Percolation of interdependent networks with limited knowledge

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Real-world networks are often not isolated and the interdependence between different networks in a complex system is as important as the topological connectivity within individual networks. We develop a theoretical framework to study the robustness of interdependent networks under attacks with limited knowledge. A node may be attacked if it is the most connected node among a given number of potential victims. This number is referred to as the attacker's knowledge level, which joins the two ends, namely, the random failure with zero knowledge and the intentional attack with full knowledge of the network. We introduce percolation models with attacks over one layer and two layers as well as mixed site-bond percolation. Along with the discontinuous phase transition, we show the existence of a critical knowledge level, which indicates a transition of network robustness under the competition between connectivity and interdependence. It is unraveled that interdependent networks can be extremely fragile to the extent that a random failure on two layers would be more deleterious than a targeted attack with full knowledge over one layer. Moreover, we find that a balanced distribution of attack knowledge on both layers tends to be most destructive if the total knowledge is a conserved quantity.

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I. INTRODUCTION

Interdependent networks have gained considerable prominence over the past decade. This is partly due to their genuine representation for many real-world networked systems such as transportation systems [1], cyber-physical systems [2], and networks of living cells [3]. As real networks are rarely isolated, the coupling and interdependence between different subnetworks or layers sow the seeds of cascading failure effect, where failures in one layer may propagate to other layers giving rise to the avalanche collapse of the entire system [4]. The pioneer work [5] on the robustness of interdependent networks models an iterative pruning process between two fully interdependent network layers on the basis of percolation theory [6]. By investigating the steady functional component of the network, referred to as the mutually connected giant component, it is revealed that interdependent networks are more vulnerable than single networks under random failures undergoing a discontinuous percolation transition. Since then percolation of interdependent networks has been extensively studied by considering, for example, different level of interdependency [7, 8], varied correlations between network layers [9, 10], targeted attacks [11] and network of networks [12]. There have also been a few comprehensive overviews on the interdependent networks and their percolation theory, see e.g. [13–15], to name a few.

Understanding vulnerability of networks to damages is of great importance for the structure and performance of various complex systems. Until recently, most research on network robustness considers random and localized failures assuming zero/local knowledge of the network topology [5, 16] or, on the other hand, targeted and optimal attacks assuming full knowledge [11, 17, 18]. In the recent work [19], an intermediate strategy called immunization under limited knowledge is introduced, where a random set of n nodes in the network is known in each iteration and only the most connected node among them is removed. If the network under consideration has N nodes, the classical random failure and targeted attack can be reproduced by the two limit scenarios $n = 1$ and $n = N$ under this framework. The giant component size and percolation threshold have been analytically derived in [19] showing that just a relatively small n (e.g. around 15) can approximately achieve the optimal attack level for both Erdős-Rényi and scale-free random networks. It is worth noting that the above percolation process uses the degree information of a limited number (but not necessarily a localized set) of nodes, and hence conceptually different from localized attack strategies [20, 21].

A versatile mirror image process of the percolation under limited knowledge has been proposed in [22], where the targeted nodes for removal can be random or most/least connected with certain probabilities. The network is shown to undergo a hybrid phase transition for the k-core organization under such pruning processes [22, 23]. An analytical framework for studying network immunization with limited knowledge is investigated in [24], where the immunity acquired may decline over time. However, all these results are established only on the basis of single networks.

To fill this gap, we here extend the single network framework [19, 22] to interdependent networks and propose three different schemes of percolation under limited knowledge over interdependent networks. The interdependent network considered here is composed by two fully interdependent network layers A and B with the same number of nodes connected by interdependency links; see Fig.1. We first consider the model in which the initial attack only happens on one layer, following the classical cascading mechanism in percolation of interde-

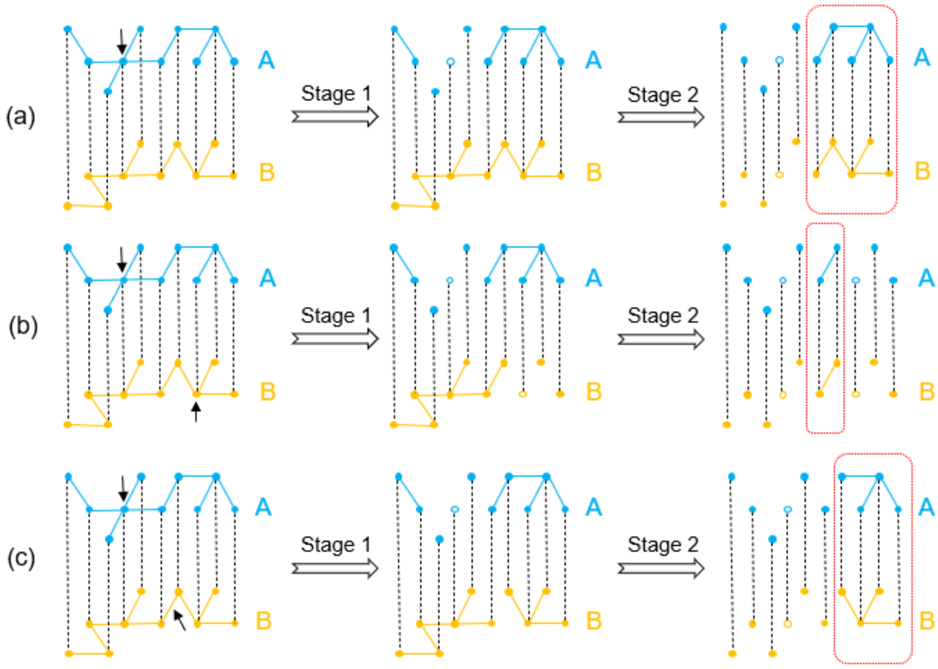

FIG. 1: Schematic of interdependent networks ($N = 10$) under percolation with limited knowledge. Assume layer A is fully known. Black arrows indicate the initially removed nodes and edges. The mutually connected giant components are indicated by red boxes. (a) Only layer A is initially attacked; (b) Both layers are initially attacked; (c) Nodes in layer A and edges in layer B are initially attacked. Solid dots are remaining nodes and hollow dots are those removed.

pendent networks [5, 6]. A discontinuous phase transition is observed for the size of mutually connected giant component under such attacks. The percolation threshold displays a growth-and-then-saturation phenomenon with respect to the knowledge level n, which is reinforced compared to the single network scenario. Interestingly, we find the competition between connectivity and interdependency may flip the network robustness as the knowledge level changes. The critical level n_c can be determined.

It is known that network layers in real interdependent networks are often of distinct characteristics and subject to different attacks. For example, modern supply chain systems are interdependent networks across regions and countries [25]. The network layers involve nodes like manufacturers, suppliers, distributors, retailers, customers etc., all of which are subject to different faults and disruption modes. Another example is the computer communication networks, which are coupled with infrastructure networks such as electric power grids. Computers and servers as nodes are prone to cyber attacks while transmission links in power grids are more likely to fail than the closure of a power station. Motivated by this, we then consider two further interdependent network percolation models, where both network layers are subject to different nodes or edges failures under limited knowledge. Note that the idea of incorporating attacks over multiple layers has been studied in some effective optimal percolation algorithms, where degrees of nodes in all layers are combined to define the strategy [17, 18]. However, failure under limited knowledge requires a different analytical approach, which we will develop in this paper.

Like our first model with attacks over one layer, we find the network undergoes an abrupt collapse under attacks over two layers. Remarkably, we show that attacks over two layers can be extremely harmful to interdependent networks to the extent that a random attack on both layers would cause more damage than a targeted attack over one layer with full knowledge. Moreover, it is unraveled that an equitable distribution of attack knowledge on both layers turns out to be most detrimental if the total knowledge is conserved.

The rest of the paper is organized as follows. Section II presents the first model, where the initial attack only happens on one layer following the classical cascading mechanism in percolation of interdependent networks. Section III deals with the second model, where the initial attack happens independently on both layers. Section IV proposes the mixed site-bond percolation, where nodes are removed in one layer and links are removed in the second one. In all these models, the analytical calculations are supported with experiments in silico. Section V concludes the paper.

II. ATTACK UNDER LIMITED KNOWLEDGE OVER ONE LAYER

Here, we introduce our first interdependent network model with two layers denoted by layer A and layer B, where the initial attack only happens on layer A. We develop a theoretical framework based on the configuration network methodology [14, 26] and compare the results with the case of single networks [19] with homogeneous and heterogeneous degree distributions.

A. Analysis for percolation of interdependent networks

Suppose the two layers A and B in the interdependent network are fully interdependent via one-to-one correspondence and both of them have N nodes. We call these N inter-layer links as interdependency links, which shall be differentiated from usual edges within each layer. Let $P^A(q) = P^A(q;0)$ and $P^B(q) = P^B(q;0)$ be the initial degree distributions at time $t = 0$, respectively, within layers A and B [26]. The percolation consists of two stages. In Stage 1, we randomly select n^A nodes from layer A at each step and remove the one with the highest degree among them. The process continues until a $1 - p^A$ fraction of nodes is removed from layer A. Next, in Stage 2, any nodes and their incident edges in layer B will be removed if the corresponding nodes in layer A are removed. Any edge connecting two nodes, whose corresponding nodes in the other layer are not in the same connected component, will also be removed. This dependency process continues back and forth between the two layers until no further deletion can happen and the final mutually connected giant component emerges (if it exists); c.f. Fig. 1. Clearly, if $n^A = 1$, it reduces to the cascading failure described in the original model [5], where a random breakdown of $1-p^A$ fraction of nodes in layer A triggers the disintegration of the whole network. In the sequel, we refer to Stage 1 as the initial targeted removal process and Stage 2 as the dependency process, respectively.

During the initial targeted removal process in layer A, technically we will assume that only nodes are deleted but their incident edges remain in the network, which facilitates the analysis below. This can be naturally understood in the setting of configuration models [28], where removing a node takes away all its incident half-edges or stubs without affecting the degrees of remaining nodes. Let $P^A(q;t)$ be the degree distribution of a random node in layer A at step $t \geq 0$ given that it remains in the network until step t. According to the above assumption, $P^A(q;t)$ changes with respect to t only due to the removal of some nodes. The corresponding cumulative distribution is denoted by $F^A(q;t) = \sum_{s=0}^{q} P^A(s;t)$, which is the probability that a randomly chosen node in layer A has degree no more than q if it stays in the resulting network at time t. Denote $F^A(q) = F^A(q;0)$. Using the maximum order statistics, the degree distribution $\hat{P}^A(q;t)$ of the removed node at step t is given by

$$\begin{aligned}\hat{P}^A(q;t) =& F^A(q;t)^{n^A} - F^A(q-1;t)^{n^A} \\ =& \Delta\left(F^A(q;t)^{n^A}\right),\end{aligned} \tag{1}$$

for $q \geq 0$, where Δ is the difference operator relevant to q. For $t \geq 0$, let $F^A(-1;t) = 0$. With one more node in layer A being removed, we have

$$N^A(q;t+1) = N^A(q;t) - \hat{P}^A(q;t), \tag{2}$$

where $N^A(q;t)$ represents the number of nodes with degree q in layer A at step t.

In the continuous limit we obtain

$$\begin{aligned}\frac{\partial N^A(q;t)}{\partial t} =& -\Delta\left(F^A(q;t)^{n^A}\right) \\ =& (N-t)\frac{\partial P^A(q;t)}{\partial t} - P^A(q;t)\end{aligned} \tag{3}$$

by using (1), (2) and the relation $N^A(q;t) = (N-t)P^A(q;t)$. Therefore,

$$\begin{aligned}\Delta\Big(& -F^A(q;t) + (N-t)\frac{\partial F^A(q;t)}{\partial t} \\ & + F^A(q;t)^{n^A}\Big) = 0.\end{aligned} \tag{4}$$

By (4) and $F^A(-1;t) = 0$ for all t, it yields for $q \geq 0$,

$$\begin{cases} (N-t)\frac{\partial F^A(q;t)}{\partial t} = F^A(q;t) - F^A(q;t)^{n^A}, \ t > 0, \\ F^A(q;0) = F^A(q). \end{cases} \tag{5}$$

When $n^A > 1$, a direct integration of (5) yields [19, 22]

$$\begin{aligned}F^A(q;t) =& \Big(1 + \left(F^A(q)^{1-n^A} - 1\right) \\ & \cdot e^{(n^A-1)\ln\left(\frac{N-t}{N}\right)}\Big)^{-\frac{1}{n^A-1}},\end{aligned} \tag{6}$$

which is equivalent to (with a slight change of notation)

$$\begin{aligned}F_p^A(q) =& \Big(1 + \left(F^A(q)^{1-n^A} - 1\right) \\ & \cdot (p^A)^{(n^A-1)}\Big)^{-\frac{1}{n^A-1}}\end{aligned} \tag{7}$$

by noting $(1-p^A)N = t$. Here, $F_p^A(q)$ is the cumulative probability that a random node has degree at most q when a $1 - p^A$ fraction of nodes is removed (leaving all incident edges intact) in layer A conditional on the node itself not being removed.

The trivial case of $n^A = 1$ can be solved directly from (5) as $F_p^A(q) = F^A(q)$, which agrees with (7) in the limit $n^A \to 1$. Therefore, for any $n^A \geq 1$, the probability of a randomly chosen node having degree q when a $1 - p^A$

fraction of nodes is removed in layer A conditional on it not being removed is expressed as

$$P_p^A(q) = \Delta F_p^A(q) = F_p^A(q) - F_p^A(q-1). \tag{8}$$

We next consider the dependency process (i.e. cascading failure) that potentially involves further removal of nodes and edges in layers A and B. Different from the above approach to tracing each stage of the initial targeted removal process, we now focus on the steady network state when all deletion processes end.

Let x be the probability that a random edge in layer A leads to the mutually connected giant component and similarly let y be the probability that a random edge in layer B leads to the mutually connected giant component. Define $R^A(q)$ to be the probability that a node in layer A is occupied in the final network given it has degree q in the original network, namely at time $t = 0$, or equivalently when a $1 - p^A$ fraction of nodes in layer A is removed. Hence,

$$\begin{aligned} x =& \Big(\sum_{q=0}^{\infty} \frac{qP^A(q)}{\langle q \rangle^A} R^A(q) \big(1 - (1-x)^{q-1}\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P^B(q) \big(1 - (1-y)^q\big) \Big), \end{aligned} \tag{9}$$

where the first term on the right-hand side means the probability that a random edge in layer A leads to an occupied node, say v, in the mutually connected giant component, and the last term means the probability that the corresponding node of v, say v' in layer B, is in the mutually connected giant component. The quantity $\langle q \rangle^A := \sum_{q=0}^{\infty} qP^A(q)$ is the mean degree in layer A. Note that the one-to-one interdependency links are random and hence v' can be viewed as a randomly chosen node in layer B.

By the conditional probability calculation, we have

$$P^A(q) R^A(q) = p^A P_p^A(q), \tag{10}$$

which is the probability that a randomly chosen node, say u, in layer A has degree q at $t = 0$ and is occupied in the final network. This can be seen by noting the following two facts: (i) If u survives the initial targeted removal process, it has the same degree at $t = 0$ and when a $1 - p^A$ fraction of nodes in layer A is removed; and (ii) If u survives the initial targeted removal process, it will also survive the dependency process.

Feeding (10) into (9) yields

$$\begin{aligned} x =& \frac{p^A}{\langle q \rangle^A} \Big(\sum_{q=0}^{\infty} qP_p^A(q) \big(1 - (1-x)^{q-1}\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P^B(q) \big(1 - (1-y)^q\big) \Big), \end{aligned} \tag{11}$$

Analogously, following a random edge in layer B we derive the following equation

$$\begin{aligned} y =& \Big(\sum_{q=0}^{\infty} \frac{qP^B(q)}{\langle q \rangle^B} \big(1 - (1-y)^{q-1}\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P^A(q) R^A(q) \big(1 - (1-x)^q\big) \Big) \\ =& \frac{p^A}{\langle q \rangle^B} \Big(\sum_{q=0}^{\infty} qP^B(q) \big(1 - (1-y)^{q-1}\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P_p^A(q) \big(1 - (1-x)^q\big) \Big), \end{aligned} \tag{12}$$

where $\langle q \rangle^B := \sum_{q=0}^{\infty} qP^B(q)$ is the mean degree in layer B. When $n^A = 1$, we have $P_p^A(q) = P^A(q)$ and the self-consistent equations (11) and (12) are equivalent to the generating function formulation derived in [5] as well as [27, Eqs. (12) and (13)].

Let P_∞ be the normalized size of the mutually connected giant component, which is also the probability that a random node in layer A or B is in the mutually connected giant component. By applying (10) again, we derive

$$\begin{aligned} P_\infty =& \Big(\sum_{q=0}^{\infty} P^A(q) R^A(q) \big(1 - (1-x)^q\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P^B(q) \big(1 - (1-y)^q\big) \Big), \\ =& p^A \Big(\sum_{q=0}^{\infty} P_p^A(q) \big(1 - (1-x)^q\big) \Big) \\ & \cdot \Big(\sum_{q=0}^{\infty} P^B(q) \big(1 - (1-y)^q\big) \Big), \end{aligned} \tag{13}$$

where the quantities x and y can be calculated from (11) and (12). In fact, the expressions (11) and (12) can be thought of as a pair of implicit functions $x = f(p^A, y)$ and $y = g(p^A, x)$. If the system undergoes a first-order phase transition, the two functions meet tangentially at the critical point $p^A = p_c^A$. By the inverse function theorem, we have

$$\frac{\partial f(p_c^A, y)}{\partial y} \cdot \frac{\partial g(p_c^A, x)}{\partial x} = 1, \tag{14}$$

which determines the percolation threshold p_c^A. In the case of single networks, it is shown that the system undergoes a second-order phase transition, where the percolation threshold p_c sits at the point where P_∞ decreases to zero [19, 22]. The limit behavior of the threshold p_c can be analytically estimated, which is not available in the present case of interdependent networks.

We remark that in the initial targeted removal process, the comparison between node degrees are based on intra-degrees, namely $P^A(q)$. Since the interdependency links are one-to-one correspondence in our model, we can produce the same results if total degrees (intra-degrees plus inter-degrees) are considered instead.

B. Results for synthetic networks

We apply the theoretical results to two classes of synthetic networks with Poisson and exponential degree distributions over layers with $N = 10^6$ nodes. For interdependent networks with two Poisson degree distributions, we have $P^A(q) = e^{-\lambda^A}(\lambda^A)^q/q!$ and $P^B(q) = e^{-\lambda^B}(\lambda^B)^q/q!$, where $q \geq 0$, $\lambda^A = \langle q \rangle^A$ and $\lambda^B = \langle q \rangle^B$. These networks are referred to as ER-ER networks, where ER stands for Erdős-Rényi by convention. For networks with two layers following exponential degree distributions, denoted by EXP-EXP, we have $P^A(q) = (1 - e^{-1/\alpha^A})e^{-q/\alpha^A}$ and $P^B(q) = (1 - e^{-1/\alpha^B})e^{-q/\alpha^B}$. Here, $q \geq 0$, $\alpha^A > 0$ and $\alpha^B > 0$ are two parameters satisfying $\alpha^A = 1/\ln(1 + 1/\langle q \rangle^A) \approx \langle q \rangle^A$ and $\alpha^B = 1/\ln(1+1/\langle q \rangle^B) \approx \langle q \rangle^B$. These networks are commonly observed in reality [28, 29] and their degree distributions facilitate closed-form analytical results through applications of generating functions [26]. We observe, for example, $P_\infty = y$ always holds in ER-ER networks and P_∞ depends nonlinearly on y in EXP-EXP networks (see Appendix A).

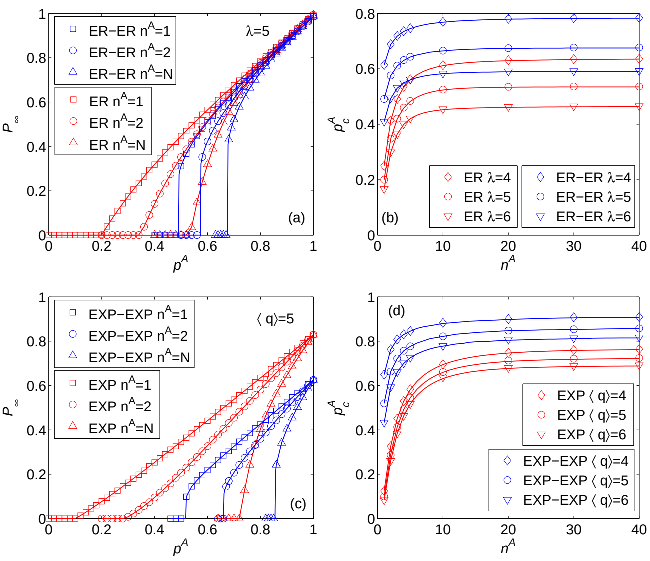

FIG. 2: P_∞ as a function of p^A with $n^A = 1, 2, N$ for (a) ER-ER networks with $\lambda := \lambda^A = \lambda^B = 5$ (blue plots) and ER networks with $\lambda = 5$ (red plots) and for (c) EXP-EXP networks with $\langle q \rangle := \langle q \rangle^A = \langle q \rangle^B = 5$ (blue plots) and EXP networks with $\langle q \rangle = 5$. Panels (b) and (d) show p_c^A as a function of n^A for the corresponding types of networks with $\lambda = 4, 5, 6$ and $\langle q \rangle = 4, 5, 6$, respectively. Solid lines are computed from theory and symbols are simulations averaged over 100 network realizations.

In Fig. 2 we compare the results for the size of mutually connected giant component P_∞ and the percolation threshold p_c^A for single networks against interdependent networks with both layers having the same degree distributions. The quantity P_∞ in Fig. 2(a) and Fig. 2(c), as an order parameter, shows that the interdependent networks undergo a discontinuous phase transition as opposed to the second-order transition in single networks under attacks with limited knowledge. As one would expect, the giant components for Poisson networks are systematically larger than those for exponential networks when comparing Fig. 2(a) and Fig. 2(c). This is due to the degree heterogeneity of exponential networks. Moreover, interdependent ER-ER networks and single ER networks posses similar values of P_∞ when the occupation probability p^A is relatively large, e.g. $p^A > 0.8$, for all knowledge level n^A. In contrast to the Poisson networks, we observe from Fig. 2(c) that interdependent EXP-EXP networks have significantly smaller giant components than the single EXP networks across the range of p^A. The interdependency prominently weakens the EXP-EXP network structure due to the existence of large-degree nodes. Even a small fraction of initial attack on layer A could easily cascade forward leading to a considerable collapse.

The percolation threshold p_c^A displayed in Fig. 2(b) and Fig. 2(d) shows a phenomenon of growth-and-then-saturation at a plateau with respect to the knowledge level n^A. For single networks, this effect has been observed in [10] and it becomes slightly more apparent in interdependent networks for both Poisson and exponential networks. For instance, a value of $n^A \approx 20$ in EXP networks with $\langle q \rangle = 4$ corresponds to a nearly full knowledge attack, whereas this number goes down to around 10 in the corresponding EXP-EXP networks.

The fragility of interdependent networks is well recognized [6, 14]. However, the crossover at $n_c^A \approx 5$ for ER-ER networks with mean degree $\lambda = 6$ and ER networks with mean degree $\lambda = 4$ displayed in Fig. 2(b) is worth noting. This crossover phenomenon implies when an interdependent network is sufficiently more connected than the single network with the same type of degree distribution, attack knowledge makes a difference. Here, the single ER network is more robust when $n^A \leq 4$ but the interdependent ER-ER network takes over when $n^A \geq 6$. The critical value $n_c^A \approx 5$ suggests a balance point of knowledge level, where the potential cascading failure (due to interdependency) cancels out the network connectivity offset (indicated by, e.g. $\Delta\lambda = 6 - 4 = 2$, the difference between the two mean degrees). When $n^A > n_c^A$, the attack is relatively less harmful over the interdependent network (given the connectivity offset) than the single network. Hence, the interdependent network is more robust in this regime. When $n^A < n_c^A$, the attack is relatively more harmful over the interdependent network (given the connectivity offset) than the single network, giving a more robust single network. The competition of the two factors with respect to knowledge level is intuitively subtle and is only revealed here through illustration.

Generally, the critical value n_c^A (if exists) can be identified numerically by solving (14) for the two networks under consideration. When $n^A = 1$ and given λ, p_c^A(ER-ER) $> p_c^A$(ER). Therefore, the critical value always exists if $\Delta\lambda$ is sufficiently large. We observe from Fig. 2(b) that ER-ER networks with $\lambda = 5$ are less robust than ER networks with $\lambda = 4$ for any n^A, and hence no crossover of robustness would occur in this case. This is because the connectivity of them is close ($\Delta\lambda = 1$) — the attack is more harmful over ER-ER networks for all n^A. Although we have taken a close look only at Poisson networks, the above analysis for the crossover phenomenon in principle also holds for networks with exponential and other types of degree distributions.

III. ATTACK UNDER LIMITED KNOWLEDGE OVER TWO LAYERS

In Section II we have assumed that the initial targeted attack only happens in layer A, namely, a $1 - p^A$ fraction of nodes is attacked in layer A under limited knowledge level n^A. Here, we consider a modified model with initial attacks happening on both layers. We observe some interesting results highlighting the vulnerability of interdependent networks under such attacks (see Section III.B).

A. Analysis for percolation of interdependent networks

Suppose that the two initial targeted removal processes occur in layer A and layer B independently. Specifically, a $1 - p^A$ fraction of nodes in layer A is removed with knowledge level n^A and a $1-p^B$ fraction of nodes in layer B is removed with knowledge level n^B. As in Section II, we establish the conditional degree distribution $P_p^A(q)$ for layer A through (7) and (8), and the corresponding conditional degree distribution for layer B can be given by

$$P_p^B(q) = \Delta F_p^B(q) = F_p^B(q) - F_p^B(q-1), \tag{15}$$

where the resulting conditional cumulative probability $F_p^B(q)$ is

$$F_p^B(q) = \Big(1 + \big(F^B(q)^{1-n^B} - 1\big) \cdot (p^B)^{(n^B-1)}\Big)^{-\frac{1}{n^B-1}} \tag{16}$$

and $F^B(q) = \sum_{s=0}^{q} P^B(s)$ is the initial cumulative probability or distribution function.

Following the cascading failure over the interdependent network, we denote by x and y the probabilities that a random edge in layer A and layer B, respectively, leads to the mutually connected giant component. Let $R^A(q)$ and $R^B(q)$ be the probabilities that a node in layer A and layer B, respectively, is occupied in the ultimate network given the node has degree q in the original network at $t = 0$. Analogous to (9) we derive

$$x = \Big(\sum_{q=0}^{\infty} \frac{qP^A(q)}{\langle q\rangle^A} R^A(q)\big(1 - (1-x)^{q-1}\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P^B(q) R^B(q)\big(1-(1-y)^q\big)\Big). \tag{17}$$

Note that the equality (10) and its counterpart expression $P^B(q)R^B(q) = p^B P_p^B(q)$ in layer B still hold. Arguing along the same line of Section II.A, we have

$$x = \frac{p^A p^B}{\langle q\rangle^A} \Big(\sum_{q=0}^{\infty} qP_p^A(q)\big(1-(1-x)^{q-1}\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P_p^B(q)\big(1-(1-y)^q\big)\Big) \tag{18}$$

and

$$\begin{aligned} y =& \Big(\sum_{q=0}^{\infty} \frac{qP^B(q)}{\langle q\rangle^B} R^B(q)\big(1-(1-y)^{q-1}\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P^A(q)R^A(q)\big(1-(1-x)^q\big)\Big) \\ =& \frac{p^A p^B}{\langle q\rangle^B} \Big(\sum_{q=0}^{\infty} qP_p^B(q)\big(1-(1-y)^{q-1}\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P_p^A(q)\big(1-(1-x)^q\big)\Big). \end{aligned} \tag{19}$$

When $p^B = 1$, we have $P_p^B(q) = P^B(q)$ by (15). Hence, the above self-consistent equations (18) and (19) reduce to (11) and (12), respectively, as one would expect.

The normalized size of the mutually connected giant component in the ultimate network can be derived as

$$\begin{aligned} P_\infty =& \Big(\sum_{q=0}^{\infty} P^A(q)R^A(q)\big(1-(1-x)^q\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P^B(q)R^B(q)\big(1-(1-y)^q\big)\Big), \\ =& p^A p^B \Big(\sum_{q=0}^{\infty} P_p^A(q)\big(1-(1-x)^q\big)\Big) \cdot \Big(\sum_{q=0}^{\infty} P_p^B(q)\big(1-(1-y)^q\big)\Big), \end{aligned} \tag{20}$$

where the quantities x and y are determined by (18) and (19). Let (18) and (19) be expressed as $x = f(p^A, p^B, y)$ and $y = g(p^A, p^B, x)$. Then the percolation thresholds p_c^A and p_c^B satisfy the surface determined by

$$\frac{\partial f(p_c^A, p_c^B, y)}{\partial y} \cdot \frac{\partial g(p_c^A, p_c^B, x)}{\partial x} = 1. \tag{21}$$

If the range of initial attack is equitable on two layers, namely $p^A = p^B$, the critical value $p_c = p_c^A = p_c^B$ can be obtained.

B. Results for synthetic networks

Here, we apply the above results on the same ER-ER networks and EXP-EXP networks as Section II.B (see Appendix B for the formulation using generating functions). Some remarks are in order.

In Fig. 3 (a) and (c), we show the evolution of the size of mutually connected giant component for ER-ER networks and EXP-EXP networks, respectively, with $n^A = n^B$. As one would expect, when the initial attacks happen on two layers, both networks collapse more quickly than the situations of attacks on only layer A at all knowledge level. The first-order phase transition phenomenon also persists. However, somewhat surprisingly, ER-ER networks turn out to be particularly susceptible to attacks on two layers to the extent that a random removal on both layers would cause more harm than a targeted attack on layer A with full knowledge (c.f. Fig. 3 (a)). This can naturally be explained by the degree homogeneity in ER-ER networks but it also sheds light on the fragility of interdependent networks against initial attacks in both layers in general. This aspect has not received sufficient attention in the literature.

The color map for the percolation threshold p_c is shown in Fig. 3 (b) for ER-ER networks and in Fig. 3(d) for EXP-EXP networks when $p^A = p^B$. We observe a quick saturation phenomenon of p_c when the knowledge level n^A or n^B increases similarly as the situation of single layer attacks in Section II.B. Although analytical comparison is difficult, the following relation is checked numerically for all cases considered here:

$$p_c(n, n) \geq p_c(n + 1, n - 1) \tag{22}$$

or equivalently $p_c(n, n) \geq p_c(n - 1, n + 1)$ by symmetry of the two layers. The monotonicity also holds naturally: $p_c(n^A, n^B) \geq p_c(n^{A\prime}, n^{B\prime})$ if $n^A \geq n^{A\prime}$ and $n^B \geq n^{B\prime}$. Combining these relations, we conclude that an even distribution of knowledge on both layers would cause the most harm when the total knowledge is bounded. This is in line with our above observation, which highlights the danger of attacks on both layers.

IV. SITE-BOND PERCOLATION IN INTERDEPENDENT NETWORKS

In the previous two sections, we have considered models with exclusively site percolation. Here, we explore a mixed site-bond percolation model, where a random edge attack happens on layer B. Specifically, assume that layer A in the interdependent network is subject to the node attack with limited knowledge whereas layer B in the network is subject to an independent random edge removal with edge occupation probability p^B. Clearly, this scenario is the counterpart for the site percolation model under attacks on two layers with $n^B = 1$ described in Section III.

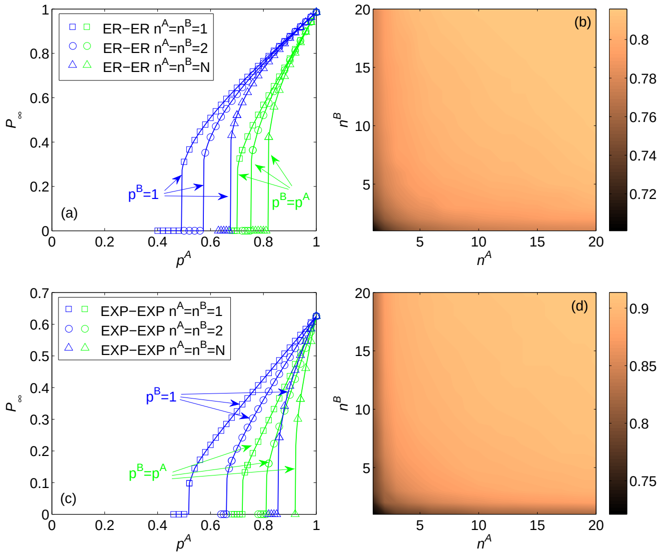

FIG. 3: P_∞ as a function of p^A with $n^A = n^B = 1, 2, N$ for (a) ER-ER networks with $\lambda^A = \lambda^B = 5$ when $p^B = 1$ (blue plots) and $p^B = p^A$ (green plots), and (c) EXP-EXP networks with $\langle q\rangle^A = \langle q\rangle^B = 5$ when $p^B = 1$ (blue plots) and $p^B = p^A$ (green plots). The critical value $p_c := p_c^A = p_c^B$ over different n^A and n^B is shown in (b) for the same ER-ER networks when $p^A = p^B$ and (d) for the same EXP-EXP networks when $p^A = p^B$. Solid lines are computed from theory and symbols/data points are simulations averaged over 100 network realizations.

As before, define x and y to be probabilities that a random edge in layer A and layer B, respectively, leads to the mutually connected giant component. Since no further node in layer B is removed outside the dependency process, following the same argument in Section II, the equation (11) still holds. However, when following a random edge in layer B to explore the mutually connected giant component, the chosen edge is present only with a probability of p^B. Therefore, we can modify the self-consistent equation (12) as

$$\begin{aligned} y =& \frac{p^A p^B}{\langle q\rangle^B}\left(\sum_{q=0}^{\infty} qP^B(q)\big(1-(1-y)^{q-1}\big)\right) \\ &\cdot\left(\sum_{q=0}^{\infty} P_p^A(q)\big(1-(1-x)^q\big)\right). \end{aligned} \tag{23}$$

Since no node in layer B is removed outside the cascading failure, it can be seen that the normalized size of the mutually connected giant component P_∞ still follows the expression (13).

Let (11) and (23) be expressed as $x = f(p^A, y)$ and $y = g(p^A, p^B, x)$. Analogously, the percolation thresholds

p_c^A and p_c^B satisfy the surface determined by

$$\frac{\partial f(p_c^A, y)}{\partial y} \cdot \frac{\partial g(p_c^A, p_c^B, x)}{\partial x} = 1. \tag{24}$$

Although the probability p^B does not explicitly appear in equations (11) and (13), both quantities x and y, and P_∞ depend on it. Apparently, we reduce to the scenario of attack on one layer in Section II when $p^B = 1$.

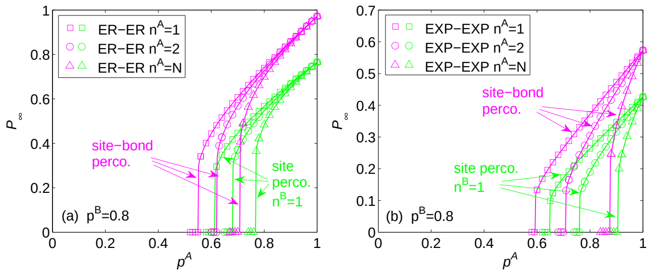

FIG. 4: P_∞ as a function of p^A with $p^B = 0.8$ and $n^A = 1, 2, N$ for (a) ER-ER networks with $\lambda^A = \lambda^B = 5$ under site percolation with $n^B = 1$ (green plots) and site-bond percolation (magnet plots), and (b) EXP-EXP networks with $\langle q\rangle^A = \langle q\rangle^B = 5$ under site percolation with $n^B = 1$ (green plots) and site-bond percolation (magnet plots). Solid lines are computed from theory and symbols are simulations averaged over 100 network realizations.

In Fig. 4, we fix the occupation probability on layer B as $p^B = 0.8$ and compare the size of mutually connected giant component P_∞ for the site percolation over two layers with $n^B = 1$ and the mixed site-bond percolation. The site percolation is significantly more harmful than site-bond percolation in terms of P_∞ and p_c^A for both ER-ER networks and EXP-EXP networks under any knowledge level n^A. Although the networks are more robust to site-bond percolation, abrupt breakdown is observed as nodes are removed from layer A. These phenomena are qualitatively consistent with the discrepancy between random site and bond percolation over other coupled networks [6, 14, 30].

V. CONCLUSION

In summary, we have considered percolation on interdependent networks with limited knowledge. Here, the cascading failure in the networks is initiated by a targeted removal process, where the most connected node among n randomly selected nodes is attacked at each step. Attack with limited knowledge offers a more detailed landscape of percolation transitioning from random failure ($n = 1$) to targeted attack with full knowledge ($n = N$). We have developed a theoretical framework to cover node attacks with limited knowledge over one layer as well as two layers. The mixed site-bond percolation over two layers has also been studied. Along with the first-order phase transition typically found in interdependent networks, we show the existence of a critical knowledge level, which signifies a crossover of network robustness when both connectivity and interdependence are taken into account. When the overall knowledge level n is fixed, a balanced partition of n for the attacks over two layers is unveiled to be most harmful. Moreover, we find that random failure in both layers can be more deleterious than targeted attack in just one layer for homogeneous interdependent networks. The paper sheds new light on the design and protection of interdependent networks under errors or malicious attacks.

Appendix A: Attack under limited knowledge over one layer in synthetic networks

In ER-ER networks, the generating functions for the degree distributions in the two layers are given by $G^A(z) = e^{\lambda^A(z-1)}$ and $G^B(z) = e^{\lambda^B(z-1)}$, where z serves as a placeholder. Analogously, let $G_p^A(z) = \sum_{q=0}^{\infty} P_p^A(q) z^q$. Using (11) and (12) we obtain

$$x = \frac{p^A}{\lambda^A}\left(G_p^{A\prime}(1) - G_p^{A\prime}(1-x)\right)\left(1 - e^{-\lambda^B y}\right) \tag{A1}$$

and

$$y = p^A\left(1 - e^{-\lambda^B y}\right)\left(1 - G_p^A(1-x)\right), \tag{A2}$$

which together determine the two probabilities x and y. By (13) and (A2) we observe that $P_\infty = y$, meaning that the probability of a random node in ER-ER network being in the mutually connected giant component is equivalent to the probability of a random edge in layer B leading to the mutually connected giant component. This is in line with the observation in [5] in the special case of random failure (i.e. $n^A = 1$).

In EXP-EXP networks, the generating functions for the degree distributions in the two layers become $G^A(z) = \left(1 - e^{-1/\alpha^A}\right)/\left(1 - ze^{-1/\alpha^A}\right)$ and $G^B(z) = \left(1 - e^{-1/\alpha^B}\right)/\left(1 - ze^{-1/\alpha^B}\right)$, where z is a placeholder. In view of (11) and (12), with some algebra we obtain

$$x = \frac{p^A y e^{-\frac{1}{\alpha^B}}\left(G_p^{A\prime}(1) - G_p^{A\prime}(1-x)\right)}{\langle q\rangle^A\left(1 - (1-y)e^{-\frac{1}{\alpha^B}}\right)} \tag{A3}$$

and

$$\begin{aligned} y =& p^A\left(1 - \frac{\left(1 - e^{-\frac{1}{\alpha^B}}\right)^2}{\left(1 - (1-y)e^{-\frac{1}{\alpha^B}}\right)^2}\right) \\ &\cdot \left(1 - G_p^A(1-x)\right), \end{aligned} \tag{A4}$$

which determine the two probabilities x and y. It follows from (13) and (A4) that

$$P_\infty = p^A(1-G_p^A(1-x))\left(\frac{ye^{-\frac{1}{\alpha^B}}}{1-(1-y)e^{-\frac{1}{\alpha^B}}}\right) = \frac{y\big(1-(1-y)e^{-\frac{1}{\alpha^B}}\big)}{2-(2-y)e^{-\frac{1}{\alpha^B}}}. \quad \text{(A5)}$$

This means $P_\infty \in [y/2, y]$, where $P_\infty \to y/2$ if $\alpha^B \to 0$ and $P_\infty \to y$ if $\alpha^B \to \infty$.

Appendix B: Attack under limited knowledge over two layers in synthetic networks

In ER-ER networks, the two quantities x and y in (18) and (19) are given by the following system

$$x = \frac{p^A p^B}{\lambda^A}\big(G_p^{A\prime}(1) - G_p^{A\prime}(1-x)\big)\big(1-G_p^B(1-y)\big) \quad \text{(B1)}$$

and

$$y = \frac{p^A p^B}{\lambda^B}\big(G_p^{B\prime}(1) - G_p^{B\prime}(1-y)\big)\big(1-G_p^A(1-x)\big) \quad \text{(B2)}$$

using the generating functions $G_p^A(z) = \sum_{q=0}^\infty P_p^A(q)z^q$ and $G_p^B(z) = \sum_{q=0}^\infty P_p^B(q)z^q$. By (20) we have

$$P_\infty = p^A p^B\big(1-G_p^A(1-x)\big)\big(1-G_p^B(1-y)\big). \quad \text{(B3)}$$

Similarly, in EXP-EXP networks the quantities x and y are determined by the system

$$x = \frac{p^A p^B}{\langle q\rangle^A}\big(G_p^{A\prime}(1) - G_p^{A\prime}(1-x)\big)\big(1-G_p^B(1-y)\big) \quad \text{(B4)}$$

and

$$y = \frac{p^A p^B}{\langle q\rangle^B}\big(G_p^{B\prime}(1) - G_p^{B\prime}(1-y)\big)\big(1-G_p^A(1-x)\big) \quad \text{(B5)}$$

The size of mutually connected giant component P_∞ follows the same expression in (B3).

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