Constrained Consensus in State-Dependent Directed Multiagent Networks

Yilun Shang

Abstract—**In this paper, constrained consensus of a group of continuous-time dynamical agents over state-dependent networks is investigated. The communication network, modulated by an asymmetric distance between agents, accommodates general directed information flows. Each agent proposes a comfortable range in a distributed manner, where they are inclined to agree on the final equilibrium state. Based on Lyapunov stability theory and robustness analysis, different conditions have been obtained to guarantee convergence within the common comfortable range when the network connectivity is fixed and time-varying. No global information is required in the proposed nonlinear control protocols. Furthermore, an opinion dynamics model has been introduced incorporating both social observer effect and bounded confidence phenomenon in the same state-dependent framework. Relaxed consensus conditions have been derived under certain symmetric assumptions. Finally, numerical examples have been presented to verify the effectiveness of the theoretical results.**

Index Terms—**Consensus, state-dependent communication, constraint, directed network, asymmetric distance.**

I. Introduction

DISTRIBUTED coordination of a system of interacting agents which cooperate with each other via a communication network has attracted considerable attention in many different fields. As a fundamental problem, consensus is to ensure that all agents in the network converge to a common value by designing a distributed strategy, which is called a consensus protocol [1]. The consensus problem has found a wide range of applications involving multiagent networks performing some collective tasks, such as flocking of birds or autonomous robots, vehicle platooning, attitude alignment and social network decision making. Since the influential theoretical works [1]–[3], a variety of linear and nonlinear consensus protocols have been designed to achieve desired consensus behaviors over fixed, switching and time-varying networks [4]–[6].

In most of the existing literature on consensus problems, the communication network among agents is assumed to be fixed or time-dependent, regulated by an external switching signal independent of the evolution of the agents. However, in practical scenarios, there are many examples where the network topology changes with the states of the agents. In swarming and flocking of birds, individuals update their headings by averaging their neighbors' headings within a certain Euclidean distance due to limited sensory capacity [7], [8]. A similar phenomenon, called homophily, is prevalent in social opinion dynamics, where individuals select their interacting neighbors by setting a threshold of their relative opinion difference [9]. Another type of state-dependent communication network resembles the law of gravity, where the weight of an interaction link weakens (but never disappears) as the distance between the states of the two agents increases. A well-known example is the Cucker-Smale model [10]. In wireless networks and mobile robotic systems, the link quality similarly deteriorates as the distance between two agents increases [11]. Besides, genome systems [12] are also regarded as state-dependent multiagent networks.

Although state-dependent multiagent networks are included in a general framework of time-varying communication graphs [13], [14], they have specific characteristics that are worth deep investigation. In fact, the communication topology is unknown a priori and cannot be solely chosen by individual agents in a state-dependent multiagent network. For example, food networks are not determined by individual species but rather by the ecological evolution and its interaction with environment; communication amongst a team of unmanned aerial vehicles is influenced by multiple factors including task objective, signal transmission mechanism, battery etc. The communication weights on edges are determined endogenously by the evolution of agents in the network, which is essentially different from time-varying graphs regulated by external signals. In [15], both continuous-time and discrete-time consensus problems have been studied over state-dependent undirected networks, where the communication weights are modulated by the distance between two adjacent agents. Sufficient conditions are proposed to guarantee consensus based upon the initial configuration of the multiagent network even when some edges may be broken due to large state differences. The work is extended to address finite-time consensus for single integrator agents over state-dependent undirected graphs containing some inherent communication links [16]. Based on the analysis of the spectrum and eigenvector of the system, the work [17] proposes a unifying framework for reaching arithmetic, geometric and harmonic average consensus, where the communication network is undirected and modulated by distance. A distributed convex optimization problem has been studied in [18] for a state-dependent undirected multiagent network. In [19], mean square consensus has been tackled for a multiagent system with white noises using event-trigged strategies. The weights of the links are state-dependent in their investigation allowing directed topology to satisfy detailed-balanced conditions. Dealing with consensus over general directed state-dependent networks is still a challenging problem. This is one of the main motivations of the current paper.

Y. Shang is with the Department of Computer and Information Sciences, Faculty of Engineering and Environment, Northumbria University, Newcastle upon Tyne, NE1 8ST, UK (e-mail: yilun.shang@northumbria.ac.uk).

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Another related issue in practical applications of consensus problems is the existence of different system state constraints. Cooperative agents in engineering or social systems often have a safe or desired range of operation [20]. Examples include the constraints of velocity and position of robots or vehicles in formation control, attitude zones for spacecraft in attitude alignment problems, and opinion or decision intervals posed by individuals in social networks. The literature for state constrained consensus problems is mainly along two different lines: hard constraints and soft constraints. The former is more restrictive in that it limits all transient states (often including the initial states) of agents within a region by using the state projections or barrier functions [21]–[24]. The latter allows the trajectories to trespass the constraint sets and only control the equilibrium points using robust consensus analysis [25]–[27]. Only state-independent networks have been examined in both categories. The constrained version of consensus is significantly more difficult than the unconstrained consensus due to the inherent nonlinearity of the constrained sets. Moreover, when the communication network becomes state-dependent, the presence of constraint sets will exert twofold influence upon both states and network topology. To our knowledge, this challenging problem has rarely been investigated in consensus literature.

In view of the above considerations, in this paper we introduce a theoretical framework for continuous-time dynamical agents to achieve constrained consensus in state-dependent communication networks. Each agent can have a constraint set that is only known to itself. Sufficient and necessary conditions are proposed to guarantee convergence when the communication network has fixed connectivity and time-varying connectivity. We further apply our results to a new class of opinion dynamics model featuring bounded confidence and observer effect. Our technique relies on the invariance principle and robustness analysis. The main novelties are summarized as follows. (1) Our framework works for both symmetric and asymmetric distance modulating functions incorporating the information of constraint sets. In the previous works (e.g. [15]–[19]) on state-dependent consensus protocols, only symmetric distance and unconstrained state are allowed. (2) Different from all the above works, the communication network considered here is a general directed graph and the detail-balanced condition [19] is not required. (3) The proposed consensus strategy is purely distributed. This is different from [17], where global information including eigenvectors of the system is essential. (4) The results fall into the category of soft constrained consensus, in which the initial states are permitted to sit outside of the individual constraint sets.

The rest of the paper is organized as follows. Some preliminaries and the model formulation is introduced in Section II. Constrained consensus over state-dependent networks is investigated in Section III for fixed connectivity and in Section IV for time-varying connectivity. In Section V, the application to an opinion dynamics model is presented. The simulation results are illustrated in Section VI. The conclusion is drawn in Section VII.

II. Problem formulation

A. Graph theory and system model

The underlying communication network of n interacting agents can be described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the node set $\mathcal{V} = \{1, 2, \cdots, n\}$ represents the agents and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ indicates the information flow among them. In particular, for a node $i \in \mathcal{V}$, the neighborhood of i is denoted by $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$, which consists of all agents that can send information to i. The adjacency matrix of $\mathcal{G}$ is denoted by $E = (e_{ij}) \in \mathbb{R}^{n\times n}$, where $e_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $e_{ij} = 0$ otherwise. Therefore, the number of neighbors (i.e. degree) for agent i is $|\mathcal{N}_i| = \sum_{j\in\mathcal{V}} e_{ij}$. A path from node i to node j is a sequence of distinct edges $(i, i_1), (i_1, i_2), \cdots, (i_k, j)$ in $\mathcal{E}$. The graph $\mathcal{G}$ is called strongly connected if for any pair of nodes i, j $(j \neq i)$, there is a path from i to j. $\mathcal{G}$ is quasi-strongly connected if there exists a node i (called root), which can reach any other node in $\mathcal{G}$. Strong connectivity implies every node is a root. If $\mathcal{G}$ is undirected, both strongly connectivity and quasi-strongly connectivity are equivalent to the connectivity. Namely, any node can be reached from any other node in a connected graph.

For each node $i \in \mathcal{V}$, let $x_i(t) \in \mathbb{R}$ be the state of i at time $t \geq 0$. The node i is associated with a constraint set $\theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, which can be chosen arbitrarily. Define a saturation function $\rho_i : \mathbb{R} \to \mathbb{R}$ by $\rho_i(z) = \underline{\theta}_i$ if $z < \underline{\theta}_i$, $\rho_i(z) = z$ if $\underline{\theta}_i \leq z \leq \overline{\theta}_i$, $\rho_i(z) = \overline{\theta}_i$ if $z > \overline{\theta}_i$. We assume each agent will transmit a disguised state to its neighbors by using the saturation function. The interval θ_i can be interpreted as, for example, comfortable opinion range in social interaction [28] and limited detection range or channel capacity for an agent in mobile networks [29].

To describe the dynamical evolution of agents, we introduce a non-negative non-increasing function $b : \mathbb{R} \to \mathbb{R}$ satisfying one of the assumptions below. Using b as a modulating function over the communication network, the weighted adjacency matrix can be defined as $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, where

$$a_{ij}(t) = e_{ij} b((x_i - \rho_j(x_j))^2). \tag{1}$$

The evolution of $x_i(t) \in \mathbb{R}$ is characterized by

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(x_j(t)) - x_i(t)), \tag{2}$$

where $i \in \mathcal{V}$ and $t \geq 0$. The saturation functions only action on the neighbors of agent i in (2). In the social interaction scenario, this means that a neighboring agent j tends to only express some mild opinions (i.e. avoid expressing extreme opinions) to agent i; see Section V for an application in opinion dynamics.

The communication weight in (1) depends on the network topology $\mathcal{G}$ and the state evolution of the system. As each neighbor j of i transmits the saturated information $\rho_j(x_j)$ to i, the function b effectively adjusts the network topology based on the distance between two adjacent states. The non-increasing property of b is used to characterize the coupling strength between nodes, which weakens as the distance between them increases. A discussion of higher-dimensional space can be found at the end of Section IV. We will also consider a specific form of b for opinion dynamics in Section V.

Remark 1. Due to the existence of the constraint set θ_i, the distance measure no longer exerts symmetric effect on agents i and j unless in the trivial case of $\theta_i = \mathbb{R}$ for all $i \in \mathcal{V}$. The previous analysis methods e.g. in [15], [16] are not applicable as they highly rely on the unconstrained conditions.

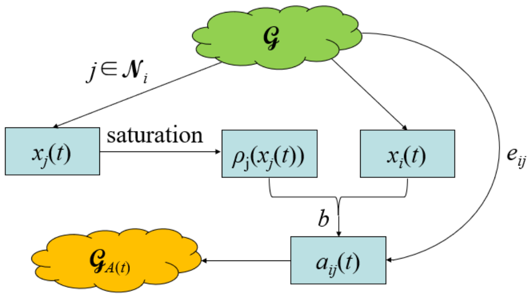

Fig. 1. Schematic of the system formulation.

Let $\mathcal{G}_{A(t)}$ be the graph commensurate with the weighted adjacency matrix $A(t)$ in the sense that $a_{ij}(t) > 0$ if and only if (j, i) is an edge in $\mathcal{G}_{A(t)}$. Clearly, $\mathcal{G}_{A(t)}$ is a subgraph of $\mathcal{G}$, regulated by the evolution of the system (2) as depicted in Fig. 1.

We assume the function b satisfies one of the following conditions.

Assumption 1. For any $z \geq 0$, $b(z) > 0$ and $b(0)$ is finite. Moreover, $b(z)$ is non-increasing and Lipschitz continuous.

Assumption 2. For any $z \geq 0$, $b(0)$ is finite, $b(z) > 0$ for $z < B$ and $b(z) = 0$ for $z \geq B$, where $B > 0$ is a constant. Moreover, $b(z)$ is non-increasing and Lipschitz continuous.

The property of b in Assumption 1 is similar to the law of gravity as the modulation function will remain positive while decline as the distance increases [14]. Assumption 2 has a cut-off point. When the difference is larger than the threshold, b becomes zero. An example of this type of b is discussed in opinion dynamics in Section V. We will see in Section IV that the parameter B is not a global information; it can be circumvented by using individual B_{ij} for $i, j \in \mathcal{V}$. The two scenarios will be analyzed in Sections III and IV, respectively.

Remark 2. The system (1) and (2) is put forward under the implicit assumption that a node can access its own true state. If this is not the case, the system may well be written as $\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(x_j(t)) - \rho_i(x_i(t)))$, where $a_{ij}(t) = e_{ij} b((\rho_i(x_i) - \rho_j(x_j))^2)$, $i, j \in \mathcal{V}$. The consensus behavior of such a system is much more straightforward to observe at least in the case of undirected network topology $\mathcal{G}$. The idea is sketched as follows. When $\mathcal{G}$ is undirected, by setting $\tilde{x}_i(t) = \rho_i(x_i(t))$, the model reduces to a symmetric diffusive system $\dot{\tilde{x}}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)\dot{\rho}_i(t)(\tilde{x}_j(t) - \tilde{x}_i(t))$, where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ is symmetric and $\dot{\rho}_i$ can be approximated by a smooth function with support set $[\underline{\theta}_i - \varepsilon, \overline{\theta}_i + \varepsilon]$ for some $\varepsilon \to 0^+$. The consensus then follows from the arguments for state-dependent consensus in [15] and a standard topology uncertainty result (e.g. [30]). To accommodate the intrinsic asymmetry as well as directedness in our system (1) and (2), a very different argument is required.

B. Robust consensus

Recall $\mathcal{G}_{A(t)}$ is a time-varying and state-dependent graph. Given a constant $\delta > 0$, an edge (j, i) is called an δ-edge of $\mathcal{G}_{A(t)}$ on time interval $[t_1, t_2)$ if $\int_{t_1}^{t_2} a_{ij}(t)dt \geq \delta$. We say $\mathcal{G}_{A(t)}$ is a quasi-strongly δ-connected graph if there is $\tau > 0$ such that for any $t \geq 0$, the δ-edges of $\mathcal{G}_{A(t)}$ on $[t, t+\tau)$ form a quasi-strongly connected graph. The following result is a special case of Theorem 4.1 in [31].

Lemma 1. *Let $\hat{t} \geq 0$ and $\delta > 0$. Consider the following system over $\mathcal{G}_{A(t)}$:*

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)(x_j(t) - x_i(t)) + y_i(t), \qquad i \in \mathcal{V}, \quad t \geq \hat{t}, \tag{3}$$

where $y_i(t)$ is piecewise continuous. Assume $\mathcal{G}_{A(t)}$ is quasi-strongly δ-connected. For any $\varepsilon > 0$, there is $\eta > 0$ such that if $\max_{i\in\mathcal{V}} \sup_{t\geq\hat{t}} |y_i(t)| \leq \eta$, then $\limsup_{t\to\infty} \max_{i,j\in\mathcal{V}} |x_i(t) - x_j(t)| \leq \varepsilon$.

III. State-dependent constrained consensus: Fixed connectivity

We consider the constrained consensus of the system (2), where the modulating function satisfies Assumption 1. This indicates the connectivity is dominated by the topology of $\mathcal{G}$ since b is always positive.

Denote by $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top \in \mathbb{R}^n$, where $\top$ represents the transpose. Let $\underline{\theta} = \max_{i\in\mathcal{V}} \underline{\theta}_i$ and $\overline{\theta} = \min_{i\in\mathcal{V}} \overline{\theta}_i$. Clearly, $\cap_{i\in\mathcal{V}}\theta_i = [\underline{\theta}, \overline{\theta}]$ if it is not empty. Our first result assumes that $\mathcal{G}$ is undirected.

Theorem 1. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (2). Suppose that $\mathcal{G}$ is undirected, connected and $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. Under Assumption 1, for any $x(0) \in \mathbb{R}^n$ there exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.*

Proof. We define two locally Lipschitz continuous functions $\phi(x(t)) = \max\{\max_{i\in\mathcal{V}} x_i(t), \overline{\theta}\}$ and $\psi(x(t)) = \min\{\min_{i\in\mathcal{V}} x_i(t), \underline{\theta}\}$. Let $V(x(t)) = \phi(x(t)) - \psi(x(t)) \geq 0$. To investigate the monotonicity of V and apply the Lyapunov theory, we first have a look at $\phi(x)$ along the system (2).

Suppose that $\phi(x(s)) > \overline{\theta}$ at some time $s \geq 0$. By the continuity, $\max_{i\in\mathcal{V}} x_i(t) > \overline{\theta}$ for $t \in [s, s+\varepsilon)$ for some $\varepsilon > 0$. Let $\mathcal{I}(t) = \{j : x_j(t) = \max_{i\in\mathcal{V}} x_i(t)\} \neq \emptyset$. For a continuous function $y(t) \in \mathbb{R}$, the upper right Dini derivative is defined as

$$\mathcal{D}^+ y(t) = \limsup_{\delta t\to 0+} \frac{y(t+\delta t) - y(t)}{\delta t}. \tag{4}$$

Differentiating $\phi(x(s))$ along the trajectory of (2) yields

$$\begin{aligned}\mathcal{D}^+\phi(x(s)) =& \mathcal{D}^+ \max_{i\in\mathcal{V}} x_i(s) = \max_{i\in\mathcal{I}(s)} \dot{x}_i(s) \\ =& \max_{i\in\mathcal{I}(s)} \sum_{j\in\mathcal{N}_i} a_{ij}(s)(\rho_j(x_j(s)) - x_i(s)).\end{aligned} \tag{5}$$

For any $\hat{i} \in \mathcal{I}(s)$, $x_{\hat{i}}(s) > \overline{\theta}$ and $x_{\hat{i}}(s) \geq x_j(s)$ for $j \in \mathcal{V}$. Since $\overline{\theta} \leq \overline{\theta}_j$ for $j \in \mathcal{V}$, we know that $\overline{\theta} < x_j(s)$ implies $x_j(s) \geq \rho_j(x_j(s))$ and that $\overline{\theta} \geq x_j(s)$ implies $\overline{\theta}_j \geq \rho_j(x_j(s))$ for $j \in \mathcal{V}$. Therefore, $\rho_j(x_j(s)) \leq x_{\hat{i}}(s)$.

Using (5) and Assumption 1 that b is a positive function, we have $\mathcal{D}^+\phi(x(s)) \leq 0$.

Suppose that $\phi(x(s)) = \overline{\theta}$ at some time s. By the definition of $\phi(x)$ and the above comment, we have $\phi(x(t)) = \overline{\theta}$ for $t \geq s$. Combining these know that $\phi(x(t))$ is non-increasing for $t \geq 0$. By applying the analogous argument to $\psi(x)$, we conclude that $\psi(x(t))$ is non-decreasing for $t \geq 0$. Consequently, $\mathcal{D}^+V(x(t)) \leq 0$ for $t \geq 0$.

Recall that $V(x(t))$ is locally Lipschitz continuous. If the vector field specified by (2) is written as $\dot{x}(t) = \varphi(x(t))$, the upper right Dini derivative of V along φ is

$$\mathcal{D}^+V(x) = \limsup_{\delta t \to 0+} \frac{V(x + \delta t \varphi(x)) - V(x)}{\delta t}. \tag{6}$$

It is known that [32, p.353] $\mathcal{D}^+V(x)|_{x=x(\hat{t})} = \mathcal{D}^+V(x(t))|_{t=\hat{t}}$. Let $\mathcal{S} = \{x \in \mathbb{R}^n : \mathcal{D}^+V(x) = 0\}$. We can show that $\mathcal{S} \subseteq [\underline{\theta}, \overline{\theta}]^n$. In fact, if this is not the case, we have some vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^\mathsf{T} \in \mathcal{S}$ and $\hat{x} \notin [\underline{\theta}, \overline{\theta}]^n$. Without loss of generality, assume that there is a node $i \in \mathcal{V}$ satisfying $\hat{x}_i = \max_{j \in \mathcal{V}} \hat{x}_j > \overline{\theta}$. Consider a solution $x(t)$ for the system (2) originating from $x(0) = \hat{x}$. Let $\hat{\mathcal{I}} = \{j \in \mathcal{V} : \hat{x}_j = \hat{x}_i\} \neq \emptyset$. By Assumption 1 and the connectivity of $\mathcal{G}$, the value of any node in $\hat{\mathcal{I}}$ will be dragged down by nodes outside of $\hat{\mathcal{I}}$ or the upper bound $\overline{\theta}$ as the system evolves. Hence, at some $s > 0$ we have $x_j(s) < \hat{x}_i$ for all $j \in \mathcal{V}$ since (2) is an averaging system and $\phi(x(t))$ is non-increasing. This means $\phi(x(s)) < \phi(x(0))$. However, any trajectory starting from $\hat{x} \in \mathcal{S}$ should remain in $\mathcal{S}$. We obtain a contradiction. This shows that $\mathcal{S} \subseteq [\underline{\theta}, \overline{\theta}]^n$ must hold true.

In view of LaSalle's invariance principle, the set of accumulation points should be contained in $\mathcal{S}$ and hence in $[\underline{\theta}, \overline{\theta}]^n$. This means $x(t) \to [\underline{\theta}, \overline{\theta}]^n$ as $t \to \infty$. By the definition of the constraint sets, for any $\varepsilon > 0$, there is some time $s > 0$ such that $|x_i(t) - \rho_i(x_i(t))| < \varepsilon$ for all $t \geq s$ and $i \in \mathcal{V}$. The multiagent system (2) can be cast as

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(t)(x_j(t) - x_i(t)) + y_i(t), \tag{7}$$

where

$$y_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\rho_j(x_j(t)) - x_j(t)) \tag{8}$$

for $t \geq 0$ and $i \in \mathcal{V}$. Hence, for any $\varepsilon > 0$, there is some time $s > 0$ such that $|y_i(t))| < \varepsilon$ for all $t \geq s$ and $i \in \mathcal{V}$. There exists a constant $c > 0$ satisfying $|x_i(t) - \rho_j(x_j(t))| \leq c$ for $t \geq 0$. By Assumption 1, $b((x_i(t) - \rho_j(x_j(t)))^2) \geq b(c^2) > 0$. Since $\mathcal{G}$ is connected, $\mathcal{G}_{A(t)}$ is quasi-strongly $b(c^2)$-connected. By Lemma 1, for any $\varepsilon > 0$, we have

$$\limsup_{t \to \infty} \max_{i,j \in \mathcal{V}} |x_i(t) - x_j(t)| \leq \varepsilon. \tag{9}$$

Letting $\varepsilon \to 0$ in (9), we know that $\lim_{t \to \infty} x_i(t) - x_j(t) = 0$ for $i, j \in \mathcal{V}$.

As we already know $\{x_i(t)\}_{t \geq 0}$ for any given $i \in \mathcal{V}$ is a bounded function with all possible accumulation points of it sit in $[\underline{\theta}, \overline{\theta}]$, let $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ be such an accumulation point (for a given i). If $\underline{\theta} = \overline{\theta}$, the theorem is proved. In the following, we assume $\underline{\theta} < \overline{\theta}$ since by assumption $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. For any $\varepsilon > 0$, there is some sufficiently large time $s > 0$ satisfying $|x_i(s) - \hat{c}| \leq \varepsilon$ for all $i \in \mathcal{V}$ by (9). We now proceed under three cases.

Case 1). Suppose that $\underline{\theta} < \hat{c} < \overline{\theta}$. In this case, we choose ε small enough so that $\underline{\theta} < \hat{c} - \varepsilon \leq x_j(s) \leq \hat{c} + \varepsilon < \overline{\theta}$ for all $j \in \mathcal{V}$. Define the consensus space $\mathcal{C} = \text{span}\{c 1_n : c \in \mathbb{R}\}$, where $1_n \in \mathbb{R}^n$ is the all-one vector. It is direct to check that the orthogonal projection of $x(t)$ over $\mathcal{C}$ is $\mathcal{P}_\mathcal{C}(x(t)) := \frac{1}{n}\big(\sum_{i \in \mathcal{V}} x_i(t)\big) 1_n$. Let $f(t) = x(t) - \mathcal{P}_\mathcal{C}(x(t)) \in \mathbb{R}^n$. If the limit of the Euclidean norm $\lim_{t \to \infty} \|f(t)\| = \lim_{t \to \infty} (f(t)^\mathsf{T} f(t))^{\frac{1}{2}} = 0$, Theorem 1 is proved in this case. What remains to show is

$$\lim_{t \to \infty} \|f(t)\| = 0. \tag{10}$$

Define a function $W(t) = \frac{1}{2} f(t)^\mathsf{T} f(t)$. Denote by $L(t) = (l_{ij}(t)) \in \mathbb{R}^{n \times n}$ the Laplacian matrix of $\mathcal{G}_{A(t)}$, where $l_{ij}(t) = -a_{ij}(t)$ for $i \neq j$ and $l_{ii}(t) = \sum_{j \in \mathcal{V}} a_{ij}(t)$. Note that $\mathcal{P}_\mathcal{C}(L(t)x(t)) = L(t)\mathcal{P}_\mathcal{C}(x(t))$. We obtain for $t \geq s$,

$$\begin{aligned}
\dot{W}(t) =& f(t)^\mathsf{T} \dot{f}(t) \\
=& (x(t) - \mathcal{P}_\mathcal{C}(x(t)))^\mathsf{T}(-L(t)x(t) + L(t)\mathcal{P}_\mathcal{C}(x(t))) \\
=& \Big(x(t) - \frac{1}{n}\Big(\sum_{i \in \mathcal{V}} x_i(t)\Big) 1_n\Big)^\mathsf{T} \\
& \cdot \Big(- L(t)x(t) + L(t)\frac{1}{n}\Big(\sum_{i \in \mathcal{V}} x_i(t)\Big) 1_n\Big) \\
=& - x(t)^\mathsf{T} L(t) x(t) \leq 0,
\end{aligned} \tag{11}$$

by using the system dynamics (2), the condition that $\underline{\theta} < x_i(s) < \overline{\theta}$ for all $i \in \mathcal{V}$ and $L(t)1_n = 1_n^\mathsf{T} L(t) = 0$. Hence, $\|f(t)\| \leq \|f(s)\|$ for $t \geq s$.

Let $\lambda(L(t)) \geq 0$ be the second smallest eigenvalue of $L(t)$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$ be the associated unit eigenvector. By direct calculation we know that $|x_i(t) - x_j(t)| \leq \sqrt{2}\|f(t)\| \leq \sqrt{2}\|f(s)\|$; see also [33, Lem. 1]. By the Rayleigh-Ritz theorem, we obtain

$$\begin{aligned}
\lambda(L(t)) =& \frac{1}{2} \sum_{i,j \in \mathcal{V}} e_{ij} b((x_i(t) - x_j(t))^2)(\gamma_i - \gamma_j)^2 \\
\geq& \frac{b(2\|f(s)\|^2)}{2} \sum_{i,j \in \mathcal{V}} e_{ij} (\gamma_i - \gamma_j)^2 \\
=& \frac{b(2\|f(s)\|^2)}{2} \gamma^\mathsf{T} L \gamma \\
\geq& \frac{b(2\|f(s)\|^2)}{2} \lambda(L),
\end{aligned} \tag{12}$$

where L is the Laplacian matrix of $\mathcal{G}$ and $\lambda(L)$ is the second smallest eigenvalue of L. By the connectivity of $\mathcal{G}$, we know that $\lambda(L) > 0$ and hence $\lambda(L(t))$ has a positive lower bound by (12). Let $H_0(L(t))$ represent the set of all eigenvectors of $L(t)$ associated with the zero eigenvalue (which is a simple eigenvalue because $\mathcal{G}_{A(t)}$ is connected in this case). A direct application of the Rayleigh-Ritz theorem shows that

$$x(t)^\mathsf{T} L(t) x(t) \geq \lambda(L(t)) \|x(t) - \mathcal{P}_{H_0(L(t))}(x(t))\|^2. \tag{13}$$

Since $\mathcal{G}_{A(t)}$ is connected, $H_0(L(t)) = \mathcal{C}$. In view of (11), (12) and (13), we have for $t \geq s$,

$$\begin{aligned}\dot{W}(t) =& - x^{\mathsf{T}}(t)L(t)x(t)\\ \leq& - \lambda(L(t))\|x(t) - \mathcal{P}_{\mathcal{C}}(x(t))\|^2\\ =& - \lambda(L(t))\|f(t)\|^2 < 0 \end{aligned} \tag{14}$$

when $\|f(t)\| \neq 0$. This indicates (10) must be true and this case is concluded.

Case 2). Suppose that $\hat{c} = \overline{\theta}$. In this case, we choose a small ε such that $\underline{\theta} < x_j(s) \leq \hat{c} + \varepsilon$ for all $j \in \mathcal{V}$. Since $\phi(x(t))$ is non-increasing and bounded from below by $\overline{\theta}$, $\phi(x(t))$ converges to some limit, say c', where $c' \geq \hat{c}$. Letting $\varepsilon \to 0$, we obtain that $c' = \hat{c}$. Applying the same argument in the beginning of this proof, we see that $\max_{i\in\mathcal{V}} x_i(t)$ is non-increasing for $t \geq s$ and hence has a limit. This limit must be c', and hence is $\hat{c}$. In light of (9), $\min_{i\in\mathcal{V}} x_i(t)$ also converges to $\hat{c}$, which concludes the proof in this case.

Case 3). Suppose that $\hat{c} = \underline{\theta}$. By symmetry, this case can be proved analogously as in Case 2). □

Remark 3. In Theorem 1 we assumed that $\mathcal{G}$ is undirected. This combined with Assumption 1 indicates that $\mathcal{G}_{A(t)}$ has a bidirectional topology, i.e., the two orientations of an edge may have different weights but they must appear and disappear simultaneously. Moreover, if the state-dependent weight between nodes i and j in (1) is redefined in a symmetric form

$$a_{ij}(t) = e_{ij} b((x_i - x_j)^2), \tag{15}$$

the above proof still applies. In this case, $\mathcal{G}_{A(t)}$ becomes undirected if $\mathcal{G}$ is undirected.

When $\mathcal{G}$ is directed, we will need to use the cut-balance condition proposed in [13]. Namely, there is a constant $c \geq 1$ such that for any $t \geq 0$ and $\emptyset \neq \mathcal{S} \subset \mathcal{V}$, the following condition holds:

$$\frac{1}{c} \sum_{i\in\mathcal{S}, j\notin\mathcal{S}} a_{ji}(t) \leq \sum_{i\in\mathcal{S}, j\notin\mathcal{S}} a_{ij}(t) \leq c \sum_{i\in\mathcal{S}, j\notin\mathcal{S}} a_{ji}(t). \tag{16}$$

Theorem 2. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (2). Suppose that $\mathcal{G}$ is directed, strongly connected and $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. If Assumption 1 holds and the cut-balance condition (16) holds for $\mathcal{G}_{A(t)}$, for any $x(0) \in \mathbb{R}^n$ there exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.*

Proof. Most part of the proof of Theorem 1 is applicable in the current situation. As $\mathcal{G}$ is strongly connected, the part essentially different from the above proof lies in Case 1). Here, $L(t)$ is no longer a symmetric matrix and the original approach is no longer applicable. However, it is shown in [13, Thm. 1] that the system

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(x_j(t) - x_i(t)), \qquad t \geq 0 \tag{17}$$

can achieve consensus as long the weights $\{a_{ij}(t)\}_{i,j\in\mathcal{V}}$ satisfy (16). Under the condition of Case 1), we consider the solution $x(t)$ starting from $t = s$ and readily conclude that $\lim_{t\to\infty} x_j(t) = \hat{c}$ for all $j \in \mathcal{V}$. This complete the proof in Case 1). The rest of the proof is largely the same. □

Remark 4. It's worth noting that the condition (16) is required for $\mathcal{G}_{A(t)}$. But under Assumption 1, this can be conveniently guaranteed by, for example, requiring the set symmetry condition on $\mathcal{G}$; c.f. [13, p.217]. Namely, for any subset $\mathcal{S} \subset \mathcal{V}$, there are two nodes $i \in \mathcal{S}$ and $j \notin \mathcal{S}$ satisfying $(i, j) \in \mathcal{E}$ if and only if there are two nodes $\hat{i} \in \mathcal{S}$ and $\hat{j} \notin \mathcal{S}$ satisfying $(\hat{j}, \hat{i}) \in \mathcal{E}$. Some simple examples satisfying the topology conditions in Theorem 2 is shown in Fig. 2. Also note that the detail-balanced condition proposed in [19] is a special case of the weighted average-preserving condition [13, Prop. 1], which further is a special case of (16).

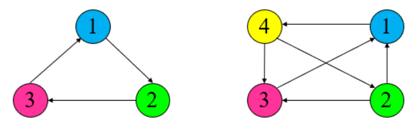

Fig. 2. Examples of strongly connected and set symmetric network $\mathcal{G}$.

Remark 5. The strongly connectivity condition of $\mathcal{G}$ in Theorem 2 cannot be generalized to quasi-strongly connectivity. Suppose that $\mathcal{G}$ is quasi-strongly connected with node i being the root. If $\mathcal{N}_i = \emptyset$ and the initial state $x_i(0) \notin [\underline{\theta}, \overline{\theta}]$, then the constrained consensus will fail since the state of i never changes. It is clear that the strongly connectivity of $\mathcal{G}$ is not only sufficient but a necessary condition, namely, any root node should be reachable.

IV. State-dependent constrained consensus: Time-varying connectivity

When the modulating function satisfies Assumption 2, the communication topology $\mathcal{G}_{A(t)}$ could be more sparse than the underlying topology of $\mathcal{G}$. This gives rise to a time-varying connectivity depending on the states of the system (2). Recall that $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathsf{T}} \in \mathbb{R}^n$, $\underline{\theta} = \max_{i\in\mathcal{V}} \underline{\theta}_i$ and $\overline{\theta} = \min_{i\in\mathcal{V}} \overline{\theta}_i$.

The following lemma characterizes the strong connectivity, which will be useful in the proof of Theorem 3.

Lemma 2. *If the number of ordered pairs of nodes not connected by a directed edge in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is less than $n-1$, then $\mathcal{G}$ is strongly connected.*

Proof. If $\mathcal{G}$ is not strongly connected, then we can divide the node set $\mathcal{V}$ into two nontrivial parts $\mathcal{V}_1$ and $\mathcal{V}_2$. Without loss of generality, we assume there is no edge from $\mathcal{V}_1$ to $\mathcal{V}_2$. Let $|\mathcal{V}_1| = k$ and $|\mathcal{V}_2| = n - k$ with $1 \leq k \leq n - 1$. Define a function $\varphi(k) = k(n - k)$ meaning the number of ordered pairs of nodes (i, j), where $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$, and $(i, j) \notin \mathcal{E}$. As the minimum of $\varphi(k)$ is attained at $k = 1$ or $k = n - 1$, we conclude that the number of ordered pairs of nodes not connected by a directed edge in $\mathcal{G}$ is at least $n - 1$. □

Theorem 3. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (2). Suppose that $\mathcal{G}$ is directed, strongly connected and $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. Assume that the cut-balance condition (16) holds for $\mathcal{G}_{A(t)}$ and*

$$\sum_{i,j\in\mathcal{V}} \int_0^{(x_i(0)-\rho_j(x_j(0)))^2} e_{ij} b(s) ds < (n-1) \int_0^B b(s) ds, \tag{18}$$

where B is given in Assumption 2. Under Assumption 2, for any $x(0) \in \mathbb{R}^n$ there exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.

Proof. We can follow the similar argument in Theorem 1 by defining two locally Lipschitz continuous functions $\phi(x(t)) = \max\{\max_{i\in\mathcal{V}} x_i(t), \overline{\theta}\}$ and $\psi(x(t)) = \min\{\min_{i\in\mathcal{V}} x_i(t), \underline{\theta}\}$. As in Theorem 1, we can show that the upper right Dini derivatives $\mathcal{D}^+V\phi(x(t)) \leq 0$ and $\mathcal{D}^+\psi(x(t)) \geq 0$ for $t \geq 0$. Consequently, define the Lyapunov candidate $V(x(t)) = \phi(x(t)) - \psi(x(t))$ and $\mathcal{D}^+V(x(t)) = \mathcal{D}^+\phi(x(t)) - \mathcal{D}^+\psi(x(t)) \leq 0$ for $t \geq 0$.

Define the set $\mathcal{S} = \{x \in \mathbb{R}^n : \mathcal{D}^+V(x) = 0\}$. We will show by a contradiction argument that $\mathcal{S} \subseteq [\underline{\theta}, \overline{\theta}]^n$. If this is not true, there is a vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^\mathsf{T} \in \mathcal{S}$ and $\hat{x} \notin [\underline{\theta}, \overline{\theta}]^n$. Without loss of generality, we assume that there exists a node $k \in \mathcal{V}$ satisfying $\hat{x}_k = \max_{j\in\mathcal{V}} \hat{x}_j > \overline{\theta}$. Consider a solution $x(t)$ for the multiagent system (2) starting from $x(0) = \hat{x}$. Let $\hat{\mathcal{I}} = \{j \in \mathcal{V} : \hat{x}_j = \hat{x}_k\} \neq \emptyset$. Recall $\mathcal{G}$ is strongly connected. We apply the energy function method [34] and define for $t \geq 0$,

$$U(x(t)) = \sum_{i,j\in\mathcal{V}} \int_0^{(x_i(t)-\rho_j(x_j(t)))^2} e_{ij} b(s)\mathrm{d}s. \tag{19}$$

Differentiating the function (19) using (1) and (2) gives rise to

$$\begin{aligned}\dot{U}(x(t)) =& 2\sum_{i,j\in\mathcal{V}} e_{ij} b((x_i(t) - \rho_j(x_j(t)))^2) \\ & \cdot (x_i(t) - \rho_j(x_j(t)))\big(\dot{x}_i(t) - \dot{x}_j(t)1_{\{x_j(t)\in[\underline{\theta}_j,\overline{\theta}_j]\}}\big) \\ =& 2\sum_{i,j\in\mathcal{V}} a_{ij}(t)(x_i(t) - \rho_j(x_j(t))) \\ & \cdot \big(\dot{x}_i(t) - \dot{x}_j(t)1_{\{x_j(t)\in[\underline{\theta}_j,\overline{\theta}_j]\}}\big) \\ =& -4\dot{x}(t)^\mathsf{T}\dot{x}(t) \leq 0. \end{aligned} \tag{20}$$

Therefore, for $t \geq 0$ we have

$$\begin{aligned} &\sum_{i,j\in\mathcal{V}} \int_0^{(x_i(t)-\rho_j(x_j(t)))^2} e_{ij} b(s)\mathrm{d}s \\ \leq& U(x(0)) \\ =& \sum_{i,j\in\mathcal{V}} \int_0^{(x_i(0)-\rho_j(x_j(0)))^2} e_{ij} b(s)\mathrm{d}s \\ <& (n-1)\int_0^B b(s)\mathrm{d}s, \end{aligned} \tag{21}$$

by using the condition (18). By Assumption 2, this indicates that there is less than $n-1$ pair of nodes that are not connected by a directed edge in $\mathcal{G}_{A(t)}$. In view of Lemma 2, $\mathcal{G}_{A(t)}$ is strongly connected for $t \geq 0$. Hence, the value of any node in $\hat{\mathcal{I}}$ will be dragged down by nodes outside of $\hat{\mathcal{I}}$ or the upper bound $\overline{\theta}$ as the system evolves. At some $s > 0$ we have $x_j(s) < \hat{x}_i$ for all $j \in \mathcal{V}$ since (2) is an averaging system and $\phi(x(t))$ is non-increasing. Therefore, $\phi(x(s)) < \phi(x(0))$. On the other hand, any trajectory starting from $\hat{x} \in \mathcal{S}$ would remain in $\mathcal{S}$. This leads to a contradiction. We then conclude $\mathcal{S} \subseteq [\underline{\theta}, \overline{\theta}]^n$ as desired.

By LaSalle's invariance principle, the set of accumulation points is contained in $\mathcal{S}$. Hence, $x(t) \to [\underline{\theta}, \overline{\theta}]^n$ as $t \to \infty$. For any $\varepsilon > 0$, there is some time $s > 0$ such that $|x_i(t) - \rho_i(x_i(t))| < \varepsilon$ for all $t \geq s$ and $i \in \mathcal{V}$. We rewrite the system (2) as

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(x_j(t) - x_i(t)) + y_i(t), \tag{22}$$

where

$$y_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(x_j(t)) - x_j(t)) \tag{23}$$

for $t \geq 0$ and $i \in \mathcal{V}$. For any $\varepsilon > 0$, there is some time $s > 0$ such that $|y_i(t))| < \varepsilon$ for all $t \geq s$ and $i \in \mathcal{V}$. There exists a constant $c > 0$ satisfying $|x_i(t) - \rho_j(x_j(t))| \leq c$ for $t \geq 0$. Since $\mathcal{G}_{A(t)}$ is strongly connected, it must be quasi-strongly δ-connected for some $\delta > 0$. It follows from Lemma 1 that for any $\varepsilon > 0$, we obtain

$$\limsup_{t\to\infty} \max_{i,j\in\mathcal{V}} |x_i(t) - x_j(t)| \leq \varepsilon. \tag{24}$$

We obtain $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ for all $i, j \in \mathcal{V}$ by choosing $\varepsilon \to 0$ in (24).

As in the proof of Theorem 1, let $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ be an accumulation point of $\{x_i(t)\}_{t\geq 0}$ for a given i. If $\underline{\theta} = \overline{\theta}$, the theorem is proved. In the following, we assume $\underline{\theta} < \overline{\theta}$. For any $\varepsilon > 0$, there is some time $s > 0$ satisfying $|x_i(s) - \hat{c}| \leq \varepsilon$ for all $i \in \mathcal{V}$ by (24). We then consider three cases as in the proof of Theorem 1. The treatment of Case 1) is analogous to Theorem 2 thanks to the cut-balance condition. The Case 2) and Case 3) can be shown exactly as in Theorem 1. The proof is complete. □

Remark 6. Note that the above Remarks 4 and 5 are still valid here. Moreover, although a common B is chosen in Assumption 2, the above proof works in the same way by taking $B = \max_{i,j\in\mathcal{V}}\{B_{ij}\}$ if the modulating function b in (1) allows heterogeneous cut-offs. Therefore, the strategy proposed here is essentially distributed.

Remark 7. In addition to the directedness consideration, the current network topology is more general in that we allow a general underlying graph $\mathcal{G}$ while previous works such as [15], [16] require that $\mathcal{G}$ is a complete undirected graph.

In the above, the state space of agents is assumed to be one-dimensional. It may be interesting in some applications to consider a higher-dimensional space with $x_i \in \mathbb{R}^N$ and a hypercube constraint $\theta_i := \prod_{k=1}^N [(\underline{\theta}_i)_k, (\overline{\theta}_i)_k] \subset \mathbb{R}^N$, where $N \geq 1$, $i \in \mathcal{V}$, and $(\cdot)_k$ represents the k-th component of an N-dimensional vector. Define a higher-dimensional saturation function $\rho_i : \mathbb{R}^N \to \mathbb{R}^N$ by $(\rho_i(z))_k = (\underline{\theta}_i)_k$ if $(z)_k < (\underline{\theta}_i)_k$, $(\rho_i(z))_k = (z)_k$ if $(\underline{\theta}_i)_k \leq (z)_k \leq (\overline{\theta}_i)_k$, $(\rho_i(z))_k = (\overline{\theta}_i)_k$ if $(z)_k > (\overline{\theta}_i)_k$ for every $k = 1, 2, \cdots, N$. Our system takes the same format of (2) as

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(x_j(t)) - x_i(t)) := g_i(x(t)) \tag{25}$$

and the coupling weight is naturally given by

$$a_{ij}(t) = e_{ij} b(\|x_i - \rho_j(x_j)\|^2), \tag{26}$$

where $x = (x_1^\top, \cdots, x_n^\top)^\top \in \mathbb{R}^{nN}$. The previous argument towards consensus is no longer applicable here in general as there can be different nodes brought in the ranges $[\max_{i\in\mathcal{V}}(\underline{\theta}_i)_k, \min_{i\in\mathcal{V}}(\overline{\theta}_i)_k]$ for differen k, which invalidates the robustness analysis. Nevertheless, we can show that the system (25) is still cooperative in the following sense.

Definition 1. [35] The system (25) is called cooperative if $w_i^1 \preceq w_i^2$ for all $i \in \mathcal{V}$ implies $x_i^1(t) \preceq x_i^2(t)$ for all $i \in \mathcal{V}$ and $t \geq 0$, where $w_i^1 = ((w_i^1)_1, \cdots, (w_i^1)_N)^\top \in \mathbb{R}^N$, $w_i^2 = ((w_i^2)_1, \cdots, (w_i^2)_N)^\top \in \mathbb{R}^N$, $x_i^1(0) = w_i^1$ and $x_i^2(0) = w_i^2$. Here, $w_i^1 \preceq w_i^2$ means $(w_i^1)_k \leq (w_i^2)_k$ for all $1 \leq k \leq N$.

It is known that (25) is cooperative if the following Kamke-Muller condition [35, Thm. 12.11] holds: For any $i \in \mathcal{V}$, if $w^1 = (w_1^{1\top}, \cdots, w_n^{1\top})^\top \preceq w^2 = (w_1^{2\top}, \cdots, w_n^{2\top})^\top$ and $w_i^1 = w_i^2$, then $g_i(w^1) \preceq g_i(w^2)$. To show the Kamke-Muller condition, we fix $i \in \mathcal{V}$. Since $w^1 \preceq w^2$, we have $\rho_j(w_j^1) \preceq \rho_j(w_j^2)$ for all $j \in \mathcal{V}$ by the definition of the saturation function. Employing $a_{ij}(t) \geq 0$ and $w_i^1 = w_i^2$, we obtain

$$\begin{aligned} g_i(w^1) &= \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(w_j^1) - w_i^1) \\ &\preceq \sum_{j\in\mathcal{N}_i} a_{ij}(t)(\rho_j(w_j^2) - w_i^2) = g_i(w^2), \end{aligned} \tag{27}$$

which concludes the proof.

V. Application to an opinion model with bounded confidence and observer effect

In this section, we introduce an opinion dynamics model featuring bounded confidence and social observer effect. In opinion dynamics with scalar opinion space, each individual holds a value, called opinion. Individuals tend to only interact and compromise with others only if their opinion difference is less than a threshold. This phenomenon is often known as an example of social homophily theory [36] and has been studied extensively in opinion dynamics literature under the name 'bounded confidence' during the last few decades [9], [34]. Another well-known psychological phenomenon affecting social opinion expression is the observer effect or Hawthorne effect, which recognizes the behavioral change of individual in social interactions in the presence of observers [37], [38]. In these situations, individuals tend to express neutral/mild opinions within certain boundaries. Although there have been numerous empirical results, analytical model has been recently introduced in [28] by incorporating a comfortable range for individual's expression opinion.

To incorporate both the bounded confidence and observer effect in our state-dependent multiagent networks, we consider the following dynamics for each agent $i \in \mathcal{V}$:

$$\begin{aligned} \dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} & e_{ij} 1_{\{|x_i(t)-\rho_j(x_j(t))|<D\}} \\ & \cdot (\rho_j(x_j(t)) - x_i(t)) \end{aligned} \tag{28}$$

for $t \geq 0$. Here, e_{ij} represents again the communication weight of the original network topology $\mathcal{G}$ and D is the bounded confidence threshold, where the two agents interact with each other only when their opinion difference is less than D. The saturation function is defined as in Section II to provide a comfortable range $[\underline{\theta}_i, \overline{\theta}_i]$ for individual i modelling the observer effect. For simplicity, we write the indicator function in (28), which strictly speaking should be replaced by a Lipschitz continuous function approximating the indicator function. In this section, we stick to this notation for ease of presentation.

It is easy to see that Assumption 2 for the modulating function is satisfied. The following result is an immediate corollary of Theorem 3.

Corollary 1. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (28). Suppose that $\mathcal{G}$ is strongly connected and $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. Assume that the cut-balance condition (16) holds for $\mathcal{G}_{A(t)}$ and*

$$\sum_{i,j\in\mathcal{V}} e_{ij} \cdot \min\{D^2, (x_i(0) - \rho_j(x_j(0)))^2\} < (n-1)D^2. \tag{29}$$

For any $x(0) \in \mathbb{R}^n$ there exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.

Proof. This can be obtained by using Theorem 3 and taking $b(s) = 1_{\{s<D^2\}}$ and $D^2 = B$. □

The condition (29) can be further relaxed if the multiagent system (28) satisfies the following symmetric condition.

Assumption 3. $\underline{\theta}_i = \underline{\theta}$ and $\overline{\theta}_i = \overline{\theta}$ for all $i \in \mathcal{V}$. There exists a constant c such that the initial states satisfy $x_i(0) + x_j(0) = c$ for any $i, j \in \mathcal{V}$ and $i + j = 1 + n$.

To establish the constrained consensus result, we need the follow lemma regarding network connectivity.

Lemma 3. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (28). Suppose that $\mathcal{G}$ is a complete graph and $n \geq 4$. Under Assumption 3, for any $t \geq 0$ if the number of ordered pairs of nodes not connected by a directed edge in $\mathcal{G}_{A(t)}$ is less than $2n - 3$, then $\mathcal{G}_{A(t)}$ is strongly connected.*

Proof. We first claim that $x_i(t)+x_j(t) = c$ for any $i+j = 1+n$ and $t \geq 0$. In fact, fix $t \geq 0$. For any $i + j = 1 + n$, using (28) and the condition that $\mathcal{G}$ is complete, we obtain

$$\begin{aligned} \dot{x}_i(t) + \dot{x}_j(t) = & \sum_{k\in\mathcal{N}_i} a_{ik}(t)(\rho_k(x_k(t)) - x_i(t)) \\ & + \sum_{l\in\mathcal{N}_j} a_{jl}(t)(\rho_l(x_l(t)) - x_j(t)), \end{aligned} \tag{30}$$

where $a_{ik}(t) = 1_{\{|x_i(t)-\rho_k(x_k(t))|<D\}}$ and $a_{jl}(t) = 1_{\{|x_j(t)-\rho_l(x_l(t))|<D\}}$. Let $\mathcal{N}_i(t)$ represent the neighborhood of agent i at time t in $\mathcal{G}_{A(t)}$. If the state symmetric condition in Assumption 3 holds at time t, namely, $x_i(t) + x_j(t) = c$ for $i + j = 1 + n$, then the neighborhoods of i and j are symmetric, i.e., for any $k \in \mathcal{N}_i(t)$ there is a $l \in \mathcal{N}_j(t)$ satisfying $k + l = 1 + n$. Moreover, the right-hand side of (30) satisfies $a_{ik}(t) = a_{jl}(t)$. This further implies $\dot{x}_i(t)+\dot{x}_j(t) = 0$. Therefore, if we start from $t = 0$, then Assumption 3 guarantees that $x_i(t) + x_j(t) = c$ for $i + j = 1 + n$ holds for all $t \geq 0$.

Fix $t \geq 0$ and suppose $\mathcal{G}_{A(t)}$ is not strongly connected. Since the states are always symmetric, the node set $\mathcal{V}$ can be divided into three nontrivial parts $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ with $|\mathcal{V}_1| = k$, $|\mathcal{V}_2| = n-2k$, and $|\mathcal{V}_3| = k$, where $k \geq 1$ We first consider $\mathcal{V}_1$

and $\mathcal{V}_2$. Without loss of generality, we assume there is no edge from $\mathcal{V}_1$ to $\mathcal{V}_2$. Define a function $\varphi(k) = k(n-2k)$ meaning the number of ordered pairs of nodes (i, j), where $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$, and (i, j) is not an edge in $\mathcal{G}_{A(t)}$. The minimum of $\varphi(k)$ is $\varphi(1) = n - 2$. Similarly, we can consider $\mathcal{V}_2$ and $\mathcal{V}_3$, which contribute $n-2$ ordered pairs. Considering $\mathcal{V}_1$ and $\mathcal{V}_3$, we obtain 1 ordered pair. Combining them together, the number of ordered pairs of nodes not connected by a directed edge in $\mathcal{G}_{A(t)}$ is at least $(n-2)+(n-2)+1 = 2n-3$. This proves Lemma 3. □

Theorem 4. *Consider the multiagent network $\mathcal{G}$ with the dynamics in (28). Suppose that $\mathcal{G}$ is a complete graph and $[\underline{\theta}, \overline{\theta}] \neq \emptyset$. Under Assumption 3, we have the following.*

(1) When $n = 2$ and $n = 3$, $\mathcal{G}_{A(0)}$ is strongly connected if and only if there exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.

(2) When $n \geq 4$, assume that the cut-balance condition (16) holds for $\mathcal{G}_{A(t)}$ and

$$\sum_{i,j\in\mathcal{V}} \min\{D^2, (x_i(0) - \rho_j(x_j(0)))^2\} < (2n-3)D^2. \quad (31)$$

There exists some $\hat{c} \in [\underline{\theta}, \overline{\theta}]$ satisfying $\lim_{t\to\infty} x_i(t) = \hat{c}$ for any $i \in \mathcal{V}$.

Proof. When $n = 2$, without loss of generality we assume $x_1(0) \leq x_2(0)$. Assume $\mathcal{G}_{A(0)}$ is strongly connected. If $\underline{\theta} \leq x_1(0) \leq x_2(0) \leq \overline{\theta}$, $\dot{x}_1(t) = 1_{\{|x_1(t)-\rho_2(x_2(t))|<D\}} \cdot (\rho_2(x_2(t)) - x_1(t)) \geq 0$. This implies $x_1(t)$ is non-decreasing. Similarly, $x_2(t)$ is non-increasing. Moreover, $x_2(t) - x_1(t) \geq 0$ holds for all $t \geq 0$. Therefore, $\mathcal{G}_{A(t)}$ is strongly connected for all $t \geq 0$. Using the same proof of Theorem 3, we obtain the constrained consensus. For the other scenarios of $x_1(0)$, $x_2(0)$, $\underline{\theta}$ and $\overline{\theta}$, the same proof also holds true.

On the other hand, suppose $\mathcal{G}_{A(0)}$ is not strongly connected. Without loss of generality, we assume $|x_1(0) - \rho_2(x_2(0))| \geq D$. Thus $\dot{x}_1(t) = 0$. We choose $x_1(0) \notin [\underline{\theta}, \overline{\theta}]$. Clearly, constrained consensus can not be achieved.

When $n = 3$, without loss of generality we assume $x_1(0) \leq x_2(0) \leq x_3(0)$. Hence, Assumption 3 implies $x_2(0) = (x_1(0) + x_3(0))/2$. The system dynamics and the analysis in Lemma 3 indicate $\dot{x}_2(t) = 0$ for $t \geq 0$. Moreover, $\dot{x}_1(t) = 1_{\{|x_1(t)-\rho_2(x_2(t))|<D\}} \cdot (\rho_2(x_2(t)) - x_1(t)) \geq 0$. This implies $x_1(t)$ is non-decreasing. Similarly, $x_2(t)$ is non-increasing. As in the case $n = 2$, we obtain the constrained consensus.

On the other hand, suppose $\mathcal{G}_{A(0)}$ is not strongly connected. Without loss of generality, we assume $|x_1(0) - \rho_2(x_2(0))| \geq D$. Thus $\dot{x}_1(t) = 0$. Since we already know that $\dot{x}_2(t) = 0$, constrained consensus can not be achieved.

When $n \geq 4$, the result can be shown analogously as Theorem 3 but now by virtue of Lemma 3. □

Remark 8. In Theorem 4, although $\mathcal{G}$ is required to be a complete graph, the actual communication topology $\mathcal{G}_{A(t)}$ can be sparser as it is regulated by the states of agents. This helps highlight the relevance of the proposed model since not each pair of agents in realistic networks will exchange information even the two agents are physically connected. We also note that this restrictive condition on the network topology is employed to relax the initial opinion configuration (29) in Corollary 1.

VI. Numerical simulations

In this section, we present some numerical examples to illustrate the theoretical results.

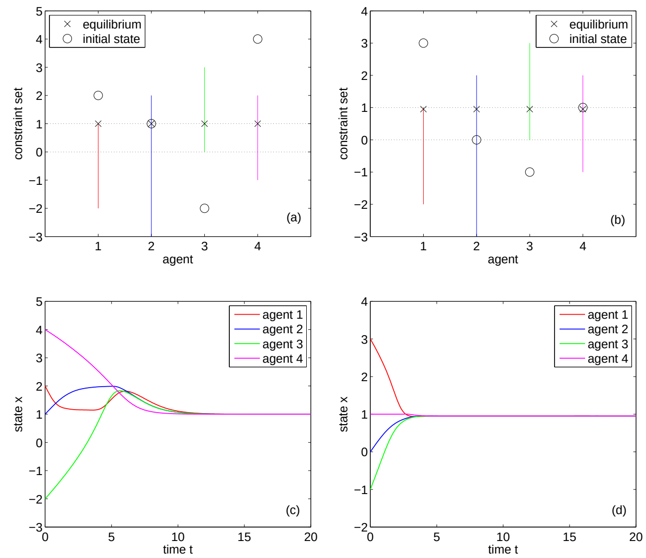

Fig. 3. Constrained consensus for Example 1: Constraint sets with initial and final states indicated by circles and crosses, respectively, for initial condition (a) $x(0) = (2, 1, -2, 4)^\mathsf{T}$ and (b) $x(0) = (3, 0, -1, 1)^\mathsf{T}$. The state trajectories in the case (a) are shown in (c) and the state trajectories in the case (b) are shown in (d).

Example 1. In this example we consider the constrained consensus with fixed connectivity. Consider the network $\mathcal{G}$ with the node set $\mathcal{V} = \{1, 2, 3, 4\}$ as shown in the right panel of Fig. 2. Take $\theta_1 = [-2, 1]$, $\theta_2 = [-3, 2]$, $\theta_3 = [0, 3]$, and $\theta_4 = [-1, 2]$. Hence, the intersection is $[\underline{\theta}, \overline{\theta}] = [0, 1]$. We choose the modulating function following the Cucker-Smale model as $b(s) = 1/(1+s)$ for $s \geq 0$. It is direct to check that the conditions in Theorem 2 are satisfied. We show the state evolution of the multiagent network (2) in Fig. 3(c) with initial condition $x(0) = (2, 1, -2, 4)^\mathsf{T}$ (see Fig. 3(a)) and Fig. 3(d) with initial condition $x(0) = (3, 0, -1, 1)^\mathsf{T}$ (see Fig. 3(b)). As one would expect from Theorem 2 that constrained consensus has been achieved.

Several observations are worth remarking. Firstly, in the case (a) the final consensus value is at 1 but in the case (b) the final consensus value is at 0.952. This means that the equilibrium can be in the interior of the set $\cap_{i\in\mathcal{V}}[\underline{\theta}_i, \overline{\theta}_i]$ as well as on its boundary. Secondly, the trajectories of the agents are allowed to trespass their respective constraint sets. For example, the initial state of agent 1 is outside the interval $[-2, 1]$ in both cases. Thirdly, the evolution does not need to be monotonic. For example, in the case (a), the agent 2 starts from $x_2(0) = 1$ and even ends in the same state but the transient trajectory deviates from 1.

Example 2. Next, we consider the constrained consensus with time-varying connectivity. Using the same network $\mathcal{G}$ as in Example, we redefine the modulating function as $b(s) = 1 - s/3$ for $s \leq 3$ and $b(s) = 0$ for $s > 3$. Choose the same initial configuration $x(0) = (3, 0, -1, 1)^\mathsf{T}$ as in Example 1. It is direct to check that the conditions in Theorem 3 are satisfied.

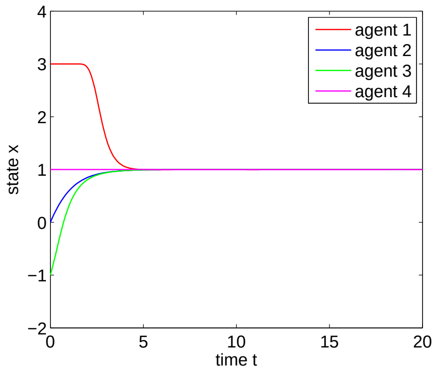

Fig. 4. Constrained consensus for Example 2: The state trajectories under the initial condition $x(0) = (3, 0, -1, 1)^\mathsf{T}$.

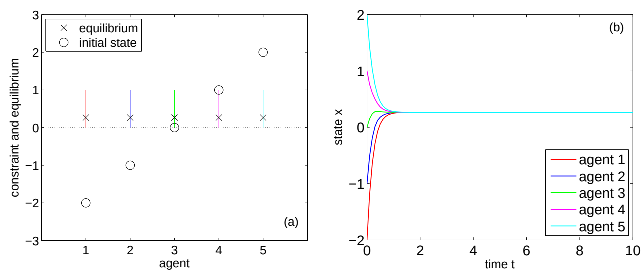

Fig. 5. Constrained consensus for Example 3: (a) Constraint sets with initial and final states indicated by circles and crosses, respectively for initial condition $x(0) = (-2, -1, 0, 1, 2)^\mathsf{T}$; (b) The state trajectories for the 5 agents.

We display in Fig. 4 the time evolution of the multiagent network (2). The constrained consensus has been achieved in line with the theoretical prediction.

Comparing Fig. 4 with Fig. 3(d), we observe that the influence of network connectivity on the time evolution on the agents. The dynamic evolution of network connectivity not only alters the transient trajectories but the final equilibrium: In Example 1 the equilibrium is 0.952 whereas in the current situation it is 1.

Example 3. In this example, we consider an opinion dynamics model satisfying Assumption 3. Let $\mathcal{G}$ be a complete graph with $n = |\mathcal{V}| = 5$ agents. Let $[\underline{\theta}, \overline{\theta}] = [0, 1]$ and consider the following system dynamics

$$\dot{x}_i(t) = \sum_{j=1}^{5} b((x_i(t) - \rho_j(x_j(t)))^2) \cdot (\rho_j(x_j(t)) - x_i(t)) \quad (32)$$

for $i \in \mathcal{V}$, where the function b is defined as follows:

$$b(s) = \begin{cases} 1, & 0 \le s < D^2 - 0.1; \\ -10 \cdot (s - D^2), & D^2 - 0.1 \le s < D^2; \\ 0, & s \ge D^2. \end{cases} \quad (33)$$

Let the initial states be $x_1(0) = -2$, $x_2(0) = -1$, $x_3(0) = 0$, $x_4(0) = 1$, $x_5(0) = 2$, and choose $D = 3$. It is direct to check the conditions in Theorem 4 are satisfied, where inequality (31) gives $54 < 63$.

We observe from Fig. 5 that constrained consensus has been achieved with the final value around 0.266. It is worth noting that although the network $\mathcal{G}$ are the initial conditions are symmetric and the constrained sets are identical, the time evolution of the states is not symmetric. This highlights the important influence of asymmetric distance modulating functions, which incorporate system states and affect the trajectories in turn.

Example 4. Finally, we consider an animal social network formed by $n = 13$ sociable weavers in Kimberley, South Africa [39]. The graph $\mathcal{G}$ shown in Fig. 6(a) is connected. Take $\theta_i = [-1, 1]$ for all $i \in \mathcal{V}$ and hence $[\underline{\theta}, \overline{\theta}] = [-1, 1]$. Choose the modulating function $b(s) = 1/(1 + s)$ for $s \ge 0$ as in Example 1. The conditions in Theorem 1 all hold. We show the state evolution of the multiagent network (2) in Fig. 6(b) with initial condition $x(0) = (-2.5, 1, 0.5, 0.2, -1.6, -1, 3, -2, -0.3, 1.5, 2, -3, 2.4)^\mathsf{T}$. As one would expect, the constrained consensus has been achieved.

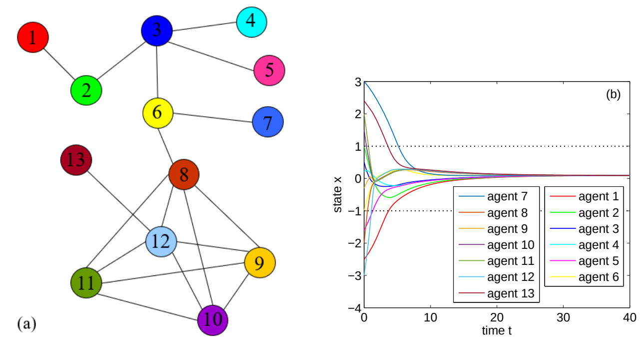

Fig. 6. (a) Social weaver network $\mathcal{G}$ in Example 4. (b) State trajectories for the agents in $\mathcal{V} = \{1, 2, \cdots, 13\}$.

Note that the network $\mathcal{G}$ can be viewed as two subgroups $\mathcal{V}_1 = \{1, 2, \cdots, 5\}$ and $\mathcal{V}_2 = \{6, 7, \cdots, 13\}$ linked by a bridge. We interestingly observe a two-stage-like consensus seeking process, where consensus tends to first reach within each group and then among both groups. It is also worth noting that the consensus speed here is much lower than the above examples even the density here is larger than that in Fig. 2(b). This is presumably due to the larger network size as more agents need to converge to a common state.

VII. Conclusion

In this paper, constrained consensus problem over state-dependent multiagent systems has been solved. The communication network depends on the system states incorporating the modulating functions in an asymmetric manner. This allows us to consider more general directed state-dependent networks. Different conditions have been obtained to ensure constrained consensus when the connectivity of the multiagent network is fixed as well as time-varying. The strategies introduced here do not rely on global information of the network. We also proposed an opinion dynamics model featuring the observer effect and bounded confidence phenomenon. Note that in the current framework, all agents are assumed to be corporative. However, in reality there may be misbehaving agents. Fault tolerance and resilient consensus in the state-dependent multiagent networks would be interesting future works.

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