Mean-Square Strong Stability and Stabilization of Discrete-time Stochastic Systems with Multiplicative Noises

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Abstract

This paper investigates the mean-square strong stability and stabilization of a discrete-time stochastic system corrupted by multiplicative noises. First, the definition of the mean-square (MS) strong stability is addressed to avoid overshoots in system dynamics, and two necessary and sufficient conditions for the MS-strong stability are derived. Moreover, the relationship between MS-strong stability and MS-stability is given. Second, some necessary and sufficient conditions of the MS-strong stabilization via state feedback and output feedback are obtained, respectively. Furthermore, analytical expressions of state feedback (SF) controller and static output feedback (SOF) controller are proposed, respectively. Finally, an equivalent design method for SOF controller and dynamic output feedback (DOF) controller is presented.

Keywords: mean-square strong stability; mean-square stability; discrete-time stochastic system; overshoot; output feedback

1. introduction

Stability (such as asymptotic and Lyapunov stability) is one of the important properties and a wide range of applications in a dynamic system which has been extensively studied, see [1]-[7]. Asymptotic or Lyapunov stability is clearly necessary for bound-ness in some sense of system variables but cannot guarantee the disappear of the overshoot in dynamic responses. Non-overshooting behaviour is a desirable property in certain applications because overshoot may destroy the system, such as, large transient voltage will destroy power systems [8]. Compared with these classical notions of stability, MS-strong stability is a stronger version of the stability, which can prohibit overshooting responses and characterize satisfactorily system operation. Some nice results were presented for the strong stability in deterministic continuous-time systems [9], discrete time systems [10] and linear delay difference systems [11, 12].

In the past years, several important results for the stability and stabilization of the discrete-time stochastic systems were reported. For instance, [13] gave the necessary and sufficient conditions for the MS-stability of discrete stochastic systems by generalized Lyapunov equations, and [14] further investigated this issue using spectrum technique. In the sequel, [15] investigated the relationships among MS-strong exponential stability, l_2 input-output stability and stochastic stability of discrete-time infinite Markov jump systems. Recently, [16] dealt with the σ-error MS-stability of semi-Markov jump linear systems. Other nice results can be found, e.g., almost sure stability and stabilization of nonlinear discrete-time stochastic systems in [17], MS-exponential stability and stabilization for stochastic control systems with discrete-time state feedbacks in [18], moment stability of nonlinear discrete stochastic systems with time-delays in [19] and the references therein. Recently, [20] studied the optimal output feedback control and stabilization problems for discrete-time multiplicative noise system with intermittent observations. [21] investigated the exponential mean-square stabilization problem for a class of discrete-time strict-feedback nonlinear systems subject to multiplicative noises. Some other nice results can be referred to [22]-[26]. However, there is really no attention on strong stability of discrete stochastic systems with multiplicative noises.

Motivated by aforementioned discussions, we deal with the MS-strong stability and stabilization for the discrete-time stochastic systems with multiplicative noises. Due to the complexity of this kind of systems, the problems under consideration are rather difficult. The main contributions of this paper are as follows: (i) The definition of the MS-strong stability is first introduced for discrete-time stochastic systems with multiplicative noises, and two necessary and sufficient conditions for MS-strong stability are derived. Furthermore, the relationship between MS-strong stability and MS-stability is revealed. (ii) Several necessary and sufficient conditions for the stabilization via SF controller and OF controller are proposed, respectively. Moreover, the analytical expressions of SF controller and SOF controller are obtained by using matrix transform technique and Finsler's Theorem. (iii) The relationship between SOF controller and DOF controller is studied, and an equivalent design method for both SOF controller and DOF controller is obtained.

The structure of this paper is as follows: In section 2, the notions of the MS-strong stability and some preliminaries are introduced. In section 3, the MS-strong stabilization via state feedback is discussed. In section 4, the MS-strong stabilization via output feedback is investigated. The paper ends by the conclusions in section 5.

Notations: R^n: the set of all real n-dimensional vectors. N_0: $N_0 = N \cup \{0\}$. $R^{n\times m}$: the set of all $n \times m$ matrices. A^T: the transpose of A. $\lambda(A)$: the eigenvalue of A. $\|x\|$: the Euclidian norm of x, where x is a vector. $\|A\|$: the Euclidian norm (the largest singular value or the spectral norm) of A where A is a matrix. $E(\cdot)$: take the mathematical expectation operation. $A>0$: A is a real symmetric positive definite matrix. $\rho(A)$: the spectral radius of A. $B^\perp$: the left annihilator of B is a matrix of maximal rank such that $B^\perp B = 0$.

2. MS-Strong Stability

In this paper, we consider the linear discrete-time stochastic system with multiplicative noises:

$$\begin{cases} x(k+1) = A_0x(k) + B_0u(k) + \sum_{i=1}^{N}[A_ix(k) + B_iu(k)]\omega_i(k), \\ x(0) = x_0 \in R^n \\ y(k) = Cx(k), \end{cases} \tag{1}$$

where $x(k){\in}R^n$ is an n-dimensional state vector, $u(k){\in}R^m$ is the control input, $y(k){\in}R^p$ is the measurement output and $\omega_i(k){\in}R$ are the sequences of real random variables which are defined on a complete probability space $(\Omega,\ \mathcal{F},\ \mu)$ and are independent wide sense stationary, second-order processes with $E[\omega_i(k)]{=}0$, $E[\omega_i(s)\omega_j(t)]{=}\delta_{ij}$ where δ_{ij} is a Kronecker function defined by $\delta_{ij}{=}1$ if $i{=}j$ while $\delta_{ij}{=}\ 0$ if $i{\neq}j$. x_0 is assumed to be deterministic and $A_i,\ B_i\ i = 0,\ 1,\cdots,\ N$ and C are constant matrices of appropriate dimension.

Definition 2.1. The autonomous system (1) (that is, $u(k){\equiv}0$) is said to be MS-strong stable, if it satisfies the following condition:

$$E\|x(k+1)\|^2 < E\|x(k)\|^2, \tag{2}$$

for all $k{\in}N_0$ and $x(k) \neq 0$.

Remark 2.1. Definition 2.1 shows that the state norm is monotonically decreasing from $E\|(0)\|^2{\neq}0$. Because system (1) is time-invariant, $E\|x(k)\|^2$ tends to zero when $k \to \infty$. Obviously, the MS-strong stability prohibits discrete-time stochastic systems to have overshoot behaviour for arbitrary initial conditions in state-space.

Remark 2.2. Definition 2.1 is different from MS-stability in [27]. The definition of MS-stability is as follows: The autonomous system (1) is said to be MS-stable if

$$\lim_{k\to\infty} E\|x(k)\|^2 = 0,$$

for $\forall x(0){\in}R^n$. According to the definition, MS-stability cannot guarantee systems to have good overshoot performance. This is because MS-stability only limits the change trend of the state when time tends to infinity, but does not restrain the state during the change process, which may produce the overshoot. Compared with MS-stability, the MS-strong stability is more strict constraint for systems' states, which effectively prohibits systems overshoot.

The following example will show that a MS-stable system may have very large overshoot.

Consider the autonomous system (1) as follows:

$$A_0 = \begin{bmatrix} 0.9 & -0.4 \\ 0 & 0.9 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0.2 & -0.6 \\ 0 & 0.2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.1 & -0.5 \\ 0 & 0.1 \end{bmatrix}.$$

According to Lemma 3 in [13],

$$\rho(\Lambda) = 0.86 < 1,$$

where

$$\Lambda = \sum_{i=0}^{N} A_i \otimes A_i,$$

therefore, the system is MS-stable. But it may have overshoot. We take $x(0) = [1 \ \ 1]'$ and the response of this system is shown in Figure 1.

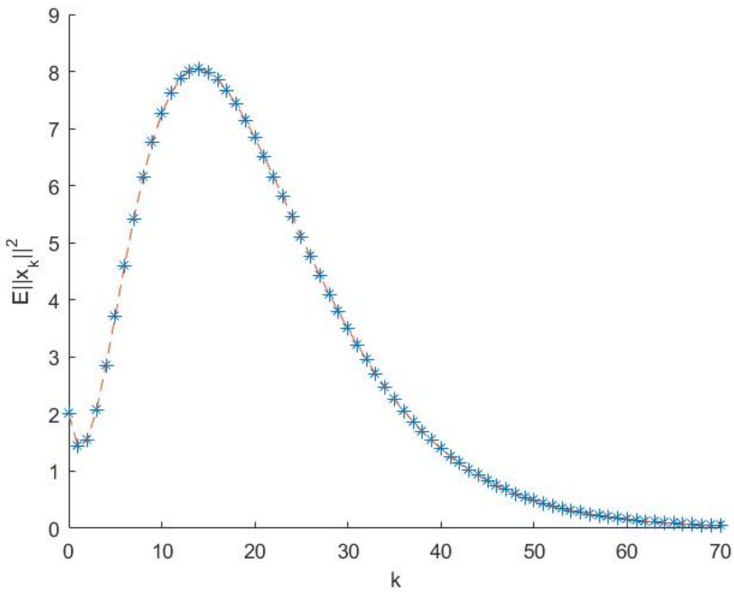

Figure 1: Responses of the autonomous system (1)

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But under the definition of MS-strong stability, the norm is strictly decreasing, so there will be no overshoot.

2.1. MS-Strong Stability Analysis

The following theorem will give some necessary and sufficient conditions for the MS-strong stability of the discrete-time stochastic system (1).

Theorem 2.1. The following three statements are equivalent:

(i) The autonomous system (1) is MS-strong stable.

(ii) $H > 0$, where

$$H = \begin{bmatrix} I & \mathscr{A}^T \\ \mathscr{A} & I \end{bmatrix} \tag{3}$$

and

$$\mathscr{A} = \begin{bmatrix} A_0^T & A_1^T & \cdots & A_N^T \end{bmatrix}^T,$$

and I is the identity matrix of the appropriate dimensions.

(iii)

$$\|\mathscr{A}\| < 1. \tag{4}$$

Proof. First, we prove the equivalence of (i) and (ii). *Necessity*: By the Definition 2.1, we can get that the MS-strong stability of the autonomous system (1) is equivalent to

$$E\|x(k+1)\|^2 - E\|x(k)\|^2 < 0. \tag{5}$$

Notice that the following sequences of inequalities:

$$\begin{aligned} & E\|x(k+1)\|^2 - E\|x(k)\|^2 < 0 \\ \Leftrightarrow\ & E[x^T(k)(\sum_{i=0}^{N} A_i^T A_i)x(k)] - E[x^T(k)x(k)] < 0 \\ \Leftrightarrow\ & E[x^T(k)(\sum_{i=0}^{N} A_i^T A_i - I)x(k)] < 0. \end{aligned} \tag{6}$$

Since $E[x^T(k)(\sum_{i=0}^{N} A_i^T A_i - I)x(k)] < 0$ holds for all $k \in N$, we have $E[x^T(0)(\sum_{i=0}^{N} A_i^T A_i - I)x(0)] < 0$ when $k=0$.

Because $x(0)$ is an n-dimensional deterministic initial state, there holds

$$x^T(0)(\sum_{i=0}^{N} A_i^T A_i - I)x(0) < 0. \tag{7}$$

Due to the arbitrariness of $x(0)$, (7) implies

$$\sum_{i=0}^{N} A_i^T A_i - I < 0.$$

Sufficiency: By the Schur's complement, we obtain that

$$H > 0 \Leftrightarrow \sum_{i=0}^{N} A_i^T A_i - I < 0. \tag{8}$$

For any $k \in N$, if $x(k) \neq 0$, then

$$\begin{aligned} & \sum_{i=0}^{N} A_i^T A_i - I < 0 \\ \Rightarrow\ & x(k)^T \Big(\sum_{i=0}^{N} A_i^T A_i - I\Big) x(k) < 0 \\ \Rightarrow\ & E\Big[x(k)^T \Big(\sum_{i=0}^{N} A_i^T A_i - I\Big) x(k)\Big] < 0 \\ \Rightarrow\ & E\|x(k+1)\|^2 - E\|x(k)\|^2 < 0. \end{aligned} \tag{9}$$

This ends the proof of the equivalence of (i) and (ii).

Next, we prove the equivalence of (ii) and (iii). Let us consider the sequences of equivalences:

$$\begin{aligned} & H > 0 \\ \Leftrightarrow\ & \sum_{i=0}^{N} A_i^T A_i - I < 0 \\ \Leftrightarrow\ & \begin{bmatrix} A_0^T & A_1^T & \dots & A_N^T \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_N \end{bmatrix} < I \\ \Leftrightarrow\ & \|\mathscr{A}\| < 1, \end{aligned} \tag{10}$$

which proves the equivalence of (ii) and (iii). In summary, the proof is completed. □

The next example will be used to verify MS-strong stability.

Example 2.1. Consider the autonomous system (1) with

$$A_0 = \begin{bmatrix} 0.2 & -0.2 \\ 0 & -0.5 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.8 & -0.2 \\ 0 & 0.4 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}.$$

By (ii) in Theorem 2.1, we obtain

$$\lambda(H) = \{0.09,\ 0.31,\ 1,\ 1,\ 1,\ 1,\ 1.69,\ 1.91\}$$

and $H > 0$. Therefore, the system is the MS-strong stable.

By (iii) in Theorem 2.1, it can be obtained that

$$\|\mathscr{A}\| = \left\| \begin{matrix} A_0 \\ A_1 \\ A_2 \end{matrix} \right\| = 0.91 < 1.$$

So the system is MS-strong stable.

We take $x(0) = [-2 \ \ 2]^T$ and the system response is shown in Figure 2.

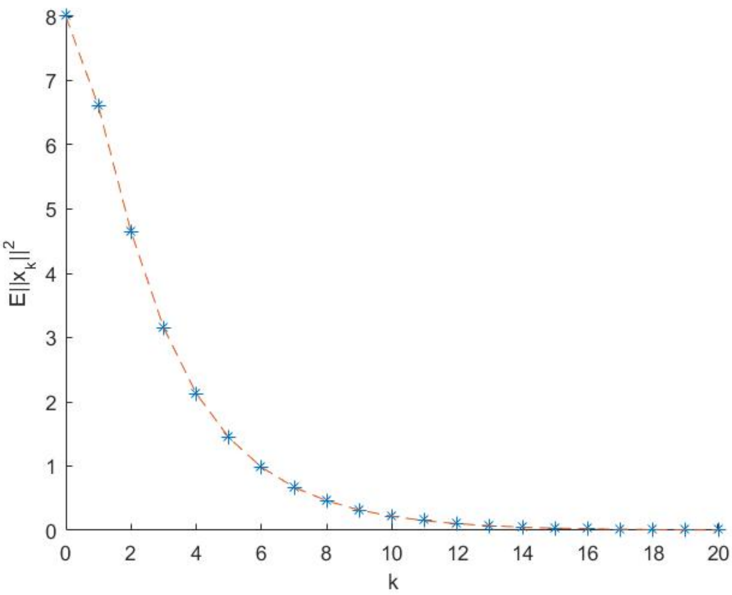

Figure 2: Responses of the autonomous system (1) in Example 2.1

2.2. Relationship between MS-strong stability and MS-stability

The MS-strong stability is a more strict constraint than MS-stability, which implies MS-stability.

Theorem 2.2. MS-strong stability of system (1) implies MS-stability of system (1).

Proof. First, let

$$M(k) \triangleq E\|x(k)\|^2 = E[x(k)^T x(k)]$$

and introduce a Lyapunov function

$$V(M(k)) = M(k) > 0.$$

By Definition 2.1, it can be seen that

$$V(M(k+1)) = M(k+1) < M(k) = V(M(k)). \tag{11}$$

Then a further standard argument from Lyapunov theorem gives that

$$\lim_{k\to\infty} E\|x(k)\|^2 = 0, \tag{12}$$

which proves the system is also MS-stable. □

Remark 2.3. Obviously, the system is MS-stable but not necessarily MS-strong stable. It is proved by that the states of the system is not necessarily strictly decreasing under definition of the MS-stability.

The following example will be used to show Theorem 2.2.

Example 2.2. Consider a system (1)($u(k)$=0) with

$$A_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}.$$

According to (iii) in Theorem 2.1, the system is the MS-strong stable since

$$\|\mathscr{A}\| = \left\| \begin{matrix} A_0 \\ A_1 \\ A_2 \end{matrix} \right\| = 0.99 < 1.$$

By Lemma 3 in [13], we obtain $\rho(\Lambda) = 0.91 < 1$. Therefore, the system is also MS-stable.

The following example shows that a system is the MS-stable but not MS-strong stable.

Example 2.3. Consider a system (1)($u(k)$=0) with

$$A_0 = \begin{bmatrix} 0.8 & -0.5 \\ 0.15 & 0.9 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & -0.8 \\ 0 & 0.1 \end{bmatrix}.$$

According to Lemma 3 in [13],

$$\rho(\Lambda) = 0.94 < 1,$$

therefore, the system is the MS-stable. But according to (iii) in Theorem 2.1, the system is not the MS-strong stable since

$$\|\mathscr{A}\| = \left\| \begin{matrix} A_0 \\ A_1 \\ A_2 \end{matrix} \right\| = 1.46 > 1.$$

We take $x(0) = [2\ \ 1]^T$ and the system response in Example 2.3 is shown in Figure 3.

3. MS-Strong Stabilization Via State Feedback

In this section, let us consider the problem of the MS-strong stabilization via state feedback.

Definition 3.1. System (1) is said to be MS-strong stabilizable, if there exists a SF controller $u(k)$=$Kx(k)$ such that

$$x(k+1) \quad = (A_0 + B_0K)x(k) + \sum_{i=1}^{N}(A_i + B_iK)x(k)\omega_i(k) \tag{13}$$

is MS-strong stable.

As a direct result of Theorem 2.1, we can get a necessary and sufficient condition for the MS-strong stabilization of system (13).

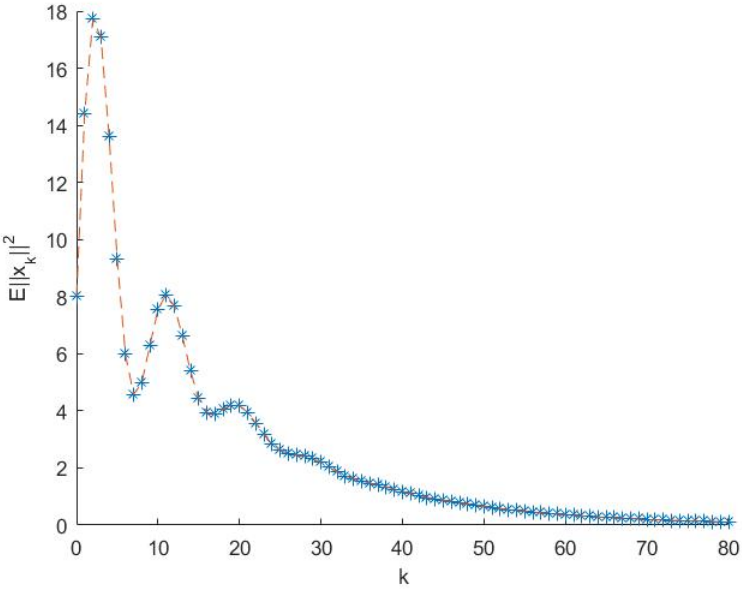

Figure 3: Responses of the autonomous system (1) in Example 2.3

Theorem 3.1. System (13) is MS-strong stabilization, if there is a SF gain matrix K such that

$$\|\mathscr{A} + \mathscr{B}K\| < 1, \tag{14}$$

where

$$\mathscr{A} = \left[A_0^T, A_1^T, \cdots, A_N^T\right]^T, \ \mathscr{B} = \left[B_0^T, B_1^T, \cdots, B_N^T\right]^T. \tag{15}$$

Next, we will give the analytical expression of state feedback gain K.

Theorem 3.2. Suppose that $\mathscr{B}$ has full column rank. Then, the system (13) is the MS-strong stabilization via state feedback if the following inequalities hold:

$$\begin{cases} \mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} > 0 \ \ (\text{if } \mathscr{B} \notin \mathrm{R}^{\mathrm{n}\times\mathrm{n}}), \\ \mathscr{B}\mathscr{B}^T > 0 \ \ (\text{if } \mathscr{B} \in \mathrm{R}^{\mathrm{n}\times\mathrm{n}}). \end{cases} \tag{16}$$

Furthermore, the SF matrix K can be obtained in the following form:

$$K = -(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A} + (\mathscr{B}^T\mathscr{B})^{-1/2}P\Theta^{1/2}, \tag{17}$$

where

$$\Theta = I_n - \mathscr{A}^T\mathscr{A} + \mathscr{A}^T\mathscr{B}(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A}, \tag{18}$$

and P is an arbitrary matrix such that $\|P\| < 1$.

Proof. The proof is divided into two steps.

Step 1: Let us prove that condition (16) is equivalent to $\Theta>0$. First $\Theta>0$ is equivalent to that there is a $U>0$ such that the following inequality sequences hold:

$$\begin{aligned} & I - \mathscr{A}^T\mathscr{A} + \mathscr{A}\mathscr{B}(\mathscr{B}^T\mathscr{B}+U)^{-1}\mathscr{B}^T\mathscr{A} > 0 \\ \Leftrightarrow\ & I - \mathscr{A}^T(I-\mathscr{B}(\mathscr{B}^T\mathscr{B}+U)^{-1}\mathscr{B}^T)\mathscr{A} > 0. \end{aligned} \tag{19}$$

By Matrix Inversion Lemma, we obtain that there is a $U>0$ such that

$$I - \mathscr{A}^T(I+\mathscr{B}U^{-1}\mathscr{B}^T)^{-1}\mathscr{A} > 0. \tag{20}$$

Then, by using Shur's complement lemma, it is also equivalent to that there is a $U > 0$ such that

$$I + \mathscr{B}U^{-1}\mathscr{B}^T - \mathscr{A}\mathscr{A}^T > 0. \tag{21}$$

In the end, according to Finsler's Theorem (pp. 42 of [28]), it is equivalent to

$$\mathscr{B}^{\perp}(I-\mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} > 0 \ \mathit{or}\ \mathscr{B}\mathscr{B}^T > 0. \tag{22}$$

Step 2: we only need to prove that $\Theta>0$ is equivalent to that the system is MS-stabilization by Theorem 3.1. *Necessity:* we first prove that the condition (14) implies $\Theta>0$. Consider the following sequences of the inequality:

$$\begin{aligned} & \|\mathscr{A}+\mathscr{B}K\| < 1 \\ \Leftrightarrow\ & (\mathscr{A}+\mathscr{B}K)^T(\mathscr{A}+\mathscr{B}K) < I \\ \Leftrightarrow\ & \mathscr{A}^T\mathscr{A} + \mathscr{A}^T\mathscr{B}K + K^T\mathscr{B}^T\mathscr{A} + K^T\mathscr{B}^T\mathscr{B}K < I \\ \Leftrightarrow\ & \mathscr{A}^T\mathscr{B}(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A} + \mathscr{A}^T\mathscr{B}K + K^T\mathscr{B}^T\mathscr{A} + K^T\mathscr{B}^T\mathscr{B}K \\ & < I - \mathscr{A}^T\mathscr{A} + \mathscr{A}^T\mathscr{B}(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A}. \end{aligned} \tag{23}$$

According to the definition of Θ in (18), (23) is equivalent to

$$(K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A})^T(\mathscr{B}^T\mathscr{B})(K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A}) < \ I - \mathscr{A}^T\mathscr{A} + \mathscr{A}^T\mathscr{B}(\mathscr{B}^T\mathscr{B})\mathscr{B}^T\mathscr{A} = \Theta. \tag{24}$$

Then, due to $\mathscr{B}$ has full column rank, we have $\mathscr{B}^T\mathscr{B} > 0$ and $(\mathscr{B}^T\mathscr{B})^{-1} > 0$. And since the left side of the above last inequality is nonnegative, $\Theta>0$ can be obtained.

Sufficiency: Due to $\Theta>0$, we have that there are $\Theta^{-\frac{1}{2}}$ and K such that (24) holds and is equivalent to the following inequality sequences:

$$\begin{aligned} & (K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A})^T(\mathscr{B}^T\mathscr{B})(K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A}) < \Theta \\ \Leftrightarrow\ & (K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A})^T((\mathscr{B}^T\mathscr{B})^T)^{\frac{1}{2}}(\mathscr{B}^T\mathscr{B})^{\frac{1}{2}}(K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A}) < (\Theta^T)^{\frac{1}{2}}\Theta^{\frac{1}{2}} \\ \Leftrightarrow\ & P^TP < 1, \quad (\text{Here, } P \triangleq (\mathscr{B}^T\mathscr{B})^{\frac{1}{2}}(K+(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A})\Theta^{-\frac{1}{2}}) \\ \Leftrightarrow\ & \|P\| < 1. \end{aligned} \tag{25}$$

Hence, it can be solved that:

$$K = -(\mathscr{B}^T\mathscr{B})^{-1}\mathscr{B}^T\mathscr{A} + (\mathscr{B}^T\mathscr{B})^{-1/2}P\Theta^{1/2}, \tag{26}$$

which ends the proof. □

Remark 3.1. In the Theorem 3.2, at the case of $A_i = 0$, $B_i = 0$, $i = 1,\ 2, \cdots,\ N$, $\mathscr{B}$ is a full rank square matrix. The system $x(k+1) = A_0x(k) + B_0u(k)$ is MS-strong stable $\Leftrightarrow$ $\mathscr{B}\mathscr{B}^T > 0$.

Remark 3.2. The matrix $\mathscr{B}^{\perp}$ can be obtained by the singular value decomposition. Let $\mathscr{B}$ has a singular value decomposition as follow:

$$\mathscr{B} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \tag{27}$$

where $\Sigma \in R^{m\times m}$ is a diagonal matrix composed by singular value and $U_1 \in R^{n\times m}$, $U_2 \in R^{n\times(n-m)}$ and $V \in R^{m\times n}$ are unitary matrix. Due to all left annihilators of $\mathscr{B}$ can be written as XU_2^T with the X representing arbitrary nonsingular matrix; therefore, $\mathscr{B}^{\perp}$ can be taken as U_2^T.

Remark 3.3. Theorem 3.2 gives the analytical form of state feedback controller under MS-strong stability, which results in higher accuracy and lower computational complexity. However, in [27], the state feedback controller with MS-stability needs to solve several matrix inequalities. When the dimension of the coefficient matrix of the system increases, it will greatly increase the complexity of calculation and reduce the accuracy of calculation.

Next, let us consider the MS-strong stabilization problem in Example 2.3 which is not MS-strong stable.

Example 3.1. Consider the state feedback control problem in Example 2.3, and take the discrete-time stochastic system (13) with:

$$A_0 = \begin{bmatrix} 0.8 & -0.5 \\ 0.15 & 0.9 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & -0.8 \\ 0 & 0.1 \end{bmatrix},$$

and

$$B_0 = B_1 = B_2 = I.$$

First, we test the condition (16) in Theorem 3.2. According to Remark 3.2, $\mathscr{B}$ has a singular value decomposition as follows:

$$U_1 = \begin{bmatrix} -0.58 & 0 \\ 0 & -0.58 \\ -0.58 & 0 \\ 0 & -0.58 \\ -0.58 & 0 \\ 0 & -0.58 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -0.58 & 0 & -0.58 & 0 \\ 0 & -0.58 & 0 & -0.58 \\ 0.79 & 0-0.21 & 0 & \\ 0 & 0.79 & 0 & -0.21 \\ -0.21 & 0 & 0.79 & 0 \\ 0 & -0.21 & 0 & 0.79 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 1.73 & 0 \\ 0 & 1.73 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, we take

$$\mathscr{B}^{\perp} = U_2^T$$

and obtain

$$\mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} = \begin{bmatrix} 0.89 & 0 & -0.12 & 0 \\ 0 & 0.85 & -0.13 & -0.19 \\ -0.12 & -0.13 & 0.76 & -0.15 \\ 0 & -0.19 & -0.15 & 0.76 \end{bmatrix},$$

which is a positive matrix because

$$\lambda(\mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T}) = \{1,\ 0.80,\ 0.46,\ 1\}.$$

Hence, by testing conditions that $\mathscr{B}$ has full column rank and

$$\mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} > 0,$$

in Theorem 3.2, the system satisfies conditions of MS-strong stabilization. Then we can take

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

without loss of generality and hence

$$\Theta = \begin{bmatrix} 0.70 & -0.16 \\ -0.16 & 0.56 \end{bmatrix}, \quad K = \begin{bmatrix} -0.13 & 0.57 \\ -0.08 & -0.19 \end{bmatrix}$$

by the (17), (18) in Theorem 3.2. And by Theorem 3.1, we can get

$$\|\mathscr{A} + \mathscr{B}K\| = 0.81 < 1.$$

Therefore, there is a SF controller K that makes the system MS-strongly stabilized.

We also take $x(0)$=[2 1]$'$ and the states of the closed-loop system in Example 3.1 is shown in Figure 4.

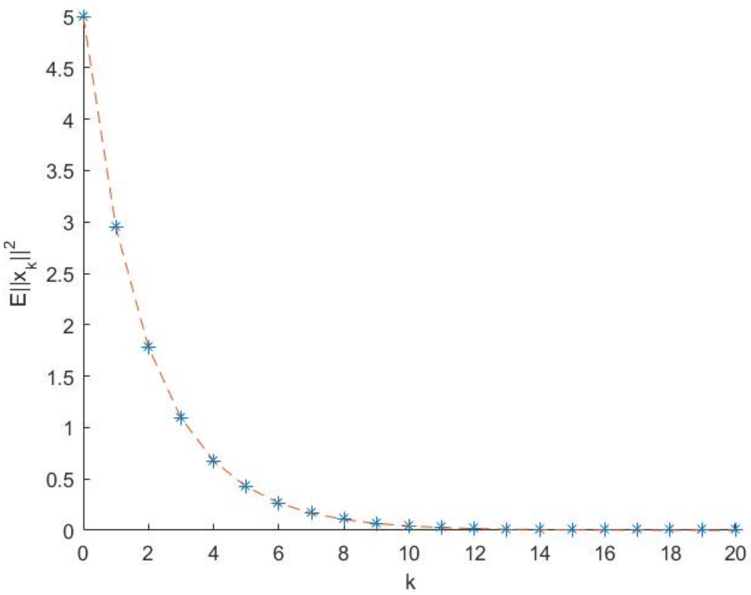

Figure 4: Responses of closed-loop system (13) in Example 3.1

4. MS-Strong Stabilization Via Output Feedback

4.1. Static Output Feedback

In this section, we consider the SOF problem of the discrete-time stochastic system (1). The system (1) is said to be MS-strongly stabilizable by SOF if there is a SOF controller

$$u(k) = Fy(k) = FCx(k),$$

such that

$$\begin{cases} x(k+1) = A_0x(k) + B_0FCx(k) + \sum_{i=1}^{N}(A_ix(k) + B_iFCx(k))\omega_i(k) \\ y(k) = Cx(k) \end{cases} \tag{28}$$

is MS-strong stable.

Similar to Theorem 2.1, we can get the following necessary and sufficient condition for MS-strong stabilization via static output feedback.

Theorem 4.1. System (28) is MS-strong stabilization, if there is a SOF gain matrix F such that

$$\|\mathscr{A} + \mathscr{B}FC\| < 1, \tag{29}$$

where $\mathscr{A}$ and $\mathscr{B}$ are given in (15).

The following theorem will give an analytical expression of SOF gain matrix F.

Theorem 4.2. Suppose that $\mathscr{B}$ has full column rank and that C has full row rank. Then, the system (28) is the MS-strong stabilization via SOF if the following conditions hold:

$$\text{(i)} \begin{cases} \mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} > 0 \quad (\text{if } \mathscr{B} \notin \mathrm{R^{n\times n}}), \\ \mathscr{B}\mathscr{B}^T > 0 \quad (\text{if } \mathscr{B} \in \mathrm{R^{n\times n}}), \end{cases} \tag{30}$$

and

$$\text{(ii)} \begin{cases} C^{T\perp}(I - \mathscr{A}^T\mathscr{A})C^{T\perp T} > 0 \quad (\text{if C} \notin \mathrm{R^{n\times n}}), \\ C^TC > 0 \quad (\text{if C} \in \mathrm{R^{n\times n}}). \end{cases} \tag{31}$$

The output feedback matrix F can be represented as follows:

$$F = -\,(\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1} + (\mathscr{B}^T\Phi\mathscr{B})^{-1/2}L\Psi^{1/2}, \tag{32}$$

where

$$\Phi = (I - \mathscr{A}\mathscr{A}^T + \mathscr{A}C^T(CC^T)^{-1}C\mathscr{A}^T)^{-1}, \tag{33}$$

$$\Psi = (CC^T)^{-1} - \{(CC^T)^{-1}C\mathscr{A}^T(\Phi - \Phi\mathscr{B}(\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi)\mathscr{A}C^T(CC^T)^{-1}\} \tag{34}$$

and L is an arbitrary matrix with appropriate dimensions such that $\|L\| < 1$.

Proof. The arguments are divided into two steps.

Step 1: Let us prove that the conditions (30) and (31) are equivalent to $\Phi>0$ and $\Psi>0$, respectively. First, $\Phi>0$ is equivalent to that there is a $V>0$ such that

$$I - \mathscr{A}\mathscr{A}^T + \mathscr{A}C^T(CC^T + V)^{-1}C\mathscr{A}^T > 0 \tag{35}$$

holds. Next, according to the Matrix Inversion Lemma, it is equivalent to that there is a $V>0$ such that

$$I - \mathscr{A}(I + C^TV^{-1}C)^{-1}\mathscr{A}^T > 0. \tag{36}$$

Then, by using Shur's complement lemma, it is also equivalent to that there is a $V>0$ such that

$$I + C^TV^{-1}C - \mathscr{A}^T\mathscr{A} > 0. \tag{37}$$

In the end, by the Finsler's Theorem(pp. 42 of [28]), it is equivalent to (30). Furthermore, the proof of (31) is similar to the above procedure.

Step 2: We only need to prove that $\Phi>0$ and $\Psi>0$ are the same things as (29). *Necessity*: By Theorem 4.1 and the properties of singular values, we will prove that the following inequality

$$\|(\mathscr{A} + \mathscr{B}FC)^T\| < 1 \tag{38}$$

implies $\Phi>0$ and $\Psi>0$. Hence, let us consider the following sequences of inequality:

$$\begin{aligned} & (\mathscr{A}+\mathscr{B}FC)(\mathscr{A}+\mathscr{B}FC)^T < I \\ \Leftrightarrow\ & \mathscr{A}\mathscr{A}^T + \mathscr{A}C^TF^T\mathscr{B}^T + \mathscr{B}FC\mathscr{A}^T + \mathscr{B}FCC^TF^T\mathscr{B}^T < I \\ \Leftrightarrow\ & \mathscr{B}FCC^TF^T\mathscr{B}^T + \mathscr{B}FC\mathscr{A}^T + \mathscr{A}C^TF^T\mathscr{B}^T + \mathscr{A}C^T(CC^T)^{-1}C\mathscr{A}^T \\ & < I - \mathscr{A}\mathscr{A}^T + \mathscr{A}C^T(CC^T)^{-1}C\mathscr{A}^T. \end{aligned} \tag{39}$$

According to the definition of Φ in (33), (39) is equivalent to

$$\begin{aligned} & (\mathscr{B}F + \mathscr{A}C^T(CC^T)^{-1})(CC^T)(\mathscr{B}F + \mathscr{A}C^T(CC^T)^{-1})^T \\ & < I - \mathscr{A}\mathscr{A}^T + \mathscr{A}C^T(CC^T)^{-1}C\mathscr{A}^T = \Phi^{-1}. \end{aligned} \tag{40}$$

Next, due to C has full row rank, we have $CC^T>0$ and $(CC^T)^{-1}>0$. Then according to Schur's complement lemma, (40) and $(CC^T)^{-1} > 0$ are equivalent to

$$(\mathscr{B}F + \mathscr{A}C^T(CC^T)^{-1})^T\Phi(\mathscr{B}F + \mathscr{A}C^T(CC^T)^{-1}) < (CC^T)^{-1} \tag{41}$$

and $\Phi>0$. And once again, we can compute this formula in a similar process to what we did before in (39)-(41), and we get that (41) is equivalent to

$$(F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1})^T(\mathscr{B}^T\Phi\mathscr{B})(F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1}) < \Psi. \tag{42}$$

Thus, $\Phi>0$ and $\Psi>0$ are true for any F satisfying (38).

Sufficiency: We suppose that $\Phi>0$ and $\Psi>0$. Hence, (42) holds and can be written as the following inequality by using the method in (25).

$$\begin{aligned} & (F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1})^T(\mathscr{B}^T\Phi\mathscr{B})(F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1}) < \Psi \\ \Leftrightarrow & \\ & (F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1})^T((\mathscr{B}^T\Phi\mathscr{B})^T)^{\frac{1}{2}}(\mathscr{B}^T\Phi\mathscr{B})^{\frac{1}{2}}(F+ \\ & (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1}) < (\Psi^T)^{\frac{1}{2}}\Psi^{\frac{1}{2}} \\ \Leftrightarrow & \\ & \|L\| < 1, \quad L = (\mathscr{B}^T\Phi\mathscr{B})^{\frac{1}{2}}(F + (\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1})\Psi^{-\frac{1}{2}}. \end{aligned} \tag{43}$$

Then, we can get

$$F = -(\mathscr{B}^T\Phi\mathscr{B})^{-1}\mathscr{B}^T\Phi\mathscr{A}C^T(CC^T)^{-1} + (\mathscr{B}^T\Phi\mathscr{B})^{-\frac{1}{2}}L\Psi^{-\frac{1}{2}}, \tag{44}$$

which proves the theorem. □

Remark 4.1. In Theorem 4.2, the dimension of the matrix L is determined by the dimensions of $\mathscr{B}$ and C. According to (43), the number of rows in L is the same as the number of columns in $\mathscr{B}$, and the number of columns in L is the same as the number of rows in C.

Example 4.1. Consider a system (28) with

$$A_0 = \begin{bmatrix} -0.15 & 1 \\ 0 & -0.06 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & -0.05 \\ 0.01 & 0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & -0.1 \\ 0 & 0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.5 & -0.6 & 1 \\ -0.6 & -0.06 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.3 & -0.1 & 0.3 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & -0.05 \\ 0 & 1 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T.$$

According to (iii) in Theorem 2.1, the the system is not the MS-strong stable by

$$\mathscr{A} = \left\| \begin{matrix} A_0 \\ A_1 \\ A_2 \end{matrix} \right\| = 1.04 > 1.$$

Next we design the static output feedback controller. First, we test the condition (16) in Theorem 3.2. According to Remark 3.2, $\mathscr{B}$ has a singular value decomposition as follow:

$$U_1^{\mathscr{B}} = \begin{bmatrix} -0.76 & -0.11 & -0.58 \\ -0.27 & 0.34 & 0.23 \\ -0.01 & -0.27 & -0.36 \\ 0.20 & 0.29 & -0.24 \\ 0.42 & -0.62 & -0.28 \\ 0.37 & 0.57 & -0.59 \end{bmatrix}, \quad U_2^{\mathscr{B}} = \begin{bmatrix} 0.11 & 0.20 & 0.16 \\ -0.31 & 0.51 & -0.63 \\ -0.26 & -0.59 & -0.62 \\ 0.82 & 0.05 & -0.38 \\ -0.05 & 0.58 & -0.11 \\ -0.39 & 0.07 & 0.19 \end{bmatrix},$$

$$\Sigma^{\mathscr{B}} = \begin{bmatrix} 1.54 & 0 & 0 \\ 0 & 1.17 & 0 \\ 0 & 0 & 0.84 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V^{\mathscr{B}} = \begin{bmatrix} 0.62 & -0.73 & -0.28 \\ 0.61 & 0.68 & -0.41 \\ -0.49 & -0.09 & -0.87 \end{bmatrix}.$$

Then, we take

$$\mathscr{B}^{\perp} = U_2^{\mathscr{B}T}$$

and obtain

$$\mathscr{B}^{\perp}(I - \mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T} = \begin{bmatrix} 0.95 & -0.04 & -0.07 \\ -0.04 & 0.97 & -0.05 \\ -0.07 & -0.05 & 0.89 \end{bmatrix},$$

which is a positive matrix because

$$\lambda(\mathscr{B}^{\perp}(I-\mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T})=\{0.82,\ 1,\ 1\}.$$

Similarly, C^T has a singular value decomposition as follows:

$$U_1^C=\begin{bmatrix}0.71\\0.71\end{bmatrix},\quad U_2^C=\begin{bmatrix}-0.71\\0.71\end{bmatrix},\quad \Sigma^C=\begin{bmatrix}1.4\\0\end{bmatrix},\quad V^C=1.$$

Then, we take

$$C^{T\perp}=U_2^{CT}$$

and obtain

$$C^{T\perp}(I-\mathscr{A}^T\mathscr{A})C^{T\perp T}=0.22>0.$$

Hence, by testing conditions that $\mathscr{B}$ has full column rank,

$$\mathscr{B}^{\perp}(I-\mathscr{A}\mathscr{A}^T)\mathscr{B}^{\perp T}>0,$$

and C has row column rank in Theorem 4.2, the system satisfies conditions of MS-strong stabilization. Then, we can take

$$L=\begin{bmatrix}0.4 & 0.6 & 0.5\end{bmatrix}^T$$

and obtain

$$F=\begin{bmatrix}0.23 & 0.38 & -0.02\end{bmatrix}^T$$

by (32), (33) in Theorem 4.2. And by Theorem 4.1, we can get

$$\|\mathscr{A}+\mathscr{B}FC\|=0.89<1.$$

Therefore, there exists a SOF controller $u(k)$=$Fy(k)$ that makes the system MS-strongly stabilized.

We take $x(0)=[3\ \ 2]^T$ and the response of the closed-loop system in Example 4.1 is shown in Figure 5.

4.2. Dynamic Output Feedback

In this subsection, we study the problem of DOF stabilization of discrete-time stochastic system.

In general, we consider the following DOF controller as follows:

$$\begin{aligned}x^c(k+1)&=\widehat{A}x^c(k)+\widehat{B}y(k)\\ u(k)&=\widehat{C}x^c(k)+\widehat{D}y(k),\end{aligned}\tag{45}$$

where the controller state x_k^c has the same dimension of x_k and $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$ are estimator gain matrices to be determined.

The following theorem illustrates the equivalent relationship between SOF controller and DOF controller.

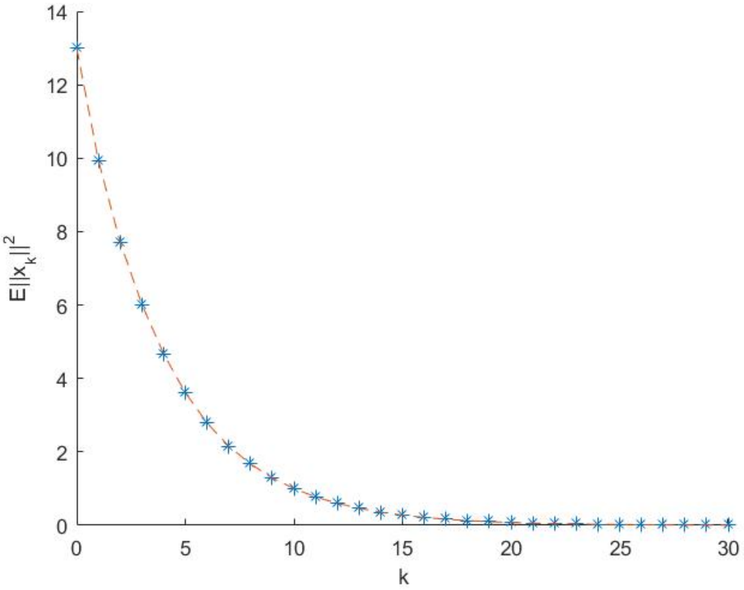

Figure 5: Responses of closed-loop system (28) in Example 4.1

Theorem 4.3. System (28) is the MS-strong stabilization via static output feedback $u(k)=Fy(k) = FCx(k)$ if and only if system (28) is the MS-strong stabilization via dynamic output feedback controller (45).

Proof. Sufficiency: We suppose that system (28) is the MS-strong stabilization via dynamic output feedback controller (45). Hence, the closed-loop system can be written as

$$\begin{bmatrix} x(k+1) \\ x^c(k+1) \end{bmatrix} = A^{c0} \begin{bmatrix} x(k) \\ x^c(k) \end{bmatrix} = \begin{bmatrix} \mathbb{A} & B_0\widehat{C} + \sum_{i=1}^{N} B_0\widehat{C}\omega_i(k) \\ \widehat{B}C & \widehat{A} \end{bmatrix} \begin{bmatrix} x(k) \\ x^c(k) \end{bmatrix}, \tag{46}$$

where $\mathbb{A} = A_0 + B_0\widehat{D}C + \sum_{i=1}^{N}(A_i + B_i\widehat{D}C)\omega_i(k)$.

By Definition 2.1 and using the method similar to the proof of Theorem 4.1 and letting $\eta(k) = [x^T(k)\ x^{cT}(k)]^T$, it can be obtained that

$$E\|\eta(k+1)\|^2 < E\|\eta(k)\|^2 \Leftrightarrow \|A^c\| < 1, \tag{47}$$

where

$$A^c = \begin{bmatrix} \mathscr{A} + \mathscr{B}\widehat{D}C & \mathscr{B}\widehat{C} \\ \widehat{B}C & \widehat{A} \end{bmatrix},$$

and definitions of $\mathscr{A}$ and $\mathscr{B}$ are the same as in (14). Then according to the properties of the singular value, it implies that

$$\|\mathscr{A} + \mathscr{B}\widehat{D}C\| < 1, \tag{48}$$

where a SOF stabilizer can be taken as $F = \widehat{D}$.

Necessity: If the system (28) is MS-strong stabilizable via SOF, according to Theorem 4.1 we can conclude that there is a SOF matrix F such that

$$\|\mathscr{A} + \mathscr{B}FC\| < 1.$$

Then, we can take $\widehat{D} = F$ and sufficiently small matrix $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, such that $\|A^c\| < 1$. Hence the system (28) is MS-strong stabilization via DOF controller (45) by the process of proof in (47). □

Remark 4.2. Under the MS-strong stability sense, the analytical solutions of the static output feedback controller and the dynamic output feedback controller are given, and also the equivalence between the static output feedback controller and dynamic output feedback controller is shown. However, under MS-stability sense, the static output feedback controller and the dynamic output feedback controller need to solve a set of matrix inequalities, and the two are not equivalent.

The following example illustrates the equivalence between SOF controller and DOF controller.

Example 4.2. Consider the DOF control problem in Example 4.1. According to Theorem 4.3, we know that there must be a DOF controller (45) that makes the system (46) MS-strongly stabilized. Furthermore, by this theorem we can take

$$\widehat{D} = F = \begin{bmatrix} 0.23, 0.38, -0.02 \end{bmatrix}^T,$$

and

$$\widehat{A} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \; \widehat{B} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \; \widehat{C} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \\ 0.1 & 0 \end{bmatrix}$$

without loss of generality. Then, we can obtain that

$$A^c = \begin{bmatrix} \mathscr{A} + \mathscr{B}\widehat{D}C & \mathscr{B}\widehat{C} \\ \widehat{B}C & \widehat{A} \end{bmatrix} = \begin{bmatrix} -0.51 & 0.64 & 0.05 & -0.18 \\ -0.16 & -0.21 & -0.06 & -0.02 \\ 0.32 & -0.03 & 0.06 & -0.03 \\ 0.20 & 0.29 & 0 & 0.15 \\ 0.43 & 0.13 & 0.1 & 0 \\ 0.38 & 0.48 & 0.01 & 0.3 \\ 0.10 & 0.10 & 0.20 & 0 \\ 0 & 0 & 0 & 0.10 \end{bmatrix}$$

and

$$\|A^c\| = 0.93 < 1.$$

Hence according to (47) of Theorem 4.3, there is a DOF controller such that the discrete-time stochastic system in Example 4.1 is the MS-stabilizable.

We take $x(0) = [3 \;\; 2]^T$ and $x^c(0) = [1 \;\; 1]^T$. The response of the closed-loop system in Example 4.2 is shown in Figure 6.

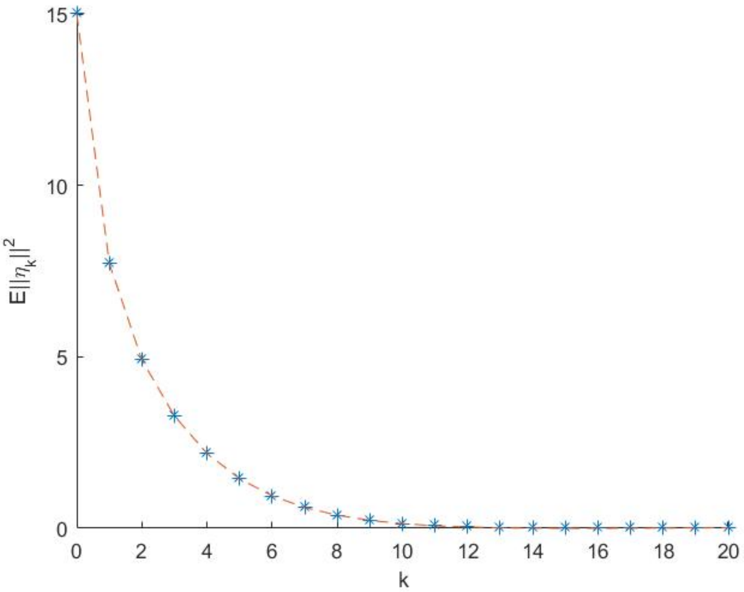

Figure 6: Responses of closed-loop system (46) in Example 4.2

5. Conclusions

In this paper, the definition of the MS-strong stability of discrete-time stochastic system has been proposed, and two necessary and sufficient conditions for the MS-strong stability have been obtained. Moreover, the relationship between the MS-strong stability and the MS-stability of the discrete-time stochastic systems have been revealed. Furthermore, the necessary and sufficient conditions guaranteeing the MS-strong stabilization via SF and SOF are derived. In addition, the analytical expressions of SF controller and SOF controller have been provided. Finally, the equivalence between the DOF and the SOF has been addressed.

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7. Conflict of Interest

The authors declare no potential conflict of interests.

8. Data Availability Statement

There is no data available to be shared.

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