in Fig. 2(d), the second components of agents 1 and 2 tend to around -0.37 and the counterpart for agents 3 and 4 is around $0.185 = -(-0.37)/2$.

Example 2. As a second example we consider scaled scaled reference tracking over an extended graph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ with an added leader agent 5 as shown in Fig. 1. The two new edges $(2, 5)$ and $(4, 5)$ have weight 1 and we take $x_5(0) = (-3, 0)^{\mathrm{T}}$. All parameters remain the same as in Example 1.

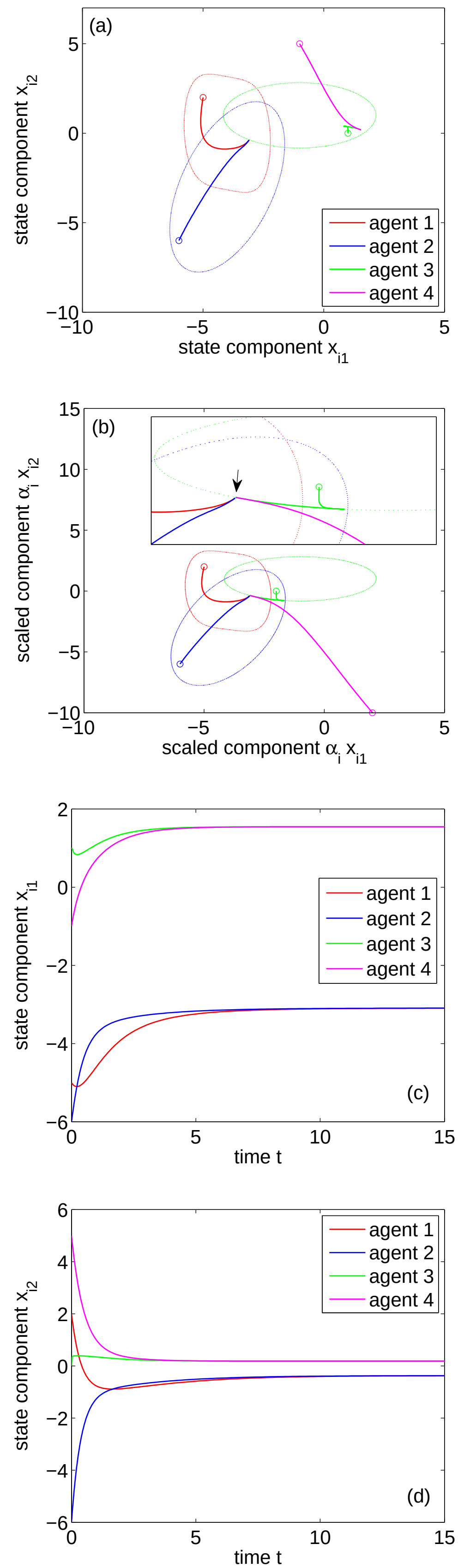

Fig. 2. Scaled consensus with constraints for Example 1: (a) State trajectories $\{x_i\}_{i=1}^4$; (b) Scaled state trajectories $\{\alpha_i x_i\}_{i=1}^4$; (c) Time evolution for the first components $\{x_{i1}(t)\}_{i=1}^4$; (d) Time evolution for the second components $\{x_{i2}(t)\}_{i=1}^4$. The scaled states of all agents tend to a common point within Ω indicated by the arrow in the inset of (b).

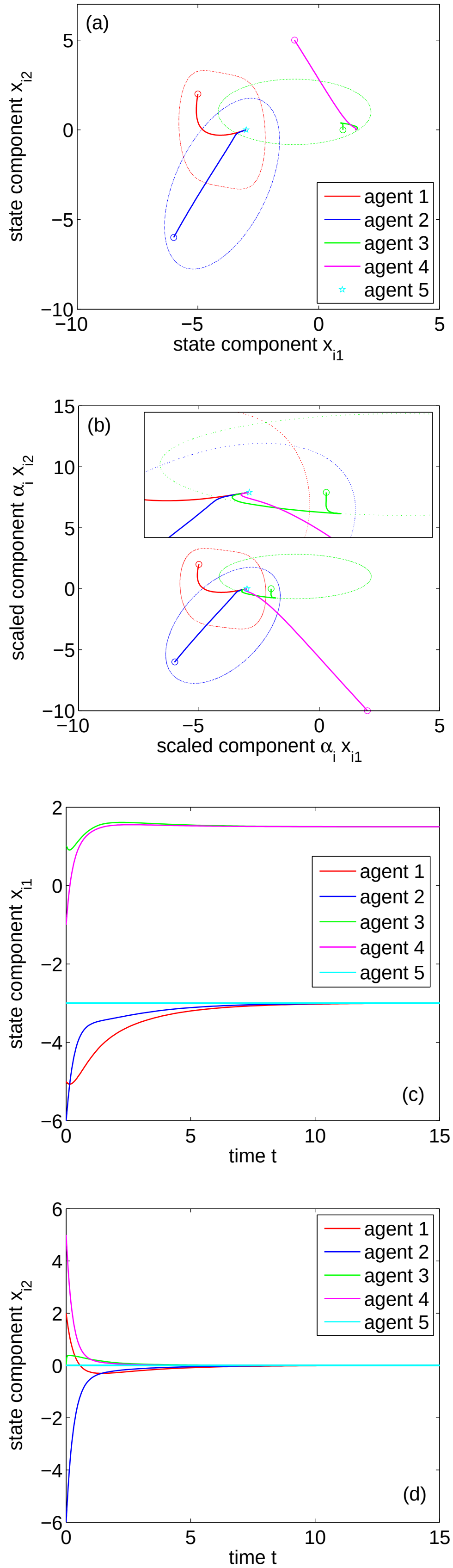

Fig. 3. Scaled reference tracking with constraints for Example 2: (a) State trajectories $\{x_i\}_{i=1}^4$; (b) Scaled state trajectories $\{\alpha_i x_i\}_{i=1}^4$; (c) Time evolution for the first components $\{x_{i1}(t)\}_{i=1}^4$; (d) Time evolution for the second components $\{x_{i2}(t)\}_{i=1}^4$. The scaled states of all followers track the static reference point within Ω indicated by the pentagram in (b).

It is straightforward to check that all conditions for Theorem 2 are met. The evolution for the multiagent system (1) and (3) with (6) is shown in Fig. 3. We observe that the scaled trajectories of all followers in $\mathcal{V}$ are led to the static reference value $x_5(0) \in \Omega$. This is further verified by the time evolution of components shown in Fig. 3(c) and Fig. 3(d). For example, $x_{51}(\infty) = -3 = x_{11}(\infty) = \alpha_3 x_{31}(\infty)/\alpha_1 = -2x_{31}(\infty) = -2 \times 1.5$.

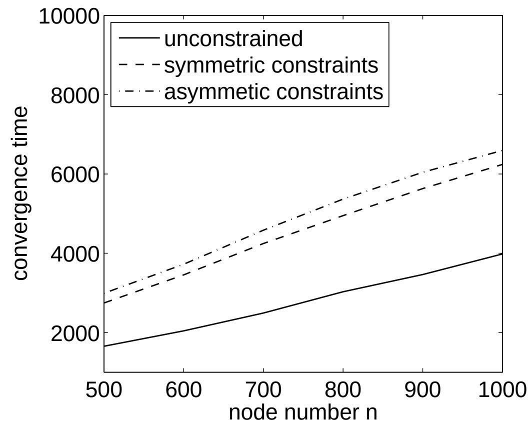

Fig. 4. Convergence time versus size of network in random graphs for Example 3.

Example 3. In this example we investigate the effect of constrains on convergence time in dense Erdős-Rényi random graphs $\mathcal{G}(n, p)$ [34] with constant edge probability $p = 0.1$ and different size n. Let $m = 2$, choose scales α_i randomly in the interval $[-2, 2]$ and each state component of agent i in the interval $[-10, 10]$ for $i \in \mathcal{V}$. One node is taken as a leader and has value $(0, 0)^{\mathrm{T}}$. Define the convergence time as $\min\{t : \|\alpha_i x_i(t)\| < \epsilon\}$ with $\epsilon = 10^{-3}$.

We take binary adjacency matrix and consider three different tracking algorithms. First, the unconstrained tracking for system (1), (3) and (6) with $\Omega_i = \mathbb{R}^2$ for all i. Second, the scaled reference tracking strategy (1), (3) and (6) with symmetric constraints, namely, each constraining set Ω_i is taken as a disk centered at $x_i(0)$ and contains the origin. Third, the scaled reference tracking strategy (1), (3) and (6) with asymmetric constraints, namely, each constraining set Ω_i is taken as an ellipse taking $x_i(0)$ and the origin as the two focal points. Fig. 4 shows the average convergence times for a sample of 20 random realizations of $\mathcal{G}(n, p)$. It takes longer to track the reference in larger graphs for both constrained and unconstrained systems. Moreover, both symmetric and asymmetric constraints hinder the tracking and the system with asymmetric constraining sets takes even longer time.

Example 4. Finally, we consider a ship steering system coordinating $n = 3$ ships with a steering station. The communication topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is shown in Fig. 5. The objective is to steer the ships 1 and 2 to a dock located at $(-1, 0)$ and the ship 3 to a dock at $(3, 0)$. To this end we apply our algorithm by sending a steering signal only to the ship 2 pointing to the mid point, the coordinate $(1, 0)$, of the two docks. The constraining sets, representing for instance safe regions, are

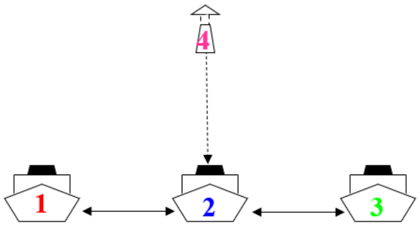

Fig. 5. Communication network $\mathcal{G}$ with $\mathcal{V} = \{1, 2, 3\}$ and extended network $\overline{\mathcal{G}}$ with $\overline{\mathcal{V}} = \mathcal{V} \cup \{4\}$ for Example 4.

assumed as follows

$$\begin{aligned} \psi_1(x_1) =&(x_{11} + 1)^2 + x_{12}^2 - 9, \\ \psi_2(x_2) =&x_{21}^2 + (x_{22} - 1)^2 - 9, \\ \psi_3(x_3) =&(x_{31} - 3)^2 + x_{32}^2 - 9. \end{aligned} \tag{34}$$

The adjacency matrix A is taken as a binary 0-1 matrix and we design the scales as $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-1, -1, 1/3, 1)$. Define $\xi_1 = \xi_2 = \xi_3 = -2$. Fig. 6 shows the trajectories for the ship steering system under initial conditions $x_1(0) = (-1, -2.5)^{\mathrm{T}}$, $x_2(0) = (2.5, 1)^{\mathrm{T}}$, $x_3(0) = (1.5, -0.8)^{\mathrm{T}}$, and $x_4(0) = (1, 0)^{\mathrm{T}}$. We observe that the ships 1 and 2 move to the coordinate $(-1, 0)$ and the ship 3 move to the coordinate $(3, 0)$ in line with the theoretical result given by Theorem 2. This example shows the success of a practical choice of the reference signal and the scales in our proposed method.

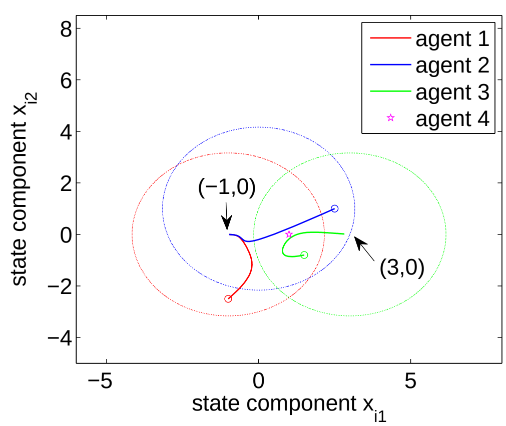

Fig. 6. State trajectories $\{x_i\}_{i=1}^3$ for the ship steering system in Example 4. The steering station 4 sends a signal to the coordinate $(1, 0)$ indicated by the pentagram.

VI. Conclusion

A model of nonlinear multiagent networks affording scaled consensus and scaled reference tracking is introduced and analyzed. The states of the agents in the network are assumed to be confined to individual convex sets. A gradient projection approach is proposed to ensure scaled convergence and suitable sufficient conditions have been developed to ensure the equilibrium is inside the intersection of constraining sets. Simulations are performed to show the scaled consensus and reference tracking as well as how constraining sets could affect the convergence over the multiagent network. The importance of the practical choice of reference signal and scales in our framework is demonstrated by a ship steering system example. Note that the current framework is based on time-invariant networks. It would be useful to extend the methods to switching or time-varying networks. In addition to the potential limitations mentioned in the paper, faster convergence rate is also an interesting future research direction.

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