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On the Sum and Spread of Reciprocal Distance Laplacian Eigenvalues of Graphs in Terms of Harary Index

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Abstract: The reciprocal distance Laplacian matrix of a connected graph G is defined as $RD^L(G) = RT(G) - RD(G)$, where $RT(G)$ is the diagonal matrix of reciprocal distance degrees and $RD(G)$ is the Harary matrix. Clearly, $RD^L(G)$ is a real symmetric matrix, and we denote its eigenvalues as $\lambda_1(RD^L(G)) \geq \lambda_2(RD^L(G)) \geq \ldots \geq \lambda_n(RD^L(G))$. The largest eigenvalue $\lambda_1(RD^L(G))$ of $RD^L(G)$, denoted by $\lambda(G)$, is called the reciprocal distance Laplacian spectral radius. In this paper, we obtain several upper bounds for the sum of k largest reciprocal distance Laplacian eigenvalues of G in terms of various graph parameters, such as order n, maximum reciprocal distance degree RT_{max}, minimum reciprocal distance degree RT_{min}, and Harary index $H(G)$ of G. We determine the extremal cases corresponding to these bounds. As a consequence, we obtain the upper bounds for reciprocal distance Laplacian spectral radius $\lambda(G)$ in terms of the parameters as mentioned above and characterize the extremal cases. Moreover, we attain several upper and lower bounds for reciprocal distance Laplacian spread $RDLS(G) = \lambda_1(RD^L(G)) - \lambda_{n-1}(RD^L(G))$ in terms of various graph parameters. We determine the extremal graphs in many cases.

Keywords: distance Laplacian matrix; reciprocal distance Laplacian matrix; Harary index; reciprocal distance Laplacian eigenvalues; reciprocal distance Laplacian spectral radius

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1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and *size* of G are $|V(G)| = n$ and $|E(G)| = m$, respectively. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident on the vertex v. Other undefined notations and terminology can be seen in [1].

The adjacency matrix $A(G) = (a_{ij})$ of G is an $n \times n$ matrix in which (i, j)-entry is equal to 1 if there is an edge between vertex v_i and vertex v_j and equal to 0 otherwise. Let $Deg(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees $d_G(v_i)$, $i = 1, 2, \ldots, n$. The positive semi-definite matrix $L(G) = Deg(G) - A(G)$ is the Laplacian matrix of G. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G, which are denoted by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$ and are ordered as $\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G)$.

In G, the *distance* between two vertices $v_i, v_j \in V(G)$, denoted by $d(v_i, v_j)$, is defined as the length of a shortest path between v_i and v_j. The diameter of G, denoted by $d(G)$, is the length of a longest path among the distance between every two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d(v_i, v_j))_{v_1, v_j \in V(G)}$.

The *transmission* $Tr_G(v_i)$ (or briefly, Tr_i if graph G is understood) of a vertex v_i is defined as the sum of the distances from v_i to all other vertices in G:

$$Tr_G(v_i) = \sum_{v_j \in V(G)} d(v_i.v_j).$$

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. In [2], Aouchiche and Hansen introduced the Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G.

The *Harary matrix* of graph G, which is also called as the reciprocal distance matrix, denoted by $RD(G)$, is an n by n matrix defined as [3]

$$RD_{ij} = \begin{cases} \frac{1}{d(v_i,v_j)} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Henceforward, we consider $i \neq j$ for $d(v_i, v_j)$.

The reciprocal distance degree of a vertex v_i, denoted by $RTr_G(v_i)$ (or shortly RT_i), is given by

$$RTr_G(v_i) = \sum_{v_j \in V(G)_{v_i \neq v_j}} \frac{1}{d(v_i, v_j)}.$$

Let $RT(G)$ be an $n \times n$ diagonal matrix defined by $RT_{ii} = RTr_G(v_i)$.

The Harary index of a graph G, denoted by $H(G)$, is defined in [3] as

$$H(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} RD_{ij} = \frac{1}{2}\sum_{v_j \in V(G)_{v_i \neq v_j}} \frac{1}{d(v_i, v_j)}.$$

Clearly,

$$H(G) = \frac{1}{2}\sum_{v_i \in V(G)} RTr_G(v_i).$$

To see more work performed on the Harary matrix, we refer the reader to [4–6] and the references therein.

In [7], the authors defined the reciprocal distance Laplacian matrix as $RD^L(G) = RT(G) - RD(G)$. Since $RD^L(G)$ is a real symmetric matrix, we can denote by

$$\lambda_1(RD^L(G)) \geq \lambda_2(RD^L(G)) \geq \ldots \geq \lambda_n(RD^L(G))$$

the eigenvalues of $RD^L(G)$. Since $RL(G)$ is a positive semidefinite matrix, we will denote the spectral radius of $RD^L(G)$ by $\lambda(G) = \lambda_1(RD^L(G))$, called the reciprocal distance Laplacian spectral radius. More work on the matrix $RD^L(G)$ can be seen in [8–11].

Let $S_k(G) = \sum_{i=1}^{k} \mu_i(G)$ be the sum of the k largest Laplacian eigenvalues of G. Several researchers have been investigating the parameter $S_k(G)$ because of its importance in dealing with many problems in the theory, for instance, Brouwer's conjecture and Laplacian energy. We refer the reader to [12–15] for recent work conducted on the graph invariant $S_k(G)$. Motivated by the parameter $S_k(G)$ of the Laplacian matrix, we define the following. For $1 \leq k \leq n-1$, let $RU_k(G)$ denote the sum of the k largest reciprocal distance Laplacian eigenvalues:

$$RU_k(G) = \sum_{i=1}^{k} \lambda_i(RD^L(G)).$$

The Laplacian spread of a graph G is defined as $LS(G) = \mu_1(G) - \mu_{n-1}(G)$, where $\mu_1(G)$ and $\mu_{n-1}(G)$ are, respectively, the largest and second smallest Laplacian eigenvalues of G. More on $LS(G)$ can be found in [16–18].

Since 0 is always a simple eigenvalue of the the reciprocal distance Laplacian matrix, we define the reciprocal distance Laplacian spread of a connected graph G such as the Laplacian spread as

$$RDLS(G) = \lambda_1(RD^L(G) - \lambda_{n-1}(RD^L(G),$$

where $\lambda_1(RD^L(G)$ and $\lambda_{n-1}(RD^L(G)$ are, respectively, the largest and second smallest reciprocal distance Laplacian eigenvalues of G.

The rest of the paper is organized as follows. In Section 2, we obtain several upper bounds for the graph invariant $RU_k(G)$ in terms of various graph parameters, such as order n, maximum reciprocal distance degree RT_{max}, minimum reciprocal distance degree RT_{min}, and Harary index $H(G)$ of G. We characterize the extremal cases corresponding to these bounds as well. As a consequence, we obtain the upper bounds for reciprocal distance Laplacian spectral radius $\lambda(G)$ in terms of the same parameters as mentioned above and determine the extremal graphs. In Section 3, we find several upper and lower bounds for reciprocal distance Laplacian spread $RDLS(G)$ in terms of various graph parameters. We characterize the extremal graphs in many cases.

2. Sum of the Reciprocal Distance Laplacian Eigenvalues

We begin with the following lemma.

Lemma 1. [7] *For any connected graph G, 0 is a simple eigenvalue of $RD^L(G)$.*

Proposition 1. *Let G be a connected graph with n vertices. Then,*

(i) $\displaystyle\sum_{i=1}^{n-1} \lambda_i(RD^L(G)) = 2H(G).$

(ii) $\displaystyle\sum_{i=1}^{n-1} \lambda_i^2(RD^L(G)) = \sum_{i=1}^{n} RT_i^2 + 2 \sum_{1\leq i<j\leq n} \frac{1}{d_{ij}^2}.$

Proof. (i) Using the fact that the sum of eigenvalues is equal to the trace of a matrix and using Lemma 1, we have

$$\sum_{i=1}^{n} \lambda_i(RD^L(G)) = \sum_{i=1}^{n-1} \lambda_i(RD^L(G)) = \sum_{i=1}^{n} RT_i = 2H(G).$$

The proof for (ii) follows arguments similar to those for (i). □

Proposition 2. *Let G be a connected graph with n vertices. Then,*

$$\sum_{1\leq i<j\leq n} \frac{1}{d_{ij}^2} \leq \frac{n(n-1)}{2}$$

with equality if and only if $G \cong K_n$.

Proof. For each $1 \leq i < j \leq n$, we have $d_{ij} \geq 1$ so that $\frac{1}{d_{ij}} \leq 1$. Thus,

$$\sum_{1\leq i<j\leq n} \frac{1}{d_{ij}^2} \leq \sum_{1\leq i<j\leq n} 1 = \binom{n}{2} = \frac{n(n-1)}{2},$$

which proves the required inequality.

Assume that the equality holds in the above inequality. Then, each $d_{ij} = 1$, whenever $1 \leq i < j \leq n$, which is only possible if $G \cong K_n$.

For the converse, we observe that the equality holds for K_n. □

Lemma 2. [19] *Let* $x = (x_1, x_2 \ldots, x_n)$ *and* $y = (y_1, y_2 \ldots, y_n)$ *be n-tuples of real numbers satisfying* $0 \le m_1 \le x_i \le M_1$, $0 \le m_2 \le y_i \le M_2$ *with* $i = 1, 2, \ldots, n$ *and* $M_1 M_2 \ne 0$. *Let* $\alpha = \frac{m_1}{M_1}$ *and* $\beta = \frac{m_2}{M_2}$. *If* $(1+\alpha)(1+\beta) \ge 2$, *then*

$$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \frac{n^2}{4}(M_1 M_2 - m_1 m_2)^2. \tag{1}$$

Let $RT_{max} = \max\{RT_i : i = 1, 2, \ldots, n\}$ and $RT_{min} = \min\{RT_i : i = 1, 2, \ldots, n\}$ be the maximum reciprocal distance degree and the minimum reciprocal distance degree of the graph G, respectively. Using Lemma 2, we obtain an upper bound for the graph invariant $\sum_{i=1}^{n} RT_i^2$ in terms of Harary index $H(G)$ and order n of graph G.

Lemma 3. *Let G be a connected graph with n vertices. Then,*

$$\sum_{i=1}^{n} RT_i^2 \le \frac{n}{4}(RT_{max} - RT_{min})^2 + \frac{4H^2(G)}{n}. \tag{2}$$

Moreover, inequality is sharp, as shown by all of the reciprocal distance degree regular graphs.

Proof. In Lemma 2, we take $x = (RT_1, RT_2, \ldots, RT_n)$, $y = (1, 1, \ldots, 1)$, $M_1 = RT_{max}$, $m_1 = RT_{min}$ and $M_2 = m_2 = 1$. With these values, it is straightforward to check that the condition $(1+\alpha)(1+\beta) \ge 2$ in Lemma 2 gets satisfied. Thus, from Inequality 1, we have

$$\begin{aligned} & \sum_{i=1}^{n} RT_i^2 \sum_{i=1}^{n} 1 - \left(\sum_{i=1}^{n} RT_i\right)^2 \le \frac{n^2}{4}(RT_{max} - RT_{min})^2 \\ \Rightarrow \quad & n \sum_{i=1}^{n} RT^2(i) - 4H^2(G) \le \frac{n^2}{4}(RT_{max} - RT_{min})^2 \\ \Rightarrow \quad & \sum_{i=1}^{n} RT_i^2 \le \frac{n}{4}(RT_{max} - RT_{min})^2 + \frac{4H^2(G)}{n}. \end{aligned}$$

Assume that G is k-reciprocal distance degree regular. Then, the left hand side of Inequality 2 becomes nk^2 and the right hand side becomes $\frac{4H^2(G)}{n} = \frac{k^2 n^2}{n} = nk^2$, which shows that the equality holds for reciprocal distance degree regular graphs. □

Now, we obtain an upper bound for the graph invariant $RU_k(G)$ in terms of various graph parameters.

Theorem 1. *Let G be a connected graph with n vertices and Harary index* $H(G)$. *For* $1 \le k \le n-2$, *we have*

$$RU_k(G) \le \frac{2H(G)k}{n-1} + \frac{\sqrt{k(n-k-1)\Big[n^2(n-1)\Big((RT_{max} - RT_{min})^2 + 4(n-1)\Big) - 16H^2(G)\Big]}}{2(n-1)\sqrt{n}}$$

with equality if and only if $G \cong K_n$. *For* $k = n-1$, *equality always holds.*

Proof. Let $RU_k(G) = R_k$. For $1 \leq k \leq n-2$, using Proposition 1 and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\Big(\lambda_{k+1}(RD^L(G)) + \ldots + \lambda_{n-1}(RD^L(G))\Big)^2 \\
&= (2H(G) - R_k)^2 \leq (n-k-1)\Big(\lambda_{k+1}^2(RD^L(G)) + \ldots + \lambda_{n-1}^2(RD^L(G))\Big) \\
&= (n-k-1)\Big(\sum_{i=1}^{n} RT_i^2 + 2\sum_{1\leq i<j\leq n}\frac{1}{d_{ij}^2} - (\lambda_1^2(RD^L(G)) + \ldots + \lambda_k^2(RD^L(G)))\Big) \\
&\leq (n-k-1)\Big(\sum_{i=1}^{n} RT_i^2 + 2\sum_{1\leq i<j\leq n} d_{ij}^2 - \frac{R_k^2}{k}\Big).
\end{aligned}$$

Further simplification gives

$$R_k^2 - \frac{4kH(G)R_k}{n-1} + \frac{4kH^2(G)}{n-1} - \frac{k(n-k-1)}{n-1}\Big(\sum_{i=1}^{n} RT_i^2 + 2\sum_{1\leq i<j\leq n}\frac{1}{d_{ij}^2}\Big) \leq 0.$$

Therefore,

$$R_k \leq \frac{2H(G)k + \sqrt{k(n-k-1)\Big[(n-1)\Big(\sum_{i=1}^{n} RT_i^2 + 2\sum_{1\leq i<j\leq n}\frac{1}{d_{ij}^2}\Big) - 4H^2(G)\Big]}}{n-1}. \tag{3}$$

Using Proposition 2, Lemma 3 in Inequality 3 and after simplifications, we have

$$R_k \leq \frac{2H(G)k}{n-1} + \frac{\sqrt{k(n-k-1)\Big[n^2(n-1)\Big((RT_{max} - RT_{min})^2 + 4(n-1)\Big) - 16H^2(G)\Big]}}{2(n-1)\sqrt{n}},$$

which proves the required inequality.

Assume that equality holds in the above inequality. Then, equality must hold simultaneously in the Cauchy–Schwarz inequality, Proposition 2, and Lemma 3, which is only possible if $G \cong K_n$.

Conversely, if $G \cong K_n$, then the left hand side of the main equality is equal to kn. After performing the necessary calculations, the right-hand side reduces to $\frac{2H(K_n)k}{n-1} + 0 = \frac{n(n-1)k}{n-1} = kn$, which proves the converse part.

Using the fact that traces of a matrix are equal to the sum of its eigenvalues and noting that $2H(G) = R_{n-1}$, we easily see that equality always holds when $k = n-1$ in the main inequality. □

Taking $k = 1$ in Theorem 1, we obtain an upper bound for the reciprocal distance Laplacian spectral radius $\lambda(G)$ of a connected graph G in terms of the maximum reciprocal distance degree RT_{max}, minimum reciprocal distance degree RT_{min}, order n, and Harary index $H(G)$.

Theorem 2. *Let G be a connected graph with n vertices and Harary index $H(G)$. Then,*

$$\lambda(G) \leq \frac{2H(G)}{n-1} + \frac{\sqrt{(n-2)\Big[n^2(n-1)\Big((RT_{max} - RT_{min})^2 + 4(n-1)\Big) - 16H^2(G)\Big]}}{2(n-1)\sqrt{n}}$$

with equality if and only if $G \cong K_n$.

Lemma 4. [20] *Let $[n] = \{1, 2, \ldots, n\}$ be the canonical n-element set, and let $[n]^{(2)}$ denote the set of two-element subsets of $[n]$, that is, the edge set of K_n. To each entry $\{i, j\} = ij$ in $[n]^{(2)}$, associate a real variable z_{ij}; then, for $n \geq 2$, and for all $z'_{ij}s$, we have*

$$\Big(\sum_{ij} z_{ij}\Big)^2 + \binom{n-1}{2} \sum_{ij} z_{ij}^2 - \frac{n-1}{2} \sum_i \Big(\sum_{j \neq i} z_{ij}\Big)^2 \geq 0.$$

Now, we obtain an upper bound for the sum of the squares of the reciprocal distance degrees in terms of the Harary index $H(G)$ and the order n of the graph G.

Lemma 5. *Let G be a connected graph with order n and having diameter d. Then*

$$\sum_i RT_i^2 \leq \frac{n(n-1)(n-2)}{2} + \frac{2H^2(G)}{n-1}$$

with equality if and only if $G \cong K_n$.

Proof. Put $\frac{1}{d_{ij}}$ for z_{ij} in Lemma 4 and observe that with each $\frac{1}{d_{ij}} \leq 1$, we have

$$\Big(\sum_{ij} \frac{1}{d_{ij}}\Big)^2 + \binom{n-1}{2} \sum_{ij} \Big(\frac{1}{d_{ij}}\Big)^2 - \frac{n-1}{2} \sum_i \Big(\sum_{j \neq i} \frac{1}{d_{ij}}\Big)^2 \geq 0$$
$$\text{or} \quad H^2(G) + \binom{n-1}{2} \sum_{ij} \Big(\frac{1}{d_{ij}}\Big)^2 - \frac{n-1}{2} \sum_i RT_i^2 \geq 0.$$

Simplifying further, we have

$$\sum_i RT_i^2 \leq \frac{2}{n-1} \frac{(n-1)(n-2)}{2} \Big(\frac{n(n-1)}{2}\Big) + \frac{2H^2(G)}{n-1}$$
$$\text{or} \quad \sum_i RT_i^2 \leq \frac{n(n-1)(n-2)}{2} + \frac{2H^2(G)}{n-1}.$$

proving the required inequality.

Assume that the equality holds in the above inequality. Then, each $\frac{1}{d_{ij}} = 1$ or $d_{ij} = 1$ which is only possible if G is the complete graph K_n.

Conversely, assume that $G \cong K_n$. Then, we observe that $H(G) = \frac{n(n-1)}{2}$ and $\sum_i RT_i^2 = n(n-1)^2$. Substituting these values in the main inequality, we see that the equality holds. □

A similar argument has been adopted in studying Estrada index [21]. Using Lemma 5, we have the following upper bound for the graph invariant $RU_k(G)$ in terms of order n and Haray index H(G). This bound seems to be more elegant than the bound in Theorem 1 since it uses relatively less number of parameters.

Theorem 3. *Let G be a connected graph with n vertices and Harary index $H(G)$. For $1 \leq k \leq n-2$, we have*

$$RU_k(G) \leq \frac{2H(G)k}{n-1} + \frac{\sqrt{k(n-k-1)\Big(n(n-1) - 2H(G)\Big)\Big(n(n-1) + 2H(G)\Big)}}{(n-1)\sqrt{2}}$$

with equality if and only if $G \cong K_n$. For $k = n-1$, equality always holds.

Proof. We proceed exactly as in Theorem 1 upto Inequality 3, then use Lemma 5 and Proposition 2, and obtain

$$R_k \leq \frac{2H(G)k + \sqrt{k(n-k-1)\left[(n-1)\left(\frac{n(n-1)(n-2)}{2} + \frac{2H^2(G)}{n-1} + n(n-1)\right) - 4H^2(G)\right]}}{n-1}.$$

Simplifying further, we have

$$R_k \leq \frac{2H(G)k + \sqrt{k(n-k-1)\left(\frac{n^2(n-1)^2}{2} - 2H^2(G)\right)}}{n-1}.$$

or

$$R_k \leq \frac{2H(G)k}{n-1} + \frac{\sqrt{k(n-k-1)\Big(n(n-1) - 2H(G)\Big)\Big(n(n-1) + 2H(G)\Big)}}{(n-1)\sqrt{2}}$$

which is the inequality in the statement of theorem.

The remaining part of the proof follows by using similar arguments as in Theorem 1. □

As a consequence of Theorem 3, we obtain the following upper bound for reciprocal distance Laplacian spectral radius $\lambda(G)$ of a connected graph G in terms of the Harary index $H(G)$ and order n of the graph G.

Theorem 4. *Let G be a connected graph with n vertices and Harary index $H(G)$. Then,*

$$\lambda(G) \leq \frac{2H(G)}{n-1} + \frac{\sqrt{(n-2)\Big(n(n-1) - 2H(G)\Big)\Big(n(n-1) + 2H(G)\Big)}}{(n-1)\sqrt{2}}$$

with equality if and only if $G \cong K_n$.

3. Reciprocal Distance Laplacian Spread

We begin this section with the following observations.

Lemma 6. [7] *Let G be a connected graph on n vertices with diameter $d = 2$. Then,*

$$\lambda_i(RD^L(G)) = \frac{n + \mu_i(G)}{2}$$

for $i = 1, 2, \ldots, n-1$. Furthermore, $\frac{n+\mu_i(G)}{2}$ and $\mu_i(G)$ both have the same multiplicity for $i = 1, 2, \ldots, n$.

A special case of the well-known *min–max theorem* is the following result.

Lemma 7. [22] *If M is a symmetric $n \times n$ matrix with eigenvalues $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$, then for any $x \in R^n$ $(x \neq 0)$,*

$$\delta_1 \geq \frac{x^T N x}{x^T x}.$$

Equality holds if and only if x is an eigenvector of M corresponding to the largest eigenvalue δ_1.

Lemma 8. [7] *If G is a graph on $n > 2$ vertices, then the multiplicity of $\lambda(G)$ is always less than or equal to $n-1$ with equality if and only if G is the complete graph.*

Lemma 9. [23] *Let G be a connected graph of order $n \geq 2$. Then, $\mu_1(G) \geq \triangle(G) + 1$, with equality if and only if $\triangle(G) = n - 1$.*

Theorem 5. *Let G be a connected graph with n vertices having Harary index $H(G)$. Then,*

$$RDSL(G) \leq \frac{\sqrt{(n-2)\Big(n(n-1) - 2H(G)\Big)\Big(n(n-1) + 2H(G)\Big)}}{\sqrt{2}} \tag{4}$$

Equality holds if and only if $G \cong K_n$.

Proof. To prove the inequality, we consider $2H(G) = \lambda_1(RD^L(G)) + \lambda_2(RD^L(G)) + \ldots + \lambda_{n-1}(RD^L(G))$, which gives $2H(G) \leq (n-2)\lambda_1(RD^L(G)) + \lambda_{n-1}(RD^L(G))$ or $\lambda_{n-1}(RD^L(G)) \geq 2W(G) - (n-2)\lambda_1(RD^L(G))$. Therefore,

$$\text{RDLS(G)} = \lambda_1(RD^L(G)) - \lambda_{n-1}(RD^L(G)) \leq \lambda_1(RD^L(G)) - 2H(G) + (n-2)\lambda_1(RD^L(G)),$$

which gives

$$RDSL(G) \leq (n-1)\lambda_1(RD^L(G)) - 2H(G). \tag{5}$$

Using Theorem 4 in Inequality 5, we have

$$\text{RDSL(G)} \leq \frac{\sqrt{(n-2)\Big(n(n-1) - 2H(G)\Big)\Big(n(n-1) + 2H(G)\Big)}}{\sqrt{2}},$$

proving the required inequality.

From Inequality 5 and Theorem 4, we see that equality holds in Inequality 4 if and only if $\lambda_1(RD^L(G)) = \lambda_2(RD^L(G)) = \ldots = \lambda_{n-2}(RD^L(G))$ and $G \cong K_n$.

Since the reciprocal distance Laplacian spectrum of K_n is $\{n^{(n-1)}, 0\}$, therefore, equality holds in Inequality 4 if and only if $G \cong K_n$. □

If we use Theorem 2 instead of Theorem 4 in the above result, we have the following theorem:

Theorem 6. *Let G be a connected graph with n vertices having Wiener index $W(G)$. Then,*

$$RDSL(G) \leq \frac{\sqrt{(n-2)\Big[n^2(n-1)\Big((RT_{max} - RT_{min})^2 + 4(n-1)\Big) - 16H^2(G)\Big]}}{2\sqrt{n}}.$$

Equality holds if and only if $G \cong K_n$.

Let $S_d = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{d}$. The following lemma gives the lower bound for the reciprocal distance Laplacian spectral radius in terms of order n, diameter d, and S_d.

Lemma 10. *Let G be a connected graph on n vertices having diameter d. Then,*

$$\lambda(G) \geq S_d + \frac{(n-d-2)}{d}.$$

Proof. Let $v_1 v_2 \ldots v_{d+1}$ be the diametral path in G such that $d_G(v_1, v_{d+1}) = d$. Consider the n-vector $x = (x_1, x_2, \ldots, x_{d-1}, x_d, x_{d+1}, \ldots, x_n)^T$ defined by

$$x_i = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = 1, d+1 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 7, we have

$$\lambda(G) \geq \frac{x^T \mathcal{D}^{\mathcal{Q}} y}{y^T y} = \frac{RT_1 + RT_{d+1}}{2} - \frac{1}{d_G(v_1, v_{d+1})} = \frac{RT_1 + RT_{d+1}}{2} - \frac{1}{d}. \tag{6}$$

It can be easily seen that

$$RT_1 \geq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{d} + \frac{(n-d-1)}{d} = S_d + \frac{(n-d-1)}{d} \tag{7}$$

Similarly,

$$RT_{d+1} \geq S_d + \frac{(n-d-1)}{d}. \tag{8}$$

On substituting inequalities 7, 8 in Inequality 6, we have

$$\lambda(G) \geq S_d + \frac{(n-d-1)}{d} - \frac{1}{d} = S_d + \frac{(n-d-2)}{d}.$$

□

Theorem 7. *Let G be a connected graph with order n having diameter d. Then,*

$$RDLS(G) \geq S_d + \frac{(n-d-2)}{d} - \frac{2H(G)}{n-1}.$$

Proof. Note that $\sum_{i=1}^{n-1} \lambda_i(RD^L(G)) = 2H(G)$. From this equality, we see that

$$\lambda_{n-1}(RD^L(G)) \leq \frac{2H(G)}{n-1}. \tag{9}$$

Using Lemma 10 and Inequality 9, we have

$$\begin{aligned} \text{RDSL(G)} &= \lambda_1(RD^L(G)) - \lambda_{n-1}(RD^L(G)) \\ &\geq S_d + \frac{(n-d-2)}{d} - \frac{2H(G)}{n-1}. \end{aligned}$$

□

Theorem 8. *Let G be a connected graph on $n \geq 3$ vertices having diameter $d \leq 2$. Then,*

$$RDLS(G) \geq \frac{n + \triangle(G) + 1}{2} - \frac{2H(G)}{n-1}. \tag{10}$$

Equality holds if and only if $d = 1$, that is, $G \cong K_n$.

Proof. First, we show that equality holds for K_n. Note that the reciprocal distance Laplacian spectrum of the complete graph K_n is $\{n^{(n-1)}, 0\}$ so that $RDLS(K_n) = \lambda_1(RD^L(K_n)) - \lambda_{n-1}(RD^L(K_n)) = n - n = 0$. Additionally, the right-hand side of Inequality 10 for K_n is equal to $\triangle(K_n) + 1 - \frac{2H(K_n)}{n-1} = n - 1 + 1 - \frac{n(n-1)}{n-1} = 0$. Thus, from the above arguments, we see that equality holds in Inequality 10 when G is a complete graph.

Now, let G be a graph with diameter $d = 2$. Using Lemma 6, we have

$$RDLS(G) = \lambda_1(RD^L(G)) - \lambda_{n-1}(RD^L(G)) = \frac{n + \mu_1}{2} - \lambda_{n-1}(RD^L(G)). \tag{11}$$

By Lemma 8, we see that Inequality 9 is strict since G is a noncomplete graph, that is, $\lambda_{n-1}(RD^L(G)) < \frac{2H(G)}{n-1}$. Using this observation with Lemma 9 in Equality 11, we have

$$RDLS(G) > \frac{n + \triangle(G) + 1}{2} - \frac{2H(G)}{n-1}.$$

□

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