Distance Laplacian spectral ordering of sun type graphs

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Abstract

Let G be a simple, connected graph of order n. Its distance Laplacian energy $DLE(G)$ is given by $DLE(G) = \sum_{i=1}^{n} \left|\rho_i^L - \frac{2W(G)}{n}\right|$, where $\rho_1^L \geq \rho_2^L \geq \cdots \geq \rho_n^L$ are the distance Laplacian eigenvalues and $W(G)$ is the Wiener index of G. Distance Laplacian eigenvalues of sun and partial sun graphs have been characterized. We order the partial sun graphs by using their second largest distance Laplacian eigenvalue. Moreover, the distance Laplacian energy of sun and partial sun graphs have been derived in this paper. These graphs are also ordered by using their distance Laplacian energies.

Keywords: Distance matrix, distance Laplacian matrix, transmission regular graph, distance Laplacian energy, sun graphs.
2020 MSC: 05C50, 05C12, 15A18.

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1. Introduction

Let $G = G(V(G), E(G))$ be a simple, connected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. Let $n = |V(G)|$ and $m = |E(G)|$, which are referred to as the order and the size of G, respectively. For a vertex $v \in V(G)$, we write $N(v)$ for the neighborhood (open) of v containing all adjacent vertices of v. The cardinality of $N(v)$ is the degree of v, which is denoted by $d_G(v)$ or simply d_v. If all vertices share the same degree, the graph is called a regular graph. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be the adjacency matrix of G, where the (i, j)-entry $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. $Deg(G) = diag(d_1, d_2, \ldots, d_n)$ with $d_i = d_G(v_i)$, $i = 1, 2, \ldots, n$ is the degree matrix. The Laplacian matrix of G is $L(G) = Deg(G) - A(G)$. The analogous matrix $Q(G) = Deg(G) + A(G)$ is called the signless Laplacian matrix. These two matrices are real positive semi-definite. We arrange their eigenvalues as $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ (Laplacian spectrum) and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ (signless Laplacian spectrum), respectively. Any unspecified terminologies regarding spectral graph theory can be found in [15, 17].

The distance between two vertices u and v, denoted by d_{uv}, is the length of a shortest path between them in G. The diameter of G is the maximum distance between all vertex pairs. Let $D(G) = (d_{uv})_{u,v\in V(G)}$ be the distance matrix of G. $Tr_G(v) = \sum_{u\in V(G)} d_{uv}$ is called the transmission of a vertex $v \in V(G)$. If $Tr_G(v) = k$ for every $v \in V(G)$, G is k-transmission regular. Let $W(G)$ be the sum of distances between any unordered pairs of vertices. The quantity is known as the transmission number or the Wiener index. We have $W(G) = \frac{1}{2} \sum_{v\in V(G)} Tr_G(v)$. Given $v_i \in V(G)$, the transmission $Tr_G(v_i)$ (or simply Tr_i) is also known as the transmission degree. For vertex v_i, its second transmission degree is defined as $T_i = \sum_{j=1}^{n} d_{ij} Tr_j$.

The diagonal matrix of vertex transmissions is denoted by $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$. Given the distance matrix of a connected graph, the associated Laplacian and its signless version have been first investigated in [4]. The distance Laplacian matrix of G is given by $D^L(G) = Tr(G) - D(G)$ and the distance signless Laplacian matrix of G is given by $D^Q(G) = Tr(G) + D(G)$.

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In 1978, Ivan Gutman [11] introduced the energy of graph, which was originated in theoretical chemistry. Let the adjacency eigenvalues of G be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The energy $E(G)$ is given by $E(G) = \sum_{i=1}^{n} |\lambda_i|$. Recent results regarding graph energy can be found in [12, 17].

The spectrum-based graph invariant has a long history in the study of chemical and mathematical literature. The concept of graph energy has been generalized to accommodate various applications by considering other matrices associated with a graph. In particular, Nikiforov [18] defined the energy of any matrix. A variety of recent results are reported in [1, 21] in this direction.

The distance and the distance Laplacian eigenvalues of G are denoted by $\rho_1^D \geq \rho_2^D \geq \ldots \geq \rho_n^D$ and $\rho_1^L \geq \rho_2^L \geq \ldots \geq \rho_n^L$, respectively. The distance energy [16] of G is defined as $DE(G) = \sum_{i=1}^{n} |\rho_i^D|$. Results regarding the distance energy can be found in [8, 2]. In [22], the energy of distance matrix $D(G)$ is generalized to the energy of distance Laplacian matrix $D^L(G)$. For a connected graph G, its distance Laplacian energy $DLE(G)$ [22] is given by

$$DLE(G) = \sum_{i=1}^{n} \left| \rho_i^L - \frac{2W(G)}{n} \right|.$$

Denote by σ the largest positive integer satisfying $\rho_\sigma^L \geq \frac{2W(G)}{n}$. Define $S_k^L(G) = \sum_{i=1}^{k} \rho_i^L$ to be the sum of k largest distance Laplacian eigenvalues. Noting $\sum_{i=1}^{n} \rho_i^L = 2W(G)$, it is proved in [8] that

$$\begin{aligned} DLE(G) = 2\left(\sum_{i=1}^{\sigma} \rho_i^L(G) - \frac{2\sigma W(G)}{n}\right) &= 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^L(G) - \frac{2jW(G)}{n}\right) \\ &= 2 \max_{1\leq j\leq n} \left(S_j^L(G) - \frac{2jW(G)}{n}\right). \end{aligned}$$

More related results regarding $DLE(G)$ are reported in the recent work [2, 6, 8, 9, 10, 14, 20].

Given a graph matrix, finding the extremal graphs with respect to a specific graph invariant associated with this matrix among a class of graphs is an interesting problem, which has attracted a constant research enthusiasm over the past few decades. For example, it is shown in [8] that the star $K_{n-1,1}$ has the minimum distance Laplacian energy among all trees of order n. A related problem is to order the graphs among a class of graphs of the same order with respect to a given spectral graph invariant (such as energy, spectral radius, etc). This problem has recently been considered by Hilal [9] for distance Laplacian matrix for trees having diameter three. These trees have been ordered in view of the distance Laplacian energies. In this paper, we aim to continue this line of research and study the distance Laplacian spectrum and the distance energy of the sun and partial sun graphs. We will order partial sun graphs taking advantage of their second largest distance Laplacian eigenvalue. Furthermore, we establish a type of ordering for sun and partial sun graphs based on their distance Laplacian energy.

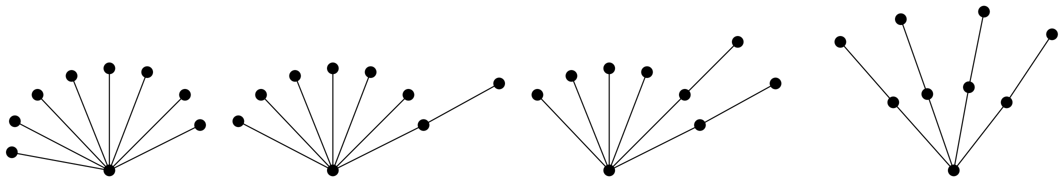
2. Distance Laplacian spectrum for sun and partial sun graphs

The *sun graph* $S_n(a, a)$ (see [13]) is a tree of order $n = 2a + 1$, containing a pendent vertices, each attached to a vertex of degree 2, and a vertex of degree a. The sun graph of order n can be viewed as obtained from the star graph $K_{1,n-1}$ by deleting $\frac{n}{2} - 1$ pendent vertices and inserting a new vertex on remaining $\frac{n}{2} - 1$ edges of the star $K_{1,n-1}$. We define a partial sun graph $S_n(a, k)$ as the tree of order $n = a + k + 1$, containing $k, 1 \leq k \leq a - 1$ pendent vertices each attached to a vertex of degree 2, $a - k$ pendent vertices each attached to a vertex of degree a and a vertex of degree a.

We note that $S_3(1, 1) \cong P_3,\ S_n(a, 0) = K_{1,a}$. Examples of some sun and partial sun type graphs of order 9 are shown in Figure (1). Sun graphs are very important, and play candidate roles for extremal graphs with respect to various graph invariants. For instance see [13, 3] and the references therein.

For, two sequences of real numbers $\{s_1, s_2, \ldots, s_n\}$ and $\{s'_1, s'_2, \ldots, s'_m\}$ with $n > m$. The latter is said to interlace the former, whenever

$$s_i \geq s'_i \geq s_{n-m+i}, \quad \text{for } i = 1, 2, \ldots, m.$$



$S_9(8,0)$ $S_9(7,1)$ $S_9(6,2)$ $S_9(4,4)$

Figure 1, Sun and partial sun graphs on 9 vertices.

The interlacing is said to be tight if there exists a positive integer $k \in [0, m]$ such that

$$s_i = s'_i \quad \text{for } i = 1, 2, \ldots, k \text{ and } s_{n-m+i} = s'_i \quad \text{for } k+1 \le i \le m.$$

Consider the following block matrix

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,s} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{s,1} & m_{s,2} & \cdots & m_{s,s} \end{pmatrix}_{n\times n},$$

such that rows and columns of M are partitioned according to a partition $S = \{S_1, S_2, \ldots, S_s\}$ of the index set $I = \{1, 2, \ldots, n\}$. The *quotient matrix* Q of M is the $s \times s$ matrix whose (i, j)-th entry is the average row (column) sum of the block $m_{i,j}$. The partition S is said to be *regular* (equitable) if each block $m_{i,j}$ of M has constant row (column) sum and in this case the matrix Q is called a *regular quotient matrix* (or equitable quotient matrix). In general, the eigenvalues of Q interlace the eigenvalues of M, while for regular (equitable) partitions [7], the interlacing is tight, that is each eigenvalue of Q is an eigenvalue of M.

Our first result is the following which fully characterizes the distance Laplacian spectrum of the sun graph $S_n(a, a)$.

Theorem 2.1. *For $a \ge 2$, let $S_n(a, a)$ be the sun graph of order $n = 2a+1$. Then the distance Laplacian spectrum of $S_n(a, a)$ is*

$$\left\{0, \frac{1}{2}\left(12a - 1 \pm \sqrt{4a^2 + 4a + 17}\right)^{a-1}, \frac{1}{2}\left(9a - 1 \pm \sqrt{9a^2 - 22a + 17}\right)\right\}.$$

Proof. Let $a \ge 2$ be a positive integer such that $n = 2a+1$ and let $\{v_1, v_2, \ldots, v_a, u_1, \ldots, u_a, w\}$ be the vertices of $S_n(a, a)$, where for $i = 1, 2, \ldots, a$, v_i's are the pendent vertices, u_i's are vertices of degree 2 and degree of w is a. Under this labelling of $S_n(a, a)$, the distance Laplacian matrix can be written as

$$\left(\begin{array}{ccccc|ccccc|c}
7a-4 & -4 & \cdots & -4 & -4 & -1 & -3 & \cdots & -3 & -3 & -2 \\
-4 & 7a-4 & \cdots & -4 & -4 & -3 & -1 & \cdots & -3 & -3 & -2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-4 & -4 & \cdots & 7a-4 & -4 & -3 & -3 & \cdots & -1 & -3 & -2 \\
-4 & -4 & \cdots & -4 & 7a-4 & -3 & -3 & \cdots & -3 & -1 & -2 \\
\hline
-1 & -3 & \cdots & -3 & -3 & 5a-3 & -2 & \cdots & -2 & -2 & -1 \\
-3 & -1 & \cdots & -3 & -3 & -2 & 5a-3 & \cdots & -2 & -2 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-3 & -3 & \cdots & -1 & -3 & -2 & -2 & \cdots & 5a-3 & -2 & -1 \\
-3 & -3 & \cdots & -3 & -1 & -2 & -2 & \cdots & -2 & 5a-3 & -1 \\
\hline
-2 & -2 & \cdots & -2 & -2 & -1 & -1 & \cdots & -1 & -1 & 3a
\end{array}\right). \tag{2.1}$$

Now, for $i = 2, 3, \ldots, a$, define a sequence of vectors over $\mathbb{R}$ as follows:

$$X_{i-1} = (-\alpha, x_{i2}, x_{i3}, \ldots, x_{ia-1}, x_{ia}, -1, x^*_{i2}, x^*_{i3}, \ldots, x^*_{ia-1}, x^*_{ia}, 0) \in \mathbb{R}^n(\mathbb{R}),$$

where

$$\alpha = \frac{1}{4}\Big(n + \sqrt{4a^2+4a+17}\Big),\ x_{ij} = \begin{cases}\alpha & \text{if } i=j\\ 0 & \text{otherwise}\end{cases},\ \text{and } x^*_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}.$$

We fist show that $X_1, X_2, \dots, X_{a-1}$ are linearly independent vectors. If not, then there exists scalers $a_1, a_2, \dots, a_{a-1}$ not all zero such that

$$a_1X_1 + a_2X_2 + \cdots + a_{a-1}X_{a-1} = \mathbf{0},$$

where $\mathbf{0}$ is the zero vector of $\mathbb{R}^n$. After, necessary calculation, we obtain

$$\left(\alpha\sum_{i=1}^{a-1} a_i, \alpha a_1, \alpha a_2, \dots, \alpha a_{a-2}, \alpha a_{a-1}, -\sum_{i=1}^{a-1} a_i, a_2, \dots, a_{a-2}, a_{a-1}, 0\right) = \mathbf{0}.$$

It follows that $a_1 = a_2 = \cdots = a_{a-1} = 0$ and this implies that $X_1, X_2, \dots, X_{a-1}$ cannot be linearly dependent vectors. Also, by the definition of eigenequation of the matrix $D^L(G)$, we have

$$D^L(S_n(a,a))X_1 = \begin{pmatrix} -(7a-4)\alpha - 4\alpha - 2\\ (7a-4)\alpha + 4\alpha + 2\\ 0\\ \vdots\\ 0\\ -2\alpha - 2 - (5a-3)\\ 2\alpha + 2 + (5a-3)\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}(14a^2+7a+8+7a\sqrt{D})\\ \frac{1}{4}(14a^2+7a+8+7a\sqrt{D})\\ 0\\ \vdots\\ 0\\ -\frac{1}{2}(12a-1+\sqrt{D})\\ \frac{1}{2}(12a-1+\sqrt{D})\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}(12a-1+\sqrt{D})\frac{1}{4}(2a+1+\sqrt{D})\\ \frac{1}{2}(12a-1+\sqrt{D})\frac{1}{4}(2a+1+\sqrt{D})\\ 0\\ \vdots\\ 0\\ -\frac{1}{2}(12a-1+\sqrt{D})\\ \frac{1}{2}(12a-1+\sqrt{D})\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{1}{4}(n+\sqrt{D})\\ \frac{1}{4}(n+\sqrt{D})\\ 0\\ \vdots\\ 0\\ -1\\ 1\\ 0\\ \vdots\\ 0\\ 0 \end{pmatrix} = \lambda X_1,$$

where $\lambda = \frac{1}{2}(12a-1+\sqrt{D})$ and $D = 4a^2+4a+17$. Thus it follows that $\frac{1}{2}\Big(12a-1+\sqrt{4a^2+4a+17}\Big)$ is the eigenvalue of $D^L(S_n(a,a))$ with the corresponding eigenvector X_1. Proceeding with a similar argument, we can show that $X_2, X_3, \dots, X_{a-1}$ are also the eigenvectors corresponding to the eigenvalue $\frac{1}{2}\Big(12a-1+\sqrt{4a^2+4a+17}\Big)$. In this manner, we obtain $a-1$ distance Laplacian eigenvalues of $D^L(S_n(a,a))$.

Again, For $i = 2, 3, \dots, a$, let

$$Y_{i-1} = (-\beta, y_{i2}, y_{i3}, \dots, y_{ia-1}, y_{ia}, -1, y^*_{i2}, y^*_{i3}, \dots, y^*_{ia-1}, y^*_{ia}, 0) \in \mathbb{R}^n(\mathbb{R}),$$

where

$$\beta = \frac{1}{4}(n - \sqrt{4a^2+4a+17}),\ y_{ij} = \begin{cases}\beta & \text{if } i=j\\ 0 & \text{otherwise}\end{cases},\ \text{and } y^*_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}.$$

It is easy to see that Y_{i-1}, with $i = 2, 3, \ldots, a$ are linearly independent vectors. Further, as above we can verify that

$$D^L(S_n(a,a))Y_1 = \begin{pmatrix} -\beta(7a) - 2 \\ \beta(7a) + 2 \\ 0 \\ \vdots \\ 0 \\ -2\beta - 2 - (5a-3) \\ 2\beta + 2 + (5a-3) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \mu Y_1,$$

where $\mu = \frac{1}{2}(12a - 1 - \sqrt{4a^2 + 4a + 17})$. Therefore, $\frac{1}{2}\Big(12a - 1\sqrt{4a^2 + 4a + 17}\Big)$ is the distance Laplacian eigenvalue of $D^L(S_n(a,a))$ with the corresponding eigenvector Y_1. Proceeding similarly, we can show that $Y_2, Y_3, \ldots, Y_{a-1}$ are also the eigenvectors corresponding to the eigenvalue $\frac{1}{2}\Big(12a - 1 - \sqrt{4a^2 + 4a + 17}\Big)$. This gives that $\frac{1}{2}\Big(12a - 1 - \sqrt{4a^2 + 4a + 17}\Big)$ is an eigenvalue of $D^L(S_n(a,a))$ with multiplicity $a-1$. This way we have obtained $2a - 2$ eigenvalues of $D^L(S_n(a,a))$. The equitable quotient matrix of Matrix (2.1) with given blocks is

$$\begin{pmatrix} 3a & -3a+2 & -2 \\ -3a+2 & 3a-1 & -1 \\ -2a & -a & 3a \end{pmatrix}. \tag{2.2}$$

The other distance Laplacain eigenvalues of $S_n(a,a)$ are the eigenvalues of (2.2) and are given below

$$\left\{0, \frac{1}{2}\Big(9a - 1 \pm \sqrt{9a^2 - 22a + 17}\Big)\right\}.$$

Thus we have completely obtained the distance Laplacian spectrum of $S_n(a,a)$. ■

The following results can be found in [5] and is helpful in finding some D^L-eigenvalues of G.

Lemma 2.2. *[5] Let G be a graph of order n. If $S = \{v_1, v_2, \ldots, v_s\}$ is the independent set of G satisfying $N(v_i) = N(v_j)$ for every $i, j \in \{1, 2, \ldots, s\}$. Then $\partial = Tr(v_i) = Tr(v_j)$ for each $i, j \in \{1, 2, \ldots, s\}$ and $\partial + 2$ is the D^L eigenvalue of G with multiplicity at least $s - 1$.*

The following result gives the distance Laplacian spectrum of the partial sun graph $S_n(a,k)$, for any $k = 1, 2, \ldots, a-1$.

Theorem 2.3. *For $a \geq 2$ and $1 \leq k \leq a-1$, let $S_n(a,k)$ be the partial sun graph of order $n = a + k + 1$. Then the distance Laplacian spectrum of $S_n(a,k)$ consists of the simple eigenvalue 0, the eigenvalues $\frac{1}{2}(5a+7k-1\pm\sqrt{(a+k)^2 + 2(a+k) + 17}\,)$ each with multiplicity $k-1$, the eigenvalue $2a + 3k + 1$ with multiplicity $a - k - 1$ and the zeros of the following polynomial*

$$x^3 - (6a + 8k)x^2 + (-5 + a + 11a^2 + 4k + 31ak + 21k^2)x + 4 + 7a - 3a^2 - 6a^3 + 12k$$
$$- 10ak - 27a^2k - 10k^2 - 39ak^2 - 18k^3.$$

Proof. The distance Laplacian eigenvalues $\frac{1}{2}(5a + 7k - 1 \pm \sqrt{(a+k)^2 + 2(a+k) + 17})$ each with multiplicity $k-1$ can be obtained with the similar process as in Theorem 2.1. Also we note that $S_n(a,k)$ has the independent set of cardinality $n-k$ sharing the same vertex. The common transmission of each vetex of this independent set is $2a + 3k - 1$, so by Lemma 2.2, we see that $2a + 3k + 1$ is the distance Laplacian eigenvalue of $S_n(a,k)$ with multiplicity $a - k - 1$. The other four distance Laplacian eigenvalues of $S_n(a,k)$ are the eigenvalues of the following equitable quotient matrix

$$\begin{pmatrix} a+2k & -k & k-a & -2k \\ -1 & 2a+k-1 & -2(a-k) & 2-3k \\ -1 & -2k & 5k+1 & -3k \\ -2 & 2-3k & -3(a-k) & 3a \end{pmatrix}. \tag{2.3}$$

The characteristic polynomial of (2.3) is

$$f(x) = x(x^3 - (6a+8k)x^2 + (-5+a+11a^2+4k+31ak+21k^2)x + 4 + 7a - 3a^2 - 6a^3 + 12k - 10ak - 27a^2k - 10k^2 - 39ak^2 - 18k^3). \tag{2.4}$$

■

We note that if $z_1 \geq z_2 \geq z_3$ are the zeros of (2.4), then we have

$$\begin{aligned}
f(a+2k+1) &= 4k - k^2 - ak < 0, \text{ for } a \geq 3,\\
f(a+2k+2) &= 2(1 - 4a + a^2 + k + ak) > 0,\\
f(2a+3k-1) &= 2a(k-2) + 2k(k-3) + 8 > 0, \text{ for } k \geq 2\\
f(2a+3k+1) &= -2(a-k)(1+a+k) < 0,\\
f(3a+3k-3) &= -6a^2 + 2ak + 16a + 2k^2 - 6k - 8\\
&= 2(k^2 - a^2) + 2a(k-a) - 2a(a-8) - 6k - 8 < 0, \text{ for } a \geq 2,\\
f(3a+3k) &= 5ak - 8a + 2k^2 - 3k + 4 = 5a\Big(k - \frac{8}{5}\Big) + 2k\Big(k - \frac{3}{2}\Big) + 4 > 0, \text{ for } k \geq 2.
\end{aligned}$$

So, by intermediate value theorem, it follows that for $a \geq 3$ and $k \geq 2$, we have $z_3 \in (a+2k+1, a+2k+2)$, $z_2 \in (2a+3k-1, 2a+3k+1)$ and $z_1 \in (3a+3k-3, 3a+3k)$.

Next we have the following consequences of Theorem 2.3, which completely describes the distance Laplacian spectrum of the graph $S_n(a,i), i = 1, 2, a-1, a-2$.

Corollary 2.4. *Let $S_n(a,1)$ with $a \geq 3$ be the sun graph of order $n = a+2$. Then the distance Laplacian spectrum of $S_n(a,1)$ consists of the eigenvalue $2a+4$ with multiplicity $n-4$, the simple eigenvalue 0 and the zeros $z_1 \geq z_2 \geq z_3$ of the following polynomial*

$$x^3 - x^2(6a+8) + x(11a^2 + 32a + 20) - 6a^3 - 30a^2 - 42a - 12,$$

where $z_1 \in (3a+3, 3a+4)$, $z_2 \in (2a+1, 2a+2)$ and $z_3 \in (a+3, a+4)$.

Proof. Let the vertex labelling of $S_n(a,1)$ be $\{u, u', v_1, v_2, \ldots, v_{a-1}, w\}$, where $d(u) = 1$, $d(u') = 2$, v_i's are pendent vertices and degree of w is a. Also, v_i's form an independent set with common transmission $2a+2$. Thus by Theorem 2.3, it follows that $2a+4$ is the distance Laplacian eigenvalue of $S_n(a,1)$ with multiplicity $a-2$. The other distance Laplacian eigenvalues of $S_n(a,1)$ are the eigenvalues of the following matrix

$$\begin{pmatrix} 3a & -1 & -3(a-1) & -2\\ -1 & 2a & -2(a-1) & -1\\ -3 & -2 & 6 & -1\\ -2 & -1 & -(a-1) & a+2 \end{pmatrix},$$

and its characteristic polynomial is $xp(x)$, where $p(x) = x^3 - x^2(6a+8) + x(11a^2 + 32a + 20) - 6a^3 - 30a^2 - 42a - 12$. Let $z_1 \geq z_2 \geq z_3$ be the zeros of $p(x)$, then by manual calculations, we obtain

$$\begin{aligned}
p(a+3) &= 3 - a < 0,\\
p(a+4) &= 2(a-2)(a-1) > 0,\\
p(2a+1) &= (a-1)^2 > 0,\\
p(2a+2) &= -2(a-2) < 0,\\
p(3a+3) &= -3(a-1) < 0,\\
p(3a+4) &= 2(a^2 + a + 2).
\end{aligned}$$

So, by intermediate value theorem, we see that $z_1 \in (3a+3, 3a+4)$, $z_2 \in (2a+1, 2a+2)$ and $z_3 \in (a+3, a+4)$. ■

Corollary 2.5. *Let $S_n(a,2)$ with $a \geq 2$ be the sun graph of order $n = a+3$. Then the distance Laplacian spectrum of $S_n(a,2)$ consists of the eigenvalue $2a+7$ with multiplicity $a-3$, the simple eigenvalue 0, the eigenvalues $\frac{1}{2}\left(5a+13 \pm \sqrt{a^2+6a+25}\right)$ and the zeros $z'_1 \geq z'_2 \geq z'_3$ of the following polynomial*

$$x^3 - x^2(6a+16) + x(11a^2+63a+87) - 6a^3 - 57a^2 - 169a - 156,$$

where $z'_1 \in (3a+5, 3a+6)$, $z'_2 \in (2a+5, 2a+6)$ and $z'_3 \in (a+5, a+6)$.

Proof. By Theorem 2.3, $2a+4$ is the distance Laplacian eigenvalue of $S_n(a,2)$ with multiplicity $a-3$. The other distance Laplacian eigenvalues of $S_n(a,2)$ are the eigenvalues of the following polynomial is

$$x(x^2 - x(5a+13) + 6a^2 + 31a + 36)p(x),$$

where $p(x) = x^3 - x^2(6a+16) + x(11a^2+63a+87) - 6a^3 - 57a^2 - 169a - 156$. Let $z'_1 \geq z'_2 \geq z'_3$ be the zeros of $p(x)$, then by manual calculations, we obtain

$$\begin{aligned}
p(a+5) &= -2(a-2) < 0,\\
p(a+6) &= 2(a^2-2a+3) > 0,\\
p(2a+5) &= 2^2 > 0,\\
p(2a+6) &= -((a-2)(a+3)) < 0,\\
p(3a+5) &= -2(a-2)(a+1) < 0,\\
p(3a+6) &= 2(a+3).
\end{aligned}$$

So, by intermediate value theorem, it follows that $z'_1 \in (3a+5, 3a+6)$ $z'_2 \in (2a+5, 2a+6)$ and $z'_3 \in (a+5, a+6)$. ■

Next, proceeding as above, we can verify the cases, $k = a-1$ and $k = a-2$, where $a \geq 3$.

Corollary 2.6. *The distance Laplacian spectrum of $S_n(a, a-1)$ consists of the eigenvalues 0, the eigenvalues $6a - 4 \pm \sqrt{a^2+4}$ each with multiplicity $a-2$, the zeros $z_1 \geq z_2 \geq z_3$ of (2.5), where $z_1 \in (6a-6, 6a-3)$, $z_2 \in (5a-4, 5a-2)$ and $z_3 \in (3a-1, 3a)$.*

$$x^3 + (8-14a)x^2 + (12-68a+63a^2)x - 44a + 136a^2 - 90a^3. \tag{2.5}$$

Corollary 2.7. *The distance Laplacian spectrum of $S_n(a, a-2)$ consists of the eigenvalues 0, the eigenvalues $\frac{1}{2}(12a - 15 \pm \sqrt{4a^2-4a+17}\ )$ each with multiplicity $a-3$, the eigenvalue $5(a-1)$ and the zeros $z_1 \geq z_2 \geq z_3$ of (2.6), where $z_1 \in (6a-9, 6a-6)$, $z_2 \in (5a-7, 5a-5)$ and $z_3 \in (3a-3, 3a-2)$.*

$$x^3 + (16-14a)x^2 + (71-141a+63a^2)x + 84 - 293a + 295a^2 - 90a^3. \tag{2.6}$$

The following result gives that ordering of the trees belonging to the family $S_n(2a-i, i)$, $2 \leq i \leq a-1$, on the basis of their second largest distance Laplacian eigenvalue $z_1(S_n(2a-i,i))$, where z_1 is the largest root of the cubic polynomial $f(x)$ given by equation 2.4 with a replaced by $2a-i$ and k by i.

Theorem 2.8. *For $2 \leq k \leq a-1$, we have $z_1(S_n(2a-(k-1), k-1)) > z_1(S_n(2a-k, k))$.*

Proof. Replacing a by $2a-k$ in the cubic polynomial $f(x)$ given by equation 2.4, we obtain

$$\begin{aligned}
f_k(x) = x^3 - (12a+2k)x^2 + (k^2 + 18ak + 44a^2 + 2a + 3k - 5)x + 4 - 6ak^2\\
- 36ka^2 - 48a^3 - 8ak - 3k^2 - 12a^2 + 5k + 14a + 4.
\end{aligned}$$

Let $z_1 = z_1(S_n(2a-k,k))$ be the largest root of the polynomial $f_k(x)$. By the discussion after the Theorem 2.3, we have $z_1 \in (3a+3k-3, 3a+3k) = (6a-3, 6a)$ as a has to be replaced by $2a-k$. It is easy to verify that

$$f_k(x) - f_{k-1}(x) = -2x^2 + (18a+2k+2)x - 12ak - 36a^2 - 2a - 6k + 8 = -2g(x),$$

where $g(x) = x^2-(9a+k+1)x+6ak+18a^2+a+3k-4$. At $x = z_1(S_n(2a-(k-1), k-1))$, we have $f_k(z_1(S_n(2a-(k-1),k-1))) = f_k(z_1(S_n(2a-(k-1),k-1))) - f_{k-1}(z_1(S_n(2a-(k-1),k-1))) = -2g(z_1(S_n(2a-(k-1),k-1)))$. It is easy to see that the function $g(z_1(S_n(2a-(k-1),k-1))) = z_1(S_n(2a-(k-1),k-1))^2-(9a+k+1)z_1(S_n(2a-(k-1),k-1))+6ak+18a^2+a+3k-4$ is increasing for $z_1(S_n(2a-(k-1),k-1)) \geq \frac{9a+k+1}{2}$. Since $z_1(S_n(2a-(k-1),k-1)) > 6a-3 > \frac{9a+k+1}{2}$, it follows that $g(z_1(S_n(2a-(k-1),k-1)))$ is an increasing function for all $6a-3 < z_1(S_n(2a-(k-1),k-1)) < 6a$. Therefore, $g(z_1(S_n(2a-(k-1),k-1))) \leq g(6a) = -5a+3k-4 < 0$. This gives that $g(z_1(S_n(2a-(k-1),k-1)))$ is negative for all $z_1(S_n(2a-(k-1),k-1)) \in (6a-3,6a)$, which further gives that $f_k(z_1(S_n(2a-(k-1),k-1)))$ is positive. Since $f_k(6a-3) < 0$ and $f_k(6a) > 0$, it follows by intermediate value theorem that the largest root of $f_k(x)$ lies in $(6a-3, z_1(S_n(2a-(k-1),k-1)))$. From this the result follows. ■

3. Distance Laplacian energy of sun type graphs

. In this section, we obtain the distance Laplacian energy of the sun and the partial sun type graphs. We will show these graphs can be ordered based on their distance Laplacian energy.

The following result gives the distance Laplacian energy of the sun graph $S_n(a,a)$.

Theorem 3.1. *For $a \geq 2$, the distance Laplacian energy of $S_n(a,a)$ is*

$$DLE(S_n(a,a)) = 6a-5+(a-1)\sqrt{4a^2+4a+17}+\sqrt{9a^2-22a+17}+\frac{10}{2a+1}.$$

Proof. The distance Laplacian spectrum of $S_n(a,a)$ is given in Theorem 2.1 and it is easy to see that its non-zero distinct eigenvalues can be ordered as

$$\begin{aligned}&\frac{1}{2}\Big(12a-1+\sqrt{4a^2+4a+17}\Big) \geq \frac{1}{2}\Big(9a-1+\sqrt{9a^2-22a+17}\Big)\\ &\geq \frac{1}{2}\Big(12a-1-\sqrt{4a^2+4a+17}\Big) \geq \frac{1}{2}\Big(9a-1-\sqrt{9a^2-22a+17}\Big).\end{aligned}$$

The average of the transmission degree sequence of $S_n(a,a)$ is

$$\frac{2W(S_n(a,a))}{n} = \frac{a(7a-4)+a(5a-3)+3a}{n} = \frac{12a^2-4a}{2a+1}.$$

Besides, if $\frac{1}{2}(12a-1-\sqrt{4a^2+4a+17}) < \frac{12a^2-4a}{2a+1}$, then we obtain $(18a-1)^2 < (2a+1)^2(4a^2+4a+17)$, which further implies that $4(4a^4+8a^3-59a^2+27a+4) > 0$, which is true for $a \geq 2$. The distance spectral radius $\frac{1}{2}(12a-1+\sqrt{4a^2+4a+17})$ is always greater than the average transmission $\frac{12a^2-4a}{2a+1}$. Lastly, it can be easily verified that $\frac{1}{2}(9a-1+\sqrt{9a^2-22a+17})$ is greater than or equal to $\frac{2W(S_n(a,a))}{n}$. Therefore, we get $\sigma = a-1+1 = a$ and by the definition, we have

$$\begin{aligned}DLE(S_n(a,a)) =& 2\Bigg(\sum_{i=1}^{a}\rho_i^L(S_n(a,a)) - \frac{2aW(S_n(a,a))}{n}\Bigg)\\ =& 2\Bigg((a-1)\frac{1}{2}\Big(12a-1+\sqrt{4a^2+4a+17}\Big)+\frac{1}{2}\Big(9a-1+\sqrt{9a^2-22a+17}\Big)\\ &-\frac{a(12a^2-4a)}{2a+1}\Bigg)\\ =& 6a-5+(a-1)\sqrt{4a^2+4a+17}+\sqrt{9a^2-22a+17}+\frac{10}{2a+1}.\end{aligned}$$

■

The following result gives the distance Laplacian energy of $S_n(a,1)$ and $S_n(a,2)$.

Theorem 3.2. *For the sun graph $S_n(a,k)$, $k=1,2$ with $a \geq 3$, the following holds.*

(i) *The distance Laplacian energy of $S_n(a,1)$ is*

$$DLE(S_n(a,1)) = 2\Big(z_1+z_2-2a-4-\frac{8}{a+2}\Big),$$

where $z_1 \in (3a+3, 3a+4)$ and $z_2 \in \Big(\frac{2a(a+3)}{a+2}, 2a+2\Big)$.

(ii) *For $a \leq 7$, the distance Laplacian energy of $S_n(a,2)$ is*

$$DLE(S_n(a,2)) = \sqrt{a^2+6a+25} + 2(z_1' + z_2') - 5a - 9 - \frac{60}{a+3},$$

where $z_1' \in (3a+5, 3a+6)$ and $z_2' \in \left(\frac{2(a^2+6a+4)}{a+3}, 2a+2\right)$.
For $a \geq 8$, the distance Laplacian energy of $S_n(a,2)$ is

$$DLE(S_n(a,2)) = \sqrt{a^2+6a+25} + 2z_1' - a + 3 - \frac{80}{a+3},$$

where $z_1' \in (3a+5, 3a+6)$.

Proof. By Corollary 2.4, the distance Laplacian spectrum of $S_n(a,1)$ consists of the eigenvalue $2a+4$ with multiplicity $n-4$, the simple eigenvalue 0 and the zeros $z_1 \geq z_2 \geq z_3$ of the following polynomial

$$p(x) = x^3 - x^2(6a+8) + x(11a^2+32a+20) - 6a^3 - 30a^2 - 42a - 12,$$

where $z_1 \in (3a+3, 3a+4)$, $z_2 \in (2a+1, 2a+2)$ and $z_3 \in (a+3, a+4)$. Also, the average of the transmission degrees is

$$\frac{2W(S_n(a,1))}{n} = \frac{3a+2a+a+2+(2a+2)(a-1)}{a+2} = \frac{2a(a+3)}{a+2} = 2a+2-\frac{4}{a+2}.$$

It is clear that z_1 is the distance Laplacian spectral radius of $S_n(a,1)$ and is always greater than or equal to $\frac{2W(S_n(a,1))}{n}$. Also, it is easy to verify the validity of $2a+4 \geq \frac{2a(a+3)}{a+2}$. Next, we show that z_2 is greater than or equal to $\frac{2W(S_n(a,1))}{n}$. By Theorem 2.4, $z_2 \in (2a+1, 2a+2)$, so if $2a+1 \geq \frac{2a(a+3)}{a+2}$, we obtain $a \leq 2$, since we already assumed $a \geq 3$. Further by manual calculations, we found that

$$p\left(\frac{2a(a+3)}{a+2}\right) = \frac{2\left(a^4+4a^3+8a^2-48\right)}{(a+2)^3} > 0 \text{ and } p(2a+2) = -2(a-2) < 0.$$

Thus $z_2 \in \left(\frac{2a(a+3)}{a+2}, 2a+2\right)$ and it follows that $z_2 \geq \frac{2W(S_n(a,1))}{n}$. Therefore, $\sigma = 1+n-4+1 = n-2 = a$ and so the distance Laplacian energy of $S_n(a,1)$ is given by

$$\begin{aligned} DLE(S_n(a,1)) =& 2\Big(\sum_{i=1}^{a} \rho_i^L(S_n(a,1)) - \frac{2aW(S_n(a,1))}{n}\Big) \\ =& 2\Big(z_1 + (2a+4)(a-2) + z_2 - \frac{a(2a(a+3))}{a+2}\Big) \\ =& 2\Big(z_1 + z_2 - 2a - 4 - \frac{8}{a+2}\Big). \end{aligned}$$

(ii) The average of the transmission degrees of $S_n(a,2)$ is $\frac{2W(S_n(a,2))}{n} = \frac{2(a^2+6a+4)}{a+3} = 2a+6-\frac{10}{a+3}$ and by Theorem 2.5, the distance Laplacian spectrum of $S_n(a,2)$ consists of the eigenvalue $2a+7$ with multiplicity $a-3$, the simple eigenvalue 0, the eigenvalues $\frac{1}{2}\left(5a+13 \pm \sqrt{a^2+6a+25}\right)$ and the zeros $z_1' \geq z_2' \geq z_3'$ of the polynomial

$$q(x) = x^3 - x^2(6a+16) + x(11a^2+63a+87) - 6a^3 - 57a^2 - 169a - 156,$$

where $z_1' \in (3a+5, 3a+6)$, $z_2' \in (2a+5, 2a+6)$ and $z_3' \in (a+5, a+6)$. Clearly, the distance Laplacian spectral radius $\frac{1}{2}(5a+13+\sqrt{a^2+6a+25})$ is always greater than or equal to $\frac{2(a^2+6a+4)}{a+3}$. Besides, $2a+7 \geq \frac{2(a^2+6a+4)}{a+3}$ gives $a+13 \geq 0$, which is true for positive a. Also $z_1' > 2a+7$, so it is greater than $\frac{2W(S_n(a,2))}{n}$. For the eigenvalue $z_2' \in (2a+5, 2a+6)$, since $\frac{2W(S_n(a,2))}{n} = 2a+6-\frac{10}{a+3} \in (2a+5, 2a+6)$, we need to calculate $q\left(\frac{2W(S_n(a,2))}{n}\right)$. By direct calculation it can ve verified that $q\left(\frac{2W(S_n(a,2))}{n}\right) = \frac{-a^5+40a^3+120a^2+245a-508}{a^3+9a^2+27a+27}$. It is now easy to verify that this last expression is negative for all $a \geq 8$ and positive for $a \leq 7$. Since $q(2a+5) > 0$ and $q(2a+6) < 0$, it follows that $z_2' \in (2a+5, \frac{2W(S_n(a,2))}{n})$, for $a \geq 8$ and $z_2' \in (\frac{2W(S_n(a,2))}{n}, 2a+6)$, for $a \leq 7$. Lastly, for the eigenvalues z_3' and $\frac{1}{2}\left(5a+13-\sqrt{a^2+6a+25}\right)$, it is clear that $z_3' < \frac{2W(S_n(a,2))}{n}$

and $\frac{1}{2}\left(5a + 13 - \sqrt{a^2 + 6a + 25}\right) < \frac{2W(S_n(a,2))}{n}$, for all a. Thus, it follows that for $a \leq 7$, we have $\sigma = 1 + a - 3 + 2 = a$ and for $a \geq 8$, we have $\sigma = 1 + a - 3 + 1 = a - 1$. Therefore, for $\sigma = a$, the distance Laplacian energy of $S_n(a, 2)$ is given by

$$\begin{aligned}
DLE(S_n(a,2)) =& 2\Big(\sum_{i=1}^{a} \rho_i^L(S_n(a,2)) - \frac{2W(S_n(a,2))}{2}\Big)\\
=& 2\Big(\frac{1}{2}(5a + 13 + \sqrt{a^2 + 6a + 25}) + (2a + 7)(a - 3) + z_1' + z_2'\\
& - \frac{2a(a^2 + 6a + 4)}{a + 3}\Big)\\
=& \sqrt{a^2 + 6a + 25} + 2(z_1' + z_2') - 5a - 9 - \frac{60}{a + 3}.
\end{aligned}$$

Similarly, for $\sigma = a - 1$, the distance Laplacian energy of $S_n(a, 2)$ is

$$DLE(S_n(a,2)) = \sqrt{a^2 + 6a + 25} + 2z_1' - a + 3 - \frac{80}{a + 3}.$$

■

The next result is an immediate consequence of above result and gives the estimates for the distance Laplacian energy of $S_n(a, 1)$ and $S_n(a, 2)$.

Corollary 3.3. *Let $S_n(a, k)$, $k = 1, 2$ be the sun graph. Then the following holds.*

(i) *For $a \geq 3$, the distance Laplacian energy of $S_n(a, 1)$ satisfies*

$$6a - \frac{16}{a + 2} < DLE(S_n(a,1)) < 6a + 4 - \frac{16}{a + 2}.$$

(ii) *For $a \geq 8$, the distance Laplacian energy of $S_n(a, 2)$ satisfies*

$$5a + 13 + \sqrt{a^2 + 6a + 25} + \frac{80}{a + 3} < DLE(S_n(a,2)) < 5a + 15 + \sqrt{a^2 + 6a + 25} + \frac{80}{a + 3}.$$

Theorem 3.4. *For $a \geq 3$ and $3 \leq k \leq a - 1$, let $S_n(a, k)$ be the partial sun graph of order $n = a + k + 1$ and let $\gamma = (a + k)^2 + 2(a + k) + 17$. Then the following holds.*

(i) *If $k \geq 7$, then*

$$DLE(S_n(a,k)) = 2z_1 + (k - 1)\sqrt{\gamma} + ak - 5a - k^2 - 4k + 1 + \frac{2k(6k - 2)}{a + k + 1}.$$

(ii) *If $3 \leq k \leq 6$, then*

$$DLE(S_n(a,k)) = 2z_1 + (k - 1)\sqrt{\gamma} - ak + a + k^2 - 8k - 5 + \frac{2(a - 1)(6k - 2)}{a + k + 1}.$$

Proof. By Theorem 2.3, the distance Laplacian spectrum of $S_n(a, k)$ consists of the simple eigenvalue 0, the eigenvalues $\frac{1}{2}(5a+7k-1 \pm \sqrt{(a + k)^2 + 2(a + k) + 17}\,)$ each with multiplicity $k-1$, the eigenvalue $2a+3k+1$ with multiplicity $a-k-1$ and the zeros of the following polynomial

$$\begin{aligned}
p_1(x) = & x^3 - (6a + 8k)x^2 + (-5 + a + 11a^2 + 4k + 31ak + 21k^2)x + 4 + 7a - 3a^2 - 6a^3 + 12k\\
& - 10ak - 27a^2k - 10k^2 - 39ak^2 - 18k^3,
\end{aligned}$$

where $z_3 \in (a + 2k + 1, a + 2k + 2)$, $z_2 \in (2a + 3k - 1, 2a + 3k + 1)$ and $z_1 \in (3a + 3k - 3, 3a + 3k)$. The average transmission degree of $S_n(a, k)$ is $\frac{2W(S_n(a,k))}{n} = \frac{2a^2+6ak+4k^2-4k}{a+k+1} = 2a + 4k - 2 - \frac{(6k-2)}{a+k+1}$. It is clear that the eigenvalue $\frac{1}{2}(5a + 7k - 1 + \sqrt{(a + k)^2 + 2(a + k) + 17}\,)$ is the distance Laplacian spectral radius of $S_n(a, k)$ and so it is always bigger than or equal to average transmission degree $\frac{2W(S_n(a,k))}{n}$. For the eigenvalue $\frac{1}{2}(5a + 7k - 1 - \sqrt{(a + k)^2 + 2(a + k) + 17}\ )$, we have $(a+k)^2 + 2(a+k)+17 \geq (a+k+1)^2$, giving that $\frac{1}{2}(5a+7k-1-\sqrt{(a + k)^2 + 2(a + k) + 17}\ ) \leq \frac{1}{2}(5a+7k-1-(a+k+1)\ ) = 2a+3k-1 < 2a+4k-2-\frac{(6k-2)}{a+k+1} = \frac{2W(S_n(a,k))}{n}$, provided that $6k-2 < (k-1)(a+k+1)$. It is easy to verify this last inequality holds for all $k \geq 4$. For $k = 3$, we have $\frac{1}{2}(5a + 7k - 1 - \sqrt{(a + k)^2 + 2(a + k) + 17}) = \frac{1}{2}(5a + 20 - \sqrt{a^2 + 8a + 32})$ and $\frac{2W(S_n(a,k))}{n} = 2a+10-\frac{16}{a+4}$. By direct calculation it can be verified that the inequality $\frac{1}{2}(5a+20-\sqrt{a^2 + 8a + 32}) < \frac{2W(S_n(a,k))}{n}$, holds in this case as well. For the eigenvalue $2a + 3k + 1$, we have $2a + 3k + 1 < 2a + 4k - 2 - \frac{(6k-2)}{a+k+1}$, giving that $6k - 2 < (k - 3)(a + k + 1)$. It is easy to verify this last inequality holds for $k \geq 7$ and $a \geq 3$. This gives that for $3 \leq k \leq 6$, we have $2a + 3k + 1 \geq \frac{2W(S_n(a,k))}{n}$ and for $k \geq 7$, we have $2a + 3k + 1 < \frac{2W(S_n(a,k))}{n}$. For the eigenvalue $z_1 \in (3a + 3k - 3, 3a + 3k)$, we have $z_1 > 3a + 3k - 3 \geq \frac{2W(S_n(a,k))}{n}$, giving that $k \leq a - 1$, which is always true. For the eigenvalue $z_3 \in (a + 2k + 1, a + 2k + 2)$, the inequality $z_3 < a + 2k + 2 < \frac{2W(S_n(a,k))}{n}$, clearly holds. Lastly, for the eigenvalue $z_2 \in (2a + 3k - 1, 2a + 3k + 1)$, we have already seen that the inequality $z_3 < 2a + 3k + 1 < \frac{2W(S_n(a,k))}{n}$ holds for all $k \geq 7$ and $a \geq 3$. For $a \geq 3$ and $3 \leq k \leq 6$, it can be easily verified that $z_3 < \frac{2W(S_n(a,k))}{n}$. Thus, it follows that for $a \geq 3$ and $k \geq 7$, we have $\sigma = k - 1 + 1 = k$, while as for $a \geq 3$ and $3 \leq k \leq 6$, we have $\sigma = a - 1$.

Therefore, for $a \geq 3$ and $k \geq 7$ the distance Laplacian energy of $S_n(a, k)$ is given by

$$\begin{aligned}DLE(S_n(a,k)) =&2\Big(\sum_{i=1}^{k}\rho_i^L(S_n(a,k)) - \frac{2kW(S_n(a,k))}{2}\Big)\\ =&2\Big(\frac{(k-1)}{2}(5a + 7a - 1 + \sqrt{\gamma}) + z_1 - \frac{k(2a^2 + 6ak + 4k^2 - 4k)}{a + k + 1}\Big)\\ =&2z_1 + (k - 1)\sqrt{\gamma} + ak - 5a - k^2 - 4k + 1 + \frac{2k(6k - 2)}{a + k + 1}.\end{aligned}$$

For $a \geq 3$ and $3 \leq k \leq 6$, if $\sigma = a - 1$, then

$$\begin{aligned}DLE(S_n(a,k)) =&2\Big(\sum_{i=1}^{a-1}\rho_i^L(S_n(a,k)) - \frac{2(a-1)W(S_n(a,k))}{2}\Big)\\ =&2\Big(\frac{1}{2}(5a + 7a - 1 + \sqrt{\gamma}) + z_1 + (a - k - 1)(2a + 3k + 1)\\ &- \frac{(a-1)(2a^2 + 6ak + 4k^2 - 4k)}{a + k + 1}\Big)\\ =&2z_1 + (k - 1)\sqrt{\gamma} - ak + a + k^2 - 8k - 5 + \frac{2(a - 1)(6k - 2)}{a + k + 1}.\end{aligned}$$

For $a \geq 3$ and $3 \leq k \leq 6$, if $\sigma = a$, then

$$\begin{aligned}DLE(S_n(a,k)) =&2\Big(\sum_{i=1}^{a}\rho_i^L(S_n(a,k)) - \frac{2aW(S_n(a,k))}{2}\Big)\\ =&2\Big(\frac{1}{2}(5a + 7a - 1 + \sqrt{\gamma}) + z_1 + z_2 + (a - k - 1)(2a + 3k + 1)\\ &- \frac{a(2a^2 + 6ak + 4k^2 - 4k)}{a + k + 1}\Big)\\ =&2(z_1 + z_2) + (k - 1)\sqrt{\gamma} - ak - 3a + k^2 - 16k - 1 + \frac{2a(6k - 2)}{a + k + 1},\end{aligned}$$

where $\gamma = (a + k)^2 + 2(a + k) + 17$. This completes the proof. ■

The final result gives the distance Laplacian energy ordering of the trees belonging to the family $S_n(2a-k, k), n = 2a+1$ and $k = 0, 1, \ldots, a$.

Theorem 3.5. *The distance Laplacian energy of the family $S_n(2a - k, k), n = 2a + 1$ satisfies the following*

$$DLE(S_n(2a, 0)) \leq DLE(S_n(2a - 1, 1)) \leq DLE(S_n(2a - 2, 2)) \leq \cdots \leq DLE(S_n(a, a)).$$

Proof. For $k \geq 7$, the distance Laplacian energy of $S_n(2a-k,k)$ can be obtained from (i) of Theorem 3.4 and is given by

$$DLE(S_n(2a-k,k)) = 2z_1 + (k-1)\sqrt{4a^2+4a+17} + 2ak - 10a - 2k^2 + k + 1 + \frac{4k(3k-1)}{n}. \tag{3.7}$$

Let $H_1 = S_n(2a-k_1,k_1)$ and $H_2 = S_n(2a-k_2,k_2)$ be any two members of the family $S_n(2a-k,k)$, with $k_2 > k_1$. Then from (3.7), we have

$$\begin{aligned}DLE(H_2) - DLE(H_1) &= 2(z_1(H_2) - z_1(H_1)) + (k_2-k_1)\sqrt{4a^2+4a+17} + 2a(k_2-k_1) \\ &\quad - 2(k_2^2 - k_1^2) + (k_2-k_1) + \frac{4}{n}\Big(k_2(3k_2-2) - k_1(3k_1-1)\Big) \\ &= 2(z_1(H_2) - z_1(H_1)) + (k_2-k_1)\Big[\sqrt{4a^2+4a+17} + 2a - 2(k_1+k_2) \\ &\quad + 1 + \frac{4}{n}(3k_1+3k_2-1)\Big].\end{aligned}$$

Since $z_1(H_1), z_1(H_2) \in (6a-3, 6a)$, it follows that $z_1(H_2) - z_1(H_1) > -3$. Also, $n = 2a+1$ gives that $\sqrt{4a^2+4a+17} = \sqrt{n^2+16} \geq n$. With this it follows that

$$DLE(H_2) - DLE(H_1) \geq -6 + (k_2-k_1)\Big[2n - 2(k_1+k_2) + \frac{4}{n}(3k_1+3k_2-1)\Big] \geq 0, \tag{3.8}$$

if $(k_2-k_1)\Big[2n - 2(k_1+k_2+\frac{3}{k_2-k_1}) + \frac{4}{n}(3k_1+3k_2-1)\Big] \geq 0$. Since, $k_2 - k_1 \geq 1$, it follows that $DLE(H_2) - DLE(H_1) \geq 0$, provided that $2n - 2(k_1+k_2+\frac{3}{k_2-k_1}) + \frac{4}{n}(3k_1+3k_2-1) \geq 0$. This last inequality holds if $2n - 2(k_1+k_2+\frac{3}{k_2-k_1}) \geq 0$, which is so if $k_1+k_2+3 \leq n$, as $k_2-k_1 \geq 1$ implies $\frac{3}{k_2-k_1} \leq 3$. Since $n = 2a+1$ and $k_1, k_2 \leq a-1$, it follows that $k_1+k_2 \leq 2a-2 = n-3$. This shows that the inequality $DLE(H_2) - DLE(H_1) \geq 0$, always holds. Thus, we have shown that the distance Laplacian energy $DLE(S_n(2a-k,k))$ of the family $S_n(2a-k,k$ is an increasing function of $k, k \geq 7$. So, we have the following ordering

$$\begin{aligned}DLE(S_n(2a-7,7)) &\leq DLE(S_n(2a-8,8)) \leq DLE(S_n(2a-9,9)) \leq \cdots \\ &\leq DLE(S_n(a+2,a-2)) \leq DLE(S_n(a+1,a-1)).\end{aligned} \tag{3.9}$$

For $3 \leq k \leq 6$, the distance Laplacian energy of $S_n(2a-k,k)$ is obtained from (ii) of Theorem 3.4 and is given by

$$\begin{aligned}DLE(S_n(2a-k,k)) &= 2z_1 + (k-1)\sqrt{4a^2+4a+17} - 2ak + 2k^2 + 2a - 9k - 5 \\ &\quad + \frac{4}{n}(3k-1)(2a-k-1) \\ &= 2z_1 + (k-1)(\sqrt{n^2+16} - n) + 2k^2 + 4k - 10 - \frac{4}{n}(3k-1)(k+2),\end{aligned} \tag{3.10}$$

where the last equality is obtained by using $n = 2a+1$. Using the fact $z_1(S_n(2a-i,i)) - z_1(S_n(2a-(i-1),i-1)) > -3$, for $i = 3,4,5,6$ together with (3.10) and proceeding by direct calculation it can be seen that

$$DLE(S_n(2a-i,i)) \geq DLE(S_n(2a-i-1,i-1)), \quad \text{for } i = 3,4,5,6. \tag{3.11}$$

Now, the distance Laplacian energy of $S_n(2a,0) = K_{2a,1}$ was obtained in [8] and is given by $DLE(S_n(2a,0)) = 12a - 10 + \frac{8}{2a+1}$. By (i) of Corollary 3.3, the distance Laplacian energy of $S_n(2a-1,1)$ satisfies $12a - 6 - \frac{16}{2a+1} < DLE(S_n(2a-1,1)) < 12a - 2 - \frac{16}{2a+1}$. Similarly, by (ii) of Corollary 3.3, the distance Laplacian energy of $S_n(2a-2,2)$ satisfies $10a + 3 + \sqrt{4a^2+4a+17} - \frac{80}{2a+1} < DLE(S_n(2a-2,2)) < 10a + 5 + \sqrt{4a^2+4a+17} - \frac{80}{2a+1}$, for $a \geq 5$. We have

$$\begin{aligned}DLE(S_n(2a-2,2)) &> 10a + 3 + \sqrt{4a^2+4a+17} - \frac{80}{2a+1} \\ &\geq 10a + 3 + 2a + 1 - \frac{80}{2a+1}\end{aligned}$$

$$= 12a + 4 - \frac{80}{2a+1}$$
$$\geq 12a - 2 - \frac{16}{2a+1} > DLE(S_n(2a-1,1)).$$

This gives that $DLE(S_n(2a-2,2)) > DLE(S_n(2a-1,1))$. Further,

$$DLE(S_n(2a-1,1)) > 12a - 6 - \frac{16}{2a+1} > 12a - 10 + \frac{8}{2a+1} = DLE(S_n(2a,0)).$$

This shows that $DLE(S_n(2a-2,2)) \geq DLE(S_n(2a-1,1)) \geq DLE(S_n(2a,0))$. Again, $z_1 \in (6a-3, 6a)$ gives

$$\begin{aligned} DLE(S_n(2a-3,3)) &= 2z_1(S_n(2a-3,3)) + 2(\sqrt{n^2+16} - n) + 20 - \frac{160}{n} \\ &\geq 12a + 14 + 2(\sqrt{n^2+16} - n) - \frac{160}{n} \\ &\geq 10a + 5 + \sqrt{4a^2+4a+17} - \frac{80}{n} > DLE(S_n(2a-2,2)), \end{aligned}$$

for all $a \geq 5$. Lastly, by Theorem 3.1, we have

$$DLE(S_n(a,a)) = 6a - 5 + (a-1)\sqrt{4a^2+4a+17} + \sqrt{9a^2-22a+17} + \frac{10}{n}.$$

Also, by (3.7) the distance Laplacian energy of $DLE(S_n(a+1,a-1))$ is given by

$$\begin{aligned} DLE(S_n(a+1,a-1)) &= 2z_1 + (a-2)\sqrt{4a^2+4a+17} - a - 19 + \frac{33}{n} \\ &< 11a + (a-2)\sqrt{4a^2+4a+17} - 19 + \frac{33}{n} \\ &\leq 6a - 5 + (a-1)\sqrt{4a^2+4a+17} + \sqrt{9a^2-22a+17} + \frac{10}{n}, \end{aligned}$$

provided that $\sqrt{9a^2-22a+17} + \sqrt{4a^2+4a+17} - 5a + 14 - \frac{23}{n} \geq 0$. Since $\sqrt{9a^2-22a+17} \geq 3(a - \frac{11}{9})$ and $\sqrt{4a^2+4a+17} \geq 2a+1$, we get the inequality $\sqrt{9a^2-22a+17} + \sqrt{4a^2+4a+17} - 5a + 14 - \frac{23}{n} \geq 0$, which always holds, giving that $DLE(S_n(a+1,a-1)) \leq DLE(S_n(a,a))$. From these observations together with (3.9) and (3.11), the result follows. This completes the proof. ■

From Theorem 3.5, it is clear that among all the trees belong to the family $S_n(2a-k,k), k = 0,1,\ldots,a$, the star $K_{2a,1}$ and the sun graph $S_n(a,a)$ attains attains the minimum and the maximum distance Laplacian energies, respectively.

4. Concluding remarks

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ matrices with complex entries. Given $M \in \mathbb{M}_n(\mathbb{C})$, let $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$ denote the singular values of M. The quantity $\|M\|_* = \sum_{i=1}^{n} \sigma_i(M)$ is called the trace norm. Given a symmetric matrix M, let $\sigma_i(M)$ be its singular values and $\lambda_i(M)$, $i = 1,2,\ldots,n$, be its eigenvalues. We have $\sigma_i(M) = |\lambda_i(M)|$. For a connected graph G, it can be seen that the trace norm of the matrix $D^L(G) - \frac{2W(G)}{n} I_n$ is equivalent to the distance Laplacian energy $DLE(G)$. Here, $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Finding the set of matrices that attain the extremal values of the trace norm among a certain class of matrices turns out to be an interesting problem. In this direction, the trace norm of matrices in the (di)graphs setting have been widely explored. We refer the readers to [19] for recent relevant results.

From this point of view, Theorem 3.5 indicates that the matrix $D^L(S_n(a,a)) - \frac{2W(S_n(a,a))}{n} I_n$ has the maximum trace norm while as the matrix $D^L(S_n(2a,0)) - \frac{2W(S_n(2a,0))}{n} I_n$ has the minimum trace norm over all the matrices belonging to the family

$$\left\{ D^L(S_n(2a-k,k)) - \frac{2W(S_n(2a-k,k))}{n} I_n : k = 0,1,2,\ldots,a \right\}.$$

Ordering graphs on the basis of their spectral norms with respect a given graph matrix is a very hard problem. This problem is so difficult that even ordering a small family of graphs with some given parameters seems impossible. For distance Laplacian matrix this problem is considered by Hilal in [9] for trees having diameter three. In [9], the author have shown that the trees of diameter three can be ordered on the basis of their distance Laplacian energies. A similar problem can considered for trees of diameter $d \geq 4$. Therefore, we leave the following problem.

Problem 1. *Let $\mathcal{T}(n, d)$ be the family of trees of order n and diameter $d \geq 4$. Is it possible to order the elements of the family $\mathcal{T}(n, d)$ on the basis of their distance Laplacian energies.*

The solution of the problem 1 for any $d \geq 4$ is clearly a very difficult job. A partial solution for some fixed values of d can even be difficult. Since, the sun type graphs $S_n(a, a)$, $a \geq 2$ and the partial sun type graphs $S_n(a, k)$, $k \geq 2$ are of diameter 4, it follows that the work in this paper provides a partial solution of the problem 1 for the case $d = 4$. In fact, we believe that the idea provided in the paper can be helpful to solve the problem 1 for the case $d = 4$, completely.
Acknowledgements. We are highly thankful to the anonymous referees for their comments and suggestions to improve the presentation of the paper.

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