On the tree-depth and tree-width in heterogeneous random graphs

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Abstract: In this note, we investigate the tree-depth and tree-width in a heterogeneous random graph obtained by including each edge e_{ij} $(i \neq j)$ of a complete graph K_n over n vertices independently with probability $p_n(e_{ij})$. When the sequence of edge probabilities satisfies some density assumptions, we show both tree-depth and tree-width are of linear size with high probability. Moreover, we extend the method to random weighted graphs with non-identical edge weights and capture the conditions under which with high probability the weighted tree-depth is bounded by a constant.

Key words: Tree-depth; tree-width; random graph; heterogeneous graph.

1. Introduction For a simple connected graph G, an elimination tree T of G is a rooted tree on the vertices of G in which G has no edges connecting two different branches in T. Note that T and G have the same sets of vertices but T does not need to be a subgraph of G. Elimination tree, firstly used by Duff [7], is one of the most important concepts in scientific computing and numerical linear algebra. It plays a pivotal role in areas including Cholesky factorization of sparse matrices, combinatorial optimization algorithms, and data structures [5, 16, 23]. Equivalently, a rooted tree T on the sets of vertices of G becomes an elimination tree of G if G is a subgraph of the closure of T, where the closure of a rooted tree T is obtained from T by adding all (and only) edges between an ancestor and its descendant. The height of a rooted tree is the number of vertices on the longest path between the root and a leaf. Tree-depth of G, denoted by $\mathrm{td}(G)$, is the minimum height of an elimination tree of G. If G is not connected, $\mathrm{td}(G)$ is defined as the maximum tree-depth among its connected components. It is known that the maximum tree-depth for a graph over n vertices is only attained by the complete graph K_n with $\mathrm{td}(K_n) = n$ and $\mathrm{td}(T) \leq \lfloor \log_2 n \rfloor + 1$ for a tree T. Moreover, the path P_n attains the upper bound among all tree graphs [8]. An example is shown in Fig. 1.

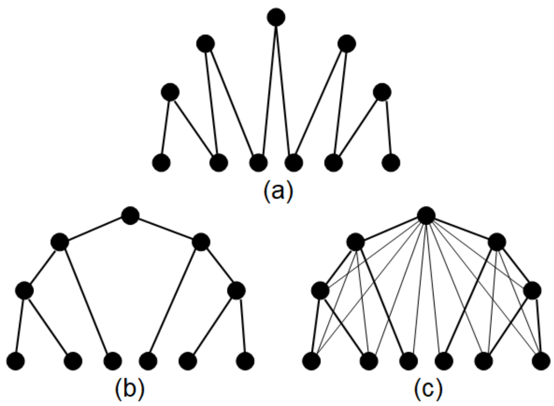

Fig. 1. Path graph $G = P_{11}$ has tree-depth $\mathrm{td}(G) = \lfloor \log_2 11 \rfloor + 1 = 4$. (a) The path G; (b) The elimination tree T of G, which has height 4; (c) The closure of T.

A related concept is the tree-width, denoted by $\mathrm{tw}(G)$, which captures the closeness of a graph relative to a tree while tree-depth captures the closeness of a graph relative to a star. Tree-width, put forward by Robertson and Seymour [20] in 1986, is a useful parameter in the parameterized complexity analysis of many graph algorithms [1, 11, 22]. A graph G has tree-width $\mathrm{tw}(G) = k$ if it is a subgraph of a k-tree with minimum k. Here, a k-tree is obtained by beginning with the complete graph K_{k+1} and repeatedly adding vertices so that each newly added vertex is adjacent to every vertex of an existing k-clique. By definition, it is clear that $\mathrm{tw}(K_n) = n - 1$ and $\mathrm{tw}(T) = 1$ for any tree T. However, determining tree-width for a general graph is NP-complete. Tree-width is related to tree-depth through the following

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inequality [2, 11]

$$\operatorname{tw}(G) \le \operatorname{td}(G) \le (1+\log_2 n)\operatorname{tw}(G). \tag{1.1}$$

Here, we are interested in the two graph invariants $\operatorname{td}(G)$ and $\operatorname{tw}(G)$ in the context of heterogeneous random graphs. Consider a complete graph K_n over the vertex set $V=\{1,2,\cdots,n\}$. Let $e_{ij}=e_{ji}$ denote the edge connecting vertices i and j for $i\ne j$. Given a set of edge probabilities $\mathbf{p}_n=\{p_n(e_{ij})\}_{1\le i<j\le n}$, the heterogeneous random graph model $G(n,\mathbf{p}_n)$ can be defined by including each edge e_{ij} of K_n independently with edge probability $p_n(e_{ij})$. Clearly, when $p_n(e_{ij})\equiv p_n$ for all i and j ($i\ne j$), we reproduce the ordinary Erdős-Rényi random graph $G(n,p_n)$. A closely related model is called the uniform random graph $G(n,m_n)$, where each graph with m_n edges occurs with the same probability. Many results of random graphs can be transferred equivalently between $G(n,p_n)$ and $G(n,m_n)$ via the mapping $p_n=m_n\binom{n}{2}^{-1}$. In the past few decades, heterogeneous random graphs are gaining traction as they well underpin complex network models [18], which often have non-trivial topological structures (such as heterogeneous degree distributions, community structure and hierarchy) eliciting fascinating phenomena in nature and technology. For a recent survey of varied random graph models and their mathematical results, we refer readers to the monograph [10]. In particular, the majority dynamics over $G(n,\mathbf{p}_n)$ has been studied in [21].

In random graphs, we say a graph property holds with high probability (w.h.p.) if the probability that all graphs holding this property occur tends to 1 as $n\to\infty$. It is shown by Kloks [13] that $G(n,m_n)$ with $m_n/n\ge c=1.18$ has linear tree-width $\operatorname{tw}(G(n,m_n))=\Theta(n)$ w.h.p. This constant c has been further improved to 1.073 in [3] and 0.5 in [14]. For $G(n,p_n)$ model, it is found in [24] that w.h.p. $\operatorname{tw}(G(n,p_n))\ge n-o(n)$ when $n\gg np_n\to\infty$. In the case of $np_n=1+\varepsilon$ for a sufficiently small $\varepsilon>0$, it is shown that $\operatorname{tw}(G(n,p_n))=n\Omega(-\varepsilon^3(\ln\varepsilon)^{-1})$ w.h.p. [6]. Tree-width has also been investigated for random intersection graphs [3] and geometric random graphs [15]. Perarnau and Serra [19] proved that $\operatorname{td}(G(n,p_n))=n-O((n/p)^{1/2})$ when $np_n\to\infty$. Tree-depth as well as tree-width of random geometric graphs has also been studied in [17].

Along the above line of research, in this short note we first study tree-depth and tree-width for dense heterogeneous random graph $G(n,\mathbf{p}_n)$ in Section 2. We then extend our approach to weighted random graphs with non-identical weight distributions in Section 3. Standard Landau asymptotic notations such as O,o,Θ and $\ll$ will be used throughout the paper by convention in random graph literature; c.f. [10].

2. Tree-depth and tree-width in heterogeneous random graphs To begin with, we define the expected neighbor density for a vertex $i\in V$ with respect to a set of vertices. Specifically, given $S\subseteq V$ and $i\notin S$ let $d_n(i,S)=|S|^{-1}\sum_{j\in S}p_n(e_{ij})$. It measures average number of neighbors of vertex i within the set S.

Theorem 1. *Suppose that there is a sequence $\{p_n\}_{n\ge1}$ and constants α and β satisfying $p_n\in(0,1)$, $0<\alpha<\frac{2}{9\ln 3}\beta$, and for all n large*

$$p_n\ge\frac{1}{\alpha n}\qquad\text{and}\qquad \min_{i\in V}\ \min_{\substack{S:\ i\notin S\\ |S|\ge n\sqrt{\frac{\alpha\ln 3}{2\beta}}}} d_n(i,S)\ge\beta p_n. \tag{2.1}$$

Then for any constant $c=c(\alpha,\beta)$ satisfying $3\sqrt{\frac{\alpha\ln 3}{2\beta}}<c\le 1$ we have

$$\mathbb{P}\left(n-\lfloor cn\rfloor\le\operatorname{td}(G(n,\boldsymbol{p}_n))\le n\right)\ge 1-e^{-\Theta(n)} \tag{2.2}$$

and similarly

$$\mathbb{P}\left(n-\lfloor cn\rfloor\le\operatorname{tw}(G(n,\boldsymbol{p}_n))\le n\right)\ge 1-e^{-\Theta(n)} \tag{2.3}$$

for all n large. Here, $\Theta(n)$ is a function of c.

Before proving Theorem 1, we present an example with non-trivial edge probabilities $\{\mathbf{p}_n\}_{n\ge1}$ satisfying the condition (2.1). Set $\alpha=1$, $\beta=10$, and $p_n=\frac{1}{n}$ for $n\ge1$. For $1\le i<j\le\lceil\frac{n}{10}\rceil$, let $p_n(e_{ij})=\frac{1}{n\ln n}$, and for any other $i<j$, let $p_n(e_{ij})=\frac{100}{n}$. Since $\sqrt{\frac{\alpha\ln 3}{2\beta}}>\frac{1}{5}$, for any $i\notin S$ and $|S|\ge\frac{n}{5}$, we have

$$\begin{aligned}
d_n(i,S)\ge&\frac{1}{|S|}\left(\frac{1}{n\ln n}\left\lceil\frac{n}{10}\right\rceil+\left(|S|-\left\lceil\frac{n}{10}\right\rceil\right)\frac{100}{n}\right)\\
\ge&\frac{5}{n}\left(\frac{1}{n\ln n}\cdot\frac{n}{10}+\left(\frac{n}{10}-1\right)\frac{100}{n}\right)\\
=&\frac{n+100(n-10)\ln n}{2n^2\ln n}\\
\ge&\frac{1+50\ln n}{2n\ln n}\\
>&\beta p_n,
\end{aligned}$$

for all $n>20$. Therefore, (2.1) holds true and it follows from (2.2) and (2.3) that, for example,

$\mathbb{P}(\min\{\mathrm{td}(G(n,\mathbf{p}_n)),\ \mathrm{tw}(G(n,\mathbf{p}_n))\} \ge 0.29n) \ge 1 - e^{-\Theta(n)}$ for all large n.

To prove Theorem 1, we need the following lemma with regard to balanced separators [13, Lem 5.3.1, Lem 6.1.2].

Lemma 1. *Let G be a graph over the vertex set V with $|V| = n$. For any number $k \in [\mathrm{tw}(G), n-4]$, G has a balanced k-partition (S, A, B) in the following sense.*

Mutually exclusive sets S, A and B satisfy $S \cup A \cup B = V$, $|S| = k+1$, $\frac{1}{3}(n-k-1) \le |A| \le |B| \le \frac{2}{3}(n-k-1)$, where S forms a separator in G meaning that no edges run between A and B.

Proof of Theorem 1. Fix any constant $c > 3\sqrt{\frac{\alpha \ln 3}{2\beta}}$. The assumption $0 < \alpha < \frac{2}{9\ln 3}\beta$ ensures $c < 1$. If $G(n,\mathbf{p}_n)$ has a balanced k-partition (S, A, B) as described in Lemma 1 with $|S| = k+1 \le (1-c)n$, then $|B| \ge |A| \ge \frac{1}{3}(n-k-1) \ge \frac{cn}{3}$. Hence, we have

$$|A||B| \ge |A|(cn - |A|) \ge \frac{2}{9}c^2n^2. \tag{2.4}$$

Define $\mathcal{E}(S, A, B)$ to be the event that $G(n,\mathbf{p}_n)$ admits a balanced k-partition (S, A, B) with $|S| = k+1 \le (1-c)n$. We obtain

$$\begin{aligned}\mathbb{P}(\mathcal{E}(S,A,B)) =& \prod_{i\in A, j\in B}(1 - p_n(e_{ij}))\\ \le& e^{-\sum_{i\in A, j\in B} p_n(e_{ij})}\\ =& e^{-\sum_{i\in A}|B| d_n(i,B)}\\ \le& e^{-p_n\beta|A|\cdot|B|}\\ \le& e^{-\frac{2}{9}p_n\beta c^2n^2},\end{aligned} \tag{2.5}$$

where in the second inequality above we used the estimate $|B| \ge \frac{cn}{3} \ge n\sqrt{\frac{\alpha\ln 3}{2\beta}}$ and (2.1), and in the last inequality we applied (2.4).

Let $\mathcal{C}$ be the collection of all balanced k-partitions (S, A, B) with $|S| = k+1 \le (1-c)n$. A simple upper bound is given by $|\mathcal{C}| \le 3^n$ since each vertex is allowed for three options in a balanced k-partition. In the light of (2.5) we can bound the probability of existing such a partition as

$$\begin{aligned}\mathbb{P}\left(\cup_{(S,A,B)\in\mathcal{C}}\mathcal{E}(S,A,B)\right) \le& \sum_{(S,A,B)\in\mathcal{C}}\mathbb{P}(\mathcal{E}(S,A,B))\\ \le& 3^n e^{-\frac{2}{9}p_n\beta c^2n^2}\\ \le& e^{n\left(\ln 3 - \frac{2\beta c^2}{9\alpha}\right)},\end{aligned} \tag{2.6}$$

where in the last inequality the assumption $p_n \ge \frac{1}{\alpha n}$ in (2.1) is utilized. Recall that $c > 3\sqrt{\frac{\alpha\ln 3}{2\beta}}$. Therefore, the probability in (2.6) is tantamount to $e^{-\Theta(n)}$. Consequently, it follows from Lemma 1 that

$$\begin{aligned}\mathbb{P}(\mathrm{tw}(G(n,\mathbf{p}_n)) \le& \lfloor(1-c)n\rfloor)\\ \le& \mathbb{P}\left(\cup_{(S,A,B)\in\mathcal{C}}\mathcal{E}(S,A,B)\right) \le e^{-\Theta(n)},\end{aligned}$$

which yields (2.3). Combining it with (1.1), we know that the result (2.2) also holds. □

By taking $\beta = 1$, $0 < \alpha < \frac{2}{9\ln 3}$, and $p_n(e_{ij}) = p_n$ for all $i < j$ in Theorem 1, we obtain the following result for homogeneous random graph $G(n, p_n)$.

Corollary 1. *Suppose that $p_n \ge \frac{1}{\alpha n}$ with $\alpha \in \left(0, \frac{2}{9\ln 3}\right)$. For any constant $c > 3\sqrt{\frac{\alpha\ln 3}{2}}$ and all n large, we have*

$$\mathbb{P}\left(n - \lfloor cn\rfloor \le \mathrm{td}(G(n,p_n)) \le n\right) \ge 1 - e^{-\Theta(n)}$$

and

$$\mathbb{P}\left(n - \lfloor cn\rfloor \le \mathrm{tw}(G(n,p_n)) \le n\right) \ge 1 - e^{-\Theta(n)}.$$

In particular, w.h.p. $\mathrm{td}(G(n,p_n)) = \Theta(n)$ and $\mathrm{tw}(G(n,p_n)) = \Theta(n)$.

These estimates are in line with previous results in [24] and [19] for dense Erdős-Rényi random graphs while enjoy more explicit convergence rate estimates.

It is also worth noting that Theorem 1 for heterogeneous random graphs is non-trivial. For instance, in the example above, we have chosen $p_n(e_{ij}) = \frac{1}{n\ln n} \ll \frac{1}{n}$, which in a homogeneous random graph will only lead to tree-depth (and tree-width) of $\Theta(\ln\ln n)$; see [19, Theorem 1.2].

3. Tree-depth in weighted random graphs In this section, we consider weighted heterogeneous random graphs by placing a random weight $w(e_{ij}) = w(e_{ji})$ on each edge e_{ij} of K_n. Given an elimination tree of G, for the longest downward path between the root and a leaf $P = (i_1, i_2, \cdots, i_\ell)$, we define $w(P) := \sum_{j=1}^{\ell-1} w(e_{i_j i_{j+1}})$ as the weight of P, i.e., $w(P)$ is the weighted height of the elimination tree. Let $\mathrm{td}^w(G) := \min_P w(P)$ be the minimum weighted height of an elimination tree of G. We call $\mathrm{td}^w(G)$ the weighted tree-depth of G. Tree-depth as a parameter has been intensively studied in some graph algorithms for weighted graphs including the fixed parameter tractable (FPT) algorithms [4, 12]. However, most of these works concern fixed graph and deterministic weights.

For every edge e_{ij} in K_n, let F_{ij} be the cumulative distribution function of the weight $w(e_{ij})$ and set

$$p_n(e_{ij}) := F_{ij}\left(\frac{1}{n}\right) = \mathbb{P}\left(w(e_{ij}) \le \frac{1}{n}\right).$$

By definition, we have $F_{ij} = F_{ji}$ for $i \neq j$. The result below shows that the weighted tree-depth is bounded above by a constant w.h.p. It is worth noting that the appropriate analogous version for tree-width is assigning weight on vertices instead of edges (see e.g. [9]), and hence is not considered here.

Theorem 2. *Assume that the sequence of cumulative distribution functions $\{F_{ij}\}_{1\le i<j\le n}$ satisfies the following two conditions:*

(i) *There is a sequence $\{p_n\}_{n\ge 1}$ and constants α and β satisfying $p_n \in (0,1)$, $0 < \alpha < \frac{2}{9\ln 3}\beta$, and for all n large the condition (2.1) holds.*

(ii) *There is a constant γ satisfying $\max_{1\le i<j\le n} \mathbb{E}w^2(e_{ij}) \le \gamma$ for all n large.*

Then we have

$$\mathbb{P}\left(\mathrm{td}^w(G(n,\boldsymbol{p}_n)) \le 1\right) \ge 1 - e^{-\Theta(n)} \tag{3.1}$$

and

$$\mathbb{E}\left(\mathrm{td}^w(G(n,\boldsymbol{p}_n))\right) \le 1 + \sqrt{\gamma}e^{-\Theta(n)} \tag{3.2}$$

for all n large. Here, $\Theta(n)$ is a function of α and β.

Proof. We say an edge e in K_n is occupied if the weight of e is less than or equal to $\frac{1}{n}$. Define $\mathcal{A}_n$ to be the event that there exists an occupied elimination tree of $G(n,\mathbf{p}_n)$ having height at least $n - \lfloor cn \rfloor$, where $c = c(\alpha,\beta)$ is determined in Theorem 1. When $\mathcal{A}_n$ occurs, each edge of the longest downward rooted path in an elimination tree has weight no more than $\frac{1}{n}$. Therefore, the sum of the weights is upper bounded by 1, namely, $\mathrm{td}^w(G(n,\mathbf{p}_n)) \le 1$. When $\mathcal{A}_n$ does not occur, the weight of any downward rooted path in an elimination tree of $G(n,\mathbf{p}_n)$ has weight no more than $\sum_{1\le i<j\le n} w(e_{ij})$. Therefore, we have

$$\mathbb{E}(\mathrm{td}^w(G(n,\mathbf{p}_n))) \le 1\cdot\mathbb{P}(\mathcal{A}_n) + \delta_n \le 1 + \delta_n, \tag{3.3}$$

where $\delta_n := \mathbb{E}\left(\sum_{1\le i<j\le n} w(e_{ij})1_{\mathcal{A}_n^c}\right)$, $1_{\mathcal{A}}$ presents the indicator function of an event $\mathcal{A}$, and $\mathcal{A}^c$ is the complement of $\mathcal{A}$.

By using the Cauchy-Schwarz inequality, we have

$$\delta_n \le \sqrt{\mathbb{E}\left(\sum_{1\le i<j\le n} w(e_{ij})\right)^2}\cdot\sqrt{\mathbb{P}(\mathcal{A}_n^c)}. \tag{3.4}$$

Notice that the inequality $ab \le (a^2+b^2)/2 < a^2 + b^2$ holds for any real numbers a and b, we have the estimate

$$\begin{aligned}\mathbb{E}\left(\sum_{1\le i<j\le n} w(e_{ij})\right)^2 &\le \binom{n}{2}\sum_{1\le i<j\le n}\mathbb{E}w^2(e_{ij})\\ &\le \binom{n}{2}^2\gamma\\ &\le \left(\frac{en}{2}\right)^4\gamma, \end{aligned} \tag{3.5}$$

where we used the condition (ii) and the fact that $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$ for any n and k (see e.g. [10, Lem 21.1]). Combining (3.4) and (3.5), we arrive at

$$\delta_n \le \frac{e^2n^2}{4}\sqrt{\gamma}e^{-\Theta(n)} = \sqrt{\gamma}e^{-\Theta(n)}$$

by using Theorem 1. Feeding this into (3.3) yields the desired estimate $\mathbb{E}\left(\mathrm{td}^w(G(n,\mathbf{p}_n))\right) \le 1 + \sqrt{\gamma}e^{-\Theta(n)}$.

Another application of Theorem 1 yields

$$\mathbb{P}(\mathrm{td}^w(G(n,\mathbf{p}_n)) > 1) \le \mathbb{P}(\mathcal{A}_n^c) \le e^{-\Theta(n)}$$

for all n large. Consequently, $\mathbb{P}\left(\mathrm{td}^w(G(n,\mathbf{p}_n)) \le 1\right) \ge 1 - e^{-\Theta(n)}$. □

For homogeneous Erdős-Rényi random graphs, we have the following result.

Corollary 2. *Let F be the common cumulative distribution function for edge weights. Assume that there are constants $a > 0$, $b > 0$, and $0 < c < 1$ satisfying $F(x) \ge ax^c$ for all $x \in (0,b)$. If there exists a constant γ satisfying $\mathbb{E}w^2(e) \le \gamma$ for any edge $e \in K_n$, we have*

$$\mathbb{P}\left(\mathrm{td}^w(G(n,p_n)) \le 1\right) \ge 1 - e^{-\Theta(n)} \tag{3.6}$$

and

$$\mathbb{E}\left(\mathrm{td}^w(G(n,p_n))\right) \le 1 + \sqrt{\gamma}e^{-\Theta(n)} \tag{3.7}$$

for all n large, where $p_n = F\left(\frac{1}{n}\right)$.

Proof. We have $p_n = F(n^{-1}) \ge an^{-c}$ for all $n > b^{-1}$. Since $c \in (0,1)$, $np_n \ge an^{1-c} \ge \alpha^{-1}$ for any $\alpha > 0$ for large n. Therefore, the condition of Corollary 1, i.e., (i) in Theorem 2 holds by taking $\beta = 1$ and $p_n(e_{ij}) \equiv p_n$. The condition (ii) in Theorem 2 also holds. Therefore, (3.6) and (3.7) follow from (3.1) and (3.2), respectively. □

Finally, we present a example of non-trivial cumulative distribution functions that satisfy the conditions (i) and (ii) in Theorem 2. For $1 \le i < j \le \lceil\frac{n}{10}\rceil$, we set

$$F_{ij}(x) = \begin{cases} 0, & x < 0; \\ x^{\frac{3}{2}}, 0 \le x \le 1; \\ 1, & x > 1; \end{cases}$$

and for any other $i < j$, set

$$F_{ij}(x) = \begin{cases} 0, & x < 0; \\ x^{\frac{1}{2}}, 0 \le x \le 1; \\ 1, & x > 1. \end{cases}$$

Therefore, for $1 \le i < j \le \lceil \frac{n}{10} \rceil$, we have $p_n(e_{ij}) = F_{ij}(n^{-1}) = n^{-\frac{3}{2}}$, and for any other $i < j$, $p_n(e_{ij}) = F_{ij}(n^{-1}) = n^{-\frac{1}{2}}$. Let $\alpha = 1$, $\beta = 10$, and $p_n = \frac{1}{n}$ for all $n \ge 1$. Since $\sqrt{\frac{\alpha \ln 3}{2\beta}} > \frac{1}{5}$, for any $i \notin S$ and $|S| \ge \frac{n}{5}$, we have

$$\begin{aligned} d_n(i,S) \ge& \frac{1}{|S|}\left(\frac{1}{n\sqrt{n}}\left\lceil\frac{n}{10}\right\rceil + \left(|S| - \left\lceil\frac{n}{10}\right\rceil\right)\frac{1}{\sqrt{n}}\right) \\ \ge& \frac{5}{n}\left(\frac{1}{n\sqrt{n}} \cdot \frac{n}{10} + \left(\frac{n}{10} - 1\right)\frac{1}{\sqrt{n}}\right) \\ \ge& \frac{6}{10\sqrt{n}} \\ >& \beta p_n, \end{aligned}$$

for all $n \ge 278$. Therefore, (i) holds true. From the distribution function $F_{ij}(x)$ it is straightforward to see that $\gamma = \frac{3}{7}$ would satisfy the condition (ii). Thus, from (3.1) and (3.2) we can conclude that $\mathbb{P}\left(\mathrm{td}^w(G(n,\mathbf{p}_n)) \le 1\right) \ge 1 - e^{-\Theta(n)}$ and $\mathbb{E}\left(\mathrm{td}^w(G(n,\mathbf{p}_n))\right) \le 1 + \sqrt{\frac{3}{7}}\, e^{-\Theta(n)}$ for all large n.

It is worth mentioning that in the above example the distribution function F_{ij} defined for $1 \le i < j \le \lceil \frac{n}{10} \rceil$ does not satisfy the assumption of distribution function in Corollary 2.

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