Concentration of rainbow k-connectivity of a multiplex random graph

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Abstract

We consider a multiplex random graph $G(n, m, p)$ with m independent color layers over a common vertex set V of order n. In each layer, the edges are independent following the Erdős-Rényi model with edge probability $p = c(\ln n + (k-1)\ln\ln n)/mn$ for some constant $c > 1$. A rainbow path in this context means a path with all edges from distinct layers. For a graph $G \in G(n, m, p)$, let $rc_k(G)$ be its rainbow k-connectivity, namely the smallest required number of layers so that any pair of vertices in G can be connected by k internally vertex-disjoint rainbow paths. We show that with high probability, $rc_k(G(n, m, p))$ is concentrated on three consecutive numbers. These numbers are at distance $\Theta(\ln n/(\ln(\ln n + (k-1)\ln\ln n))^2)$ to the diameter of the graph.

Keywords:
Rainbow connectivity, rainbow path, multiplex network, random graph.
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1. Introduction

Connectivity is an emblematic notion in both graph theory and network science. To strengthen conventional connectivity, Chartrand et al. [1, 2] firstly put forward the rainbow k-connectivity as an important property for secure network communication. The rainbow k-connectivity here is an edge connectivity associated with a graph edge coloring problem. Given an integer $k \geq 1$, an edge-colored graph G is called rainbow k-connected if any pair of vertices are connected by k internally vertex-disjoint rainbow paths, where a rainbow path is a path all of whose edges carry different colors. Clearly, a rainbow k-connected graph is also k-connected; conversely, any k-connected graph admits a trivial coloring that makes it rainbow k-connected. Hence, one can define the rainbow k-connectivity of G, denoted $rc_k(G)$, as the minimum number of colors needed to make G rainbow k-connected. In literature, $rc_1(G)$ is also called rainbow connectivity or rainbow connection number. A basic observation is that $rc_k(G) = 1$ if and only if $k = 1$ and G is a complete graph [2]. For a connected graph G of order n, $rc_1(G) \leq n - 1$ and the equality is attained if and only if G is a tree [1]. Recall that $rc_1(G) \geq \mathrm{diam}(G)$, where $\mathrm{diam}(G)$ is the diameter of G. It then follows from [3] and [4] that $\frac{3n}{d_{\min}+1} \leq rc_1(G) \leq \frac{3n}{d_{\min}+1} + 3$ for any connected graph G of order n and minimum degree $d_{\min} \geq 3$. More developments for variants of rainbow k-connectivity of different graph families can be found e.g. in the surveys [5, 6].

Moving away from deterministic graphs, the work [7] shows that, in the binomial random graph model $G(n, p)$, the sharp threshold for the monotonic property $rc_1(G(n, p)) \leq 2$ takes the form of $p = \sqrt{\ln n/n}$. This result is extended in [8] to show that $p = (\ln n)^{1/d}/n^{1-1/d}$ is a sharp threshold for the property $rc_k(G(n, p)) \leq d$ if $d \geq 2$ and $k = O(\ln n)$. At the connectivity threshold $p = (\ln n + \omega(n))/n$ with $\omega(n) = o(\ln n)$, it is shown that the rainbow connectivity $rc_1(G(n, p))$ is asymptotically equivalent to the trivial lower bound $\text{diam}(G(n, p))$ [9]. Results of similar flavor have also been investigated in other random graph models including random regular graphs [10, 11], random bipartite graphs [12, 13], and randomly perturbed graphs [14, 15].

Inspired by recent development from network science in diversifying edge types [16, 17], we in this paper approach the rainbow k-connectivity by considering an alternative random network with multiple layers. Specifically, given a set V of n vertices, we consider a simple multiplex random graph model $G(n, m, p)$. It has $m \geq 1$ network layers each of which is an independent copy of the binomial random graph $G(n, p)$. For $1 \leq l \leq m$, we denote by G_l the l-th layer of $G(n, m, p)$ and often refer to it as the layer of color l. The layers are a decomposition by the color of the edges in $G(n, m, p)$. For a graph G sampled from the model $G(n, m, p)$, viewed as the union $G = \cup_{l=1}^{m} G_l$ over the common vertex set V, all conventional rainbow-related notions can be transferred to it naturally. For example, given an integer $k \geq 1$, G is called rainbow k-connected if any pair of vertices in V are connected by k internally vertex-disjoint paths, each of which is a path containing edges from distinct layers. The rainbow k-connectivity of G, denoted again by $rc_k(G)$, is the minimum number of layers needed to make G rainbow k-connected.

This multiplex approach to cope with rainbow-related concepts has recently attracted increasing research attention in both deterministic and random settings. In [18], it is proved that if all layers are fixed graphs with $n = m$ and minimum degree at least $n/2$, then the multiplex graph G admits a rainbow Hamilton cycle, in which each edge comes from exactly a distinct layer. The method has been generalized to show the existence of many rainbow Hamilton cycles with restrictions on maximum and minimum degrees [19]. Rainbow subgraph containment problems have been studied in [20, 21]. Some tight sufficient conditions for having a subgraph H consisting of m edges each of which from exactly one layer are established. In the $G(n, n-1, p)$ model, it is revealed that $p = c \ln n/n^2$ with $c = 2$ is a sharp threshold for containing a rainbow spanning tree which consists of exactly one edge in each layer [22]. Rainbow connectivity threshold is determined at $p = c \ln n/mn$ for $c > 1$, which is concentrated on at most three values [23]. For more recent advances on combinatorial objects such as matching and extremal structure in line with the multiplex graph framework, we refer the readers to [24, 25, 26].

Following the above line of research and noting that rainbow k-connectivity is much less probed compared to the basic case of $k = 1$, in this paper we study $rc_k(G(n, m, p))$ for any given integer $k \geq 1$. A classical result for k-connectivity in $G(n, p)$ states that [27] $p = (\ln n + (k-1) \ln \ln n)/n$ is the sharp threshold for a graph being k-connected for any integer $k \geq 1$; see also [28]. Our main result below shows that $rc_k(G(n, m, p))$ takes one of three consecutive integers at the supercritical regime $p = c(\ln n + (k-1) \ln \ln n)/n$ for some $c > 1$. We say a sequence of events $\mathcal{E}_n$ happens with high probability (w.h.p.) if the probability it holds tends to 1 as $n \to \infty$.

Theorem 1. *Let $k \geq 1$ be an integer. If $p = \frac{c(\ln n+(k-1)\ln\ln n)}{mn}$ for some constant $c > 1$, then w.h.p. the rainbow k-connectivity of multiplex random graph $G(n, m, p)$ is concentrated on the following three values: $\hat{m}$, $\hat{m}+1$, $\hat{m}+2$, where* $\hat{m} = \lfloor \frac{\ln n}{\ln(c/e)+\ln(\ln n+(k-1)\ln\ln n)} + \frac{1}{2} + 0.49\frac{\ln\ln\ln n}{\ln\ln n} \rfloor$.

The concentration-type results for random graph models have been well recognized. In the $G(n, p)$ model, for instance, some important graph-theoretic parameters such as chromatic number [29, 30, 31], domination number [32, 33], and diameter [34, 35], have been shown to concentrate on just several numbers at some regimes. Our result adds to the wealth of literature in this direction.

We will prove Theorem 1 by tightening the gap between a lower bound and an upper bound for the number of color layers in $G(n, m, p)$, combining ideas from [23] and [27]. The lower bound (Section 2) is relatively straightforward employing the first moment method and the upper bound (Section 3) involves a repetition of breadth-first searches to explore vertices. We will also present numerical results in support of our analytical results in Section 4.

2. The lower bound for $rc_k(G(n, m, p))$

To fix the notations, we will adopt the standard Landau asymptotics to compare positive sequences $\{x_n\}$ and $\{y_n\}$. Specifically, $x_n = o(y_n)$ or $x_n \ll y_n$ means the relation $\lim_{n\to\infty} x_n/y_n = 0$, whereas $x_n = O(y_n)$ means that there is a constant $c > 0$ satisfying $x_n \le cy_n$ for all large enough n. The relationship $x_n = O(y_n)$ is equivalent to $y_n = \Omega(x_n)$. If both $x_n = O(y_n)$ and $y_n = O(x_n)$ hold, we write $x_n = \Theta(y_n)$. Also denote the set $[m] = \{1, 2, \cdots, m\}$ for any integer $m \ge 1$.

The following result indicates that $rc_k(G(n, m, p)) > \frac{\ln n}{\ln(c/e)+\ln(\ln n+(k-1)\ln\ln n)} - \frac{1}{2} + 0.49 \cdot \frac{\ln\ln\ln n}{\ln\ln n}$ w.h.p. Hence $rc_k(G(n, m, p)) \ge \hat{m}$ w.h.p., which leads to the lower half of Theorem 1.

Proposition 1. *Let $k \ge 1$ be an integer. Suppose that $p = \frac{c(\ln n+(k-1)\ln\ln n)}{mn}$ for some constant $c > 1$. If*

$$m \le \frac{\ln n}{\ln(\frac{c}{e}) + \ln(\ln n + (k-1)\ln\ln n)} - \frac{1}{2} + \left(\frac{1}{2} - \varepsilon_n\right)\frac{\ln\ln\ln n}{\ln\ln n} \tag{1}$$

for some ε_n satisfying $\frac{\ln\ln\ln\ln n}{\ln\ln\ln n} \ll \varepsilon_n = o(1)$, then w.h.p. $G(n, m, p)$ is not rainbow k-connected.

Proof. Fix two distinct vertices $v_1, v_2 \in V$ and let X be the random variable counting the number of sets of k internally vertex-disjoint rainbow paths between v_1 and v_2 in $G(n, m, p)$. The strategy is to show the expectation $\mathrm{E}(X) = o(1)$ and then apply the first moment method; see e.g. [37].

Suppose a set of k internally vertex-disjoint rainbow paths between v_1 and v_2 over V has path lengths $l_1, l_2, \cdots, l_k$, respectively. By definition, $1 \le l_i \le m$ for $i \in [k]$. The total number of such path sets can be calculated as

$$\begin{aligned}
&(n-2)(n-3)\cdots(n-l_1)\binom{m}{l_1}l_1!\cdot(n-l_1-1)\cdots(n-l_1-l_2+1)\binom{m}{l_2}l_2!\\
&\cdot(n-l_1-l_2)\cdots(n-l_1-l_2-l_3+2)\binom{m}{l_3}l_3!\cdots\\
&\cdot(n-l_1-l_2-\cdots-l_{k-1}+k-3)\cdots(n-l_1-l_2-\cdots-l_k+k-1)\binom{m}{l_k}l_k!\cdot\frac{1}{k!}\\
\le& n^{l_1+l_2+\cdots+l_k-k}\cdot\frac{(m!)^k}{k!}.
\end{aligned} \tag{2}$$

Considering the random edges in $G(n, m, p)$, the expected number of such path sets is bounded from the above by

$$\begin{aligned}
n^{l_1+l_2+\cdots+l_k-k}\frac{(m!)^k}{k!}p^{l_1+l_2+\cdots+l_k} \le& \frac{(np)^{l_1+\cdots+l_k}}{n^k k!}(m^{m+\frac{1}{2}}e^{1-m})^k\\
=& \frac{1}{k!}\prod_{i=1}^{k}\left(\frac{1}{n}\Big(\frac{c(\ln n+(k-1)\ln\ln n)}{e}\Big)^{l_i}\Big(\frac{m}{e}\Big)^{m+\frac{1}{2}-l_i}e^{\frac{3}{2}}\right),
\end{aligned} \tag{3}$$

where we have employed (2), the definition of p, and Stirling's formula $m! \le m^{m+1/2}e^{1-m}$ $(m \ge 1)$.

Define $r := m - \frac{\ln n}{\ln(c/e)+\ln(\ln n+(k-1)\ln\ln n)}$. Therefore, we have $n = \big(\frac{c(\ln n+(k-1)\ln\ln n)}{e}\big)^{m-r}$ and

$$\begin{aligned} r \le & -\frac{1}{2} + \frac{\frac{1}{2}\ln\ln\ln n - \ln\ln\ln\ln n}{\ln(\frac{c}{e}) + \ln(\ln n + (k-1)\ln\ln n)} \\ = & -\frac{1}{2} + \frac{\ln\big(\sqrt{\ln\ln n}\cdot e^{-\ln\ln\ln\ln n}\big)}{\ln\big(\frac{c}{e}(\ln n + (k-1)\ln\ln n)\big)}, \end{aligned} \tag{4}$$

where we have employed (1) and our choice of ε_n. By taking the sum over all possible path lengths, we obtain

$$\begin{aligned} \mathrm{E}(X) \le & \frac{1}{k!}\prod_{i=1}^{k}\Bigg(\sum_{l_i=1}^{m}\Big(\frac{c(\ln n + (k-1)\ln\ln n)}{e}\Big)^{l_i-m+r}\Big(\frac{m}{e}\Big)^{m+\frac{1}{2}-l_i}e^{\frac{3}{2}}\Bigg) \\ \le & \frac{1}{k!}\prod_{i=1}^{k}\Bigg(e^{\frac{3}{2}}\sqrt{\frac{\ln n}{\ln\ln n}}\Big(\frac{c(\ln n + (k-1)\ln\ln n)}{e}\Big)^{r} \\ & \cdot\sum_{l_i=1}^{m}\Big(\frac{e\ln n}{c(\ln\ln n)(\ln n + (k-1)\ln\ln n)}\Big)^{m-l_i}\Bigg) \\ \le & \frac{1}{k!}\Bigg(2e^{\frac{3}{2}}\sqrt{\frac{\ln n}{\ln\ln n}}\Big(\frac{c(\ln n + (k-1)\ln\ln n)}{e}\Big)^{r}\Bigg)^{k} \end{aligned} \tag{5}$$

in the light of (3) and $m \le (e\ln n)/\ln\ln n$. Feeding (4) into (5), we have

$$\mathrm{E}(X) \le \frac{1}{k!}\Bigg(2e^{2-\ln\ln\ln\ln n}\sqrt{\frac{\ln n}{c(\ln n + (k-1)\ln\ln n)}}\Bigg)^{k} = o(1) \tag{6}$$

as $n \to \infty$.

Invoking Markov's inequality, we obtain $\mathrm{P}(X \ge 1) \le \mathrm{E}(X) = o(1)$. This concludes the proof of Proposition 1. □

For any pair of distinct vertices $v_1, v_2 \in V$, let Y count the number of edges between them in $G(n,m,p)$. Clearly, in the regime of Proposition 1,

$$\mathrm{P}(Y \ge 1) = 1 - \Big(1 - \frac{c(\ln n + (k-1)\ln\ln n)}{mn}\Big)^{m} = (1 - o(1))c\frac{\ln n + (k-1)\ln\ln n}{n}. \tag{7}$$

This implies that $G(n,m,p)$ is k-connected w.h.p. It follows from [34] that $\mathrm{diam}(G(n,m,p))$ is concentrated around $\frac{\ln n}{\ln((1-o(1))c)+\ln(\ln n+(k-1)\ln\ln n)}$ (within 4 integers). Therefore,

$$0 \le \hat{m} - \mathrm{diam}(G(n,m,p)) = \Theta\Big(\frac{\ln n}{(\ln(\ln n + (k-1)\ln\ln n))^2}\Big), \tag{8}$$

which shows that the lower bound given by Proposition 1 is close to the diameter.

3. The upper bound for $rc_k(G(n,m,p))$

To meet the other half of Theorem 1, we have the following result.

Proposition 2. *Let $k \geq 1$ be an integer. Suppose that $p = \frac{c(\ln n+(k-1)\ln\ln n)}{mn}$ for some constant $c > 1$. If*

$$m \geq \frac{\ln n}{\ln(\frac{c}{e}) + \ln(\ln n + (k-1)\ln\ln n)} + \frac{3}{2} + \frac{\sqrt{\ln\ln\ln n}}{\ln\ln n}, \tag{9}$$

then w.h.p. $G(n, m, p)$ is rainbow k-connected.

Combining Propositions 1 and 2, we see that the distance between the two bounds in (1) and (9) is less than 2, which readily gives rise to Theorem 1.

In the rest of the paper, we will set $r := \lceil \ln\ln\ln\ln n \rceil$ and often resort to the simpler bound $m \geq \frac{\ln n}{2\ln\ln n}$ (in view of (9)). Proposition 2 will be proved through a series of lemmas.

Lemma 1. *Suppose that $G(n, m, p)$ is w.h.p. rainbow k-connected whenever*

$$\frac{\ln n}{\ln(\frac{c}{e}) + \ln(\ln n + (k-1)\ln\ln n)} + \frac{3}{2} + \frac{\sqrt{\ln\ln\ln n}}{\ln\ln n} \leq m \leq \frac{3\ln n}{\ln(\ln n + (k-1)\ln\ln n)}. \tag{10}$$

Then $G(n, m, p)$ is also w.h.p. rainbow k-connected for $m > \frac{3\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$.

Proof. Assume $m > \frac{3\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$. Let $q := \min\{2^\theta : \theta \in \mathbb{N},\ \frac{m}{2^\theta} < \frac{3\ln n}{\ln(\ln n+(k-1)\ln\ln n)}\}$, where $\mathbb{N}$ is the set of non-negative integers. Clearly, q is well-defined and $\frac{m}{q} \geq \frac{3\ln n}{2\ln(\ln n+(k-1)\ln\ln n)}$.

Consider a multiplex random graph H consisting of $\lfloor m/q \rfloor$ layers: $H_0 = \cup_{l=1}^{q} G_l$, $H_1 = \cup_{l=q+1}^{2q} G_l$, $\cdots$, $H_{\lfloor m/q \rfloor - 1} = \cup_{l=(\lfloor m/q \rfloor - 1)q+1}^{\lfloor m/q \rfloor q} G_l$. Each of these layers is a random graph in $G(n, \hat{p})$ model with $\hat{p} = 1 - (1-p)^q$. Let $\hat{c} = c(1 - 2q/m) > 1$. We have

$$\begin{aligned}\hat{p}\Big\lfloor \frac{m}{q} \Big\rfloor &\geq (1-(1-p)^q)\Big(\frac{m}{q} - 1\Big) \geq (qp - q^2p^2)\Big(\frac{m}{q} - 1\Big) \\ &\geq pm(1-qp) - qp \geq pm - 2qp = \frac{\hat{c}(\ln n + (k-1)\ln\ln n)}{n},\end{aligned} \tag{11}$$

where we have used the estimates $(1-p)^q \leq e^{-qp} \leq 1 - qp + q^2p^2$ in the second inequality and $pm \leq 1$ in the last inequality.

Employing the comment at the beginning of the proof, we observe

$$\begin{aligned}\Big\lfloor \frac{m}{q} \Big\rfloor \geq \frac{m}{q} - 1 &\geq \frac{3\ln n}{2\ln(\ln n + (k-1)\ln\ln n)} - 1 \\ &\geq \frac{\ln n}{\ln(\frac{\hat{c}}{e}) + \ln(\ln n + (k-1)\ln\ln n)} + \frac{3}{2} + \frac{\sqrt{\ln\ln\ln n}}{\ln\ln n}\end{aligned} \tag{12}$$

and $\lfloor m/q \rfloor \leq m$. In the light of (11) and (12), we can apply the assumption in Lemma 1 to H. Therefore, w.h.p. H is rainbow k-connected. This implies $G(n, m, p)$ is k-connected w.h.p. since $\lfloor m/q \rfloor q \leq m$. □

Based on Lemma 1, we in the following only focus on the range

$$\frac{\ln n}{2\ln\ln n} \leq m \leq \frac{3\ln n}{\ln(\ln n + (k-1)\ln\ln n)}. \tag{13}$$

A key construction to prove Proposition 2 is to explore vertices in a potential rainbow path in the random multiplex graph $G(n, m, p)$ through a breadth-first search. Let $\ell \in \mathbb{N}$. Take a set of vertices $W \subseteq V$, a vertex $v \in W$, and a set of colors $C \subseteq [m]$. Define the set $L_{\ell,W}^{C}(v)$ be the set of vertices u in W satisfying (i) u is connected to v in the induced subgraph of $G(n,m,p)$ over W via a rainbow path of length ℓ consisting of edges all within the color layers C, and (ii) u cannot be connected to v in the induced subgraph of $G(n,m,p)$ over W via a rainbow path of length $\ell-1$ consisting of edges all within the color layers C. In other words, we can carry out a breadth-first search in the induced subgraph over W starting from the 0th level $L^C_{0,W}(v)=\{v\}$ to explore vertices that can be reached within distance ℓ (via rainbow paths using edges in G_l ($l\in C$)) level by level for $\ell=1,2,3,\cdots$. We can explore $L^C_{\ell+1,W}(v)$ based on the vertices in the set $L^C_{\ell,W}(v)$ — simply extending a rainbow path by one more step within W using a color in C that has not been utilized in that particular path. The set $L^C_{\ell,W}(v)$ contains vertices in the ℓth level in the search process over $G(n,m,p)$ for $\ell\in\mathbb{N}$. By construction, $L^C_{\ell,W}(v)$ is a random set, with randomness stemming from the random edge occurrences in $G(n,m,p)$.

We summarize some Chernoff bounds [36, pp. 64-67] in the following lemma; see also [37, 38].

Lemma 2. *Let X be the sum of some independent indicator random variables taking values 0 and 1. If $\mu\le \mathrm{E}(X)$ and $0<\varepsilon<1$, then $\mathrm{P}(X\le(1-\varepsilon)\mu)\le\big(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\big)^{\mu}$ and $\mathrm{P}(X\le(1-\varepsilon)\mu)\le e^{-\varepsilon^2\mu/2}$. If $\mu\ge\mathrm{E}(X)$ and $\varepsilon>0$, then $\mathrm{P}(X\ge(1+\varepsilon)\mu)\le\big(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\big)^{\mu}$.*

The next result bounds the maximum degree in all layers of $G(n,m,p)$ by applying a Chernoff's bound.

Lemma 3. *For $l\in[m]$, let $d_{\max}(G_l)$ be the maximum degree of G_l. Then $\max_{l\in[m]}d_{\max}(G_l)<\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}$ w.h.p.*

Proof. Fix $l\in[m]$ and let the degree of a vertex $v\in V$ in G_l be $d_l(v)$. Using (13) and the definition of p, we have

$$\mathrm{E}\,d_l(v)=p(n-1)\le\frac{c(\ln n+(k-1)\ln\ln n)}{m}\le 2c\ln\ln n+1:=\mu \tag{14}$$

for all large n. Set $\varepsilon:=\frac{\ln n}{(\ln\ln n)^2}-1>0$. A quick application of Lemma 2 yields

$$\begin{aligned}\mathrm{P}\Big(d_l(v)\ge\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big)&\le e^{\mu\varepsilon-\mu(1+\varepsilon)\ln(1+\varepsilon)}\le e^{\mu\varepsilon(1-\ln\varepsilon)}\\&\le e^{(2c\ln\ln n+1)\cdot\frac{\ln n}{(\ln\ln n)^2}\left(1-\ln\left(\frac{\ln n}{2(\ln\ln n)^2}\right)\right)}\\&\le e^{\frac{2c\ln n}{\ln\ln n}(1-\ln\ln n+\ln 2+2\ln\ln\ln n)}=o(n^{-2}).\end{aligned}\tag{15}$$

Therefore,

$$\mathrm{P}\Big(\max_{l\in[m]}d_{\max}(G_l)\ge\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big)\le mn\cdot o(n^{-2})=o(1),\tag{16}$$

which concludes the proof. □

Without loss of generality, we will assume the inequality in Lemma 3 always holds in view of the chain rule in probability (since our goal is to prove the 'w.h.p.' statement in Proposition 2).

The next lemma roughly says that even with fewer color layers and fewer vertices, the 1st level explored by the above breadth-first search is sufficiently abundant. Recall $r=\lceil(\ln\ln\ln\ln n)^{1/2}\rceil$.

Lemma 4. *For any $U\subseteq V$ with $|U|\le\frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$, there exists some constant $\alpha=\alpha(c,U)>0$ such that w.h.p. for any vertex $v\in V\backslash U$ and any color set $C\subseteq[m]$ with $|C|\le r+1$, we have $\big|L^{[m]\backslash C}_{1,V\backslash U}(v)\big|\ge\alpha(\ln n+(k-1)\ln\ln n)$.*

Proof. Given the set U, fix a vertex $v \in V\backslash U$. We only consider the induced subgraph of $G(n,m,p)$ over $V\backslash U$. It is easy to see for sufficiently large n, the following is true:

$$\begin{aligned}
\mathrm{E}\left(\left|L_{1,V\backslash U}^{[m]}(v)\right|\right) =&(n-1-|U|)(1-(1-p)^m) \geq (n-1-|U|)(pm-p^2m^2) := \mu \\
\geq&\Big(n-1-\frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}\Big) \\
&\cdot\Big(\frac{c(\ln n+(k-1)\ln\ln n)}{n}-\Big(\frac{c(\ln n+(k-1)\ln\ln n)}{n}\Big)^2\Big) \\
\geq&\Big(\frac{c+1}{2}\Big)(\ln n+(k-1)\ln\ln n),
\end{aligned} \tag{17}$$

where we have noted $0 < pm < 1$ and $c > 1$.

For any $\varepsilon \in (0,1)$, using Lemma 2 we obtain

$$\mathrm{P}\left(\left|L_{1,V\backslash U}^{[m]}(v) < (1-\varepsilon)\mu\right) \leq e^{\mu(-\varepsilon-(1-\varepsilon)\ln(1-\varepsilon))}. \tag{18}$$

When ε is sufficiently close to 1, $-\varepsilon-(1-\varepsilon)\ln(1-\varepsilon)$ is sufficiently close to -1. Combining (17) and (18), we obtain

$$\begin{aligned}
\mathrm{P}\left(\left|L_{1,V\backslash U}^{[m]}(v) < (1-\varepsilon)\mu\right) \leq& e^{\left(\frac{c+1}{2}\right)(\ln n+(k-1)\ln\ln n)\cdot(-\varepsilon-(1-\varepsilon)\ln(1-\varepsilon))} \\
\leq& n^{\left(\frac{c+1}{2}\right)\cdot(-\varepsilon-(1-\varepsilon)\ln(1-\varepsilon))} = o(n^{-1}),
\end{aligned} \tag{19}$$

where we have chosen a constant ε sufficiently close to 1. Taking the union bound over all vertices in $V\backslash U$ and noting (17) and $c > 1$, we know that w.h.p. $\left|L_{1,V\backslash U}^{[m]}(v)\right| \geq (1-\varepsilon)(\ln n+(k-1)\ln\ln n)$ holds for any $v \in V\backslash U$.

In view of Lemma 3, we can choose a constant $0 < \alpha < 1-\varepsilon$ such that w.h.p.

$$\begin{aligned}
\left|L_{1,V\backslash U}^{[m]\backslash C}(v)\right| \geq&(1-\varepsilon)(\ln n+(k-1)\ln\ln n)-|C|\Big(\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big) \\
\geq&\alpha(\ln n+(k-1)\ln\ln n)
\end{aligned} \tag{20}$$

holds for any $v \in V\backslash U$ and any color set $C \subseteq [m]$ satisfying $|C| \leq r+1$. □

Similarly as commented before, we will assume the statement in Lemma 4 always holds without loss of generality. The quantity $\alpha = \alpha(c,U)$ determined here will be used throughout the paper.

The next result shows that without going further than level r in the breadth-first search, we will explore sufficient vertices even with fewer vertices and one less color layer in $G(n,m,p)$.

Lemma 5. *For any $U \subseteq V$ with $|U| \leq \frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$, w.h.p. it holds that for any vertex $v \in V\backslash U$, there exits $\hat{\ell} \leq r$ such that*

$$\left|L_{\hat{\ell},V\backslash U}^{[m-1]}(v)\right| \geq \frac{2\alpha c^{r-1}}{3}(\ln n+(k-1)\ln\ln n)^r. \tag{21}$$

Proof. Given the set U, fix a vertex $v \in V\backslash U$. We consider a breadth-first search starting from the vertex v in the induced subgraph of $G(n,m,p)$ over $V\backslash U$. If at some level $\ell \leq r$, $\left|L_{\ell,V\backslash U}^{[m-1]}(v)\right| \geq c^r(\ln n+(k-1)\ln\ln n)^r$, then (21) is true since $0 < \alpha < 1$ and $c > 1$. Otherwise, the total explored vertices until level r cannot be more than $rc^r(\ln n+(k-1)\ln\ln n)^r$. Thus, during the breadth-first search until level r, there are always at least $n-\frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}-rc^r(\ln n+(k-1)\ln\ln n)^r$ vertices that are not explored yet.

Define a sequence $\{\rho_\ell\}_{\ell\ge 0}$ with $\rho_0=1$, $\rho_1=\alpha(\ln n+(k-1)\ln\ln n)$, and for $1\le \ell\le r-1$,

$$\begin{aligned}\rho_{\ell+1}=&\rho_\ell\cdot pn(m-\ell-1)(1-pm)\\&\cdot\Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big)\cdot(1-\theta_\ell),\end{aligned}\tag{22}$$

where $\theta_1=\theta_2=(\ln n)^{-1/4}$ and $\theta_\ell=(\ln n)^{-3/4}$ for $3\le\ell\le r-1$. Through a direct calculation, we obtain

$$\begin{aligned}\rho_\ell=&\alpha(\ln n+(k-1)\ln\ln n)(pn)^{\ell-1}\frac{(m-2)!}{(m-\ell-1)!}(1-pm)^{\ell-1}\\&\cdot\Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big)^{\ell-1}\prod_{i=1}^{\ell-1}(1-\theta_i)\end{aligned}\tag{23}$$

for $1\le\ell\le r$. In particular,

$$\begin{aligned}\rho_r=&\alpha(\ln n+(k-1)\ln\ln n)(pn)^{r-1}\frac{(m-2)!}{(m-r-1)!}(1-pm)^{r-1}\\&\cdot\Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big)^{r-1}\cdot\prod_{i=1}^{r-1}(1-\theta_i)\\\ge&\alpha(\ln n+(k-1)\ln\ln n)(pnm)^{r-1}e^{-\frac{2r^2}{m}}(1-pm)^{r-1}\\&\cdot\Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big)^{r-1}\cdot\prod_{i=1}^{r-1}(1-\theta_i),\end{aligned}\tag{24}$$

where we have used the following estimate:

$$\frac{(m-2)!}{(m-r-1)!}\ge(m-r)^{r-1}=m^{r-1}\Big(1-\frac{r}{m}\Big)^{r-1}\ge m^{r-1}(e^{-\frac{2r}{m}})^{r-1}\ge m^{r-1}e^{-\frac{2r^2}{m}}.\tag{25}$$

Here, the second inequality in (25) can be seen as a result of the following claim by taking $x_n=r/m$ and $\delta_n=1/x_n$.

Claim: For any $\delta_n>1/2$ and a sequence $0\le x_n=o(1)$, we have $1-x_n\ge e^{-x_n-\delta_n x_n^2}$.

Proof of the Claim. The claim can be proved by examining Taylor's expansion of the function $\phi(x):=1-x-e^{-x-\delta x^2}$ at $x=0$. In fact, we have $\phi(x)=(\delta-1/2)x^2+(1/6-\delta)x^3+O(x^4)$. The result follows immediately. □

In (24), it is easy to see that $e^{-\frac{2r^2}{m}}\to 1$ and all the three multiplicative terms following it also tend to 1 as $n\to\infty$. Hence, $\rho_r\ge\frac{2\alpha c^{r-1}}{3}(\ln n+(k-1)\ln\ln n)^r$ for all large n.

Next, we will show that for $0\le\ell\le r$

$$\left|L^{[m-1]}_{\ell,V\setminus U}(v)\right|\ge\rho_\ell,\tag{26}$$

which will then conclude the proof of Lemma 5.

The bound in (26) can be proved based on induction. Firstly, we know $\left|L^{[m-1]}_{0,V\setminus U}(v)\right|=\rho_0=1$ and $\left|L^{[m-1]}_{1,V\setminus U}(v)\right|\ge\rho_1$ by Lemma 4. Suppose (26) holds for levels up to ℓ, and we estimate for the level $\ell+1$, where $1 \le \ell \le r-1$. Taking into account of our breadth-first search process, for each vertex $w_1 \in L^{[m-1]}_{\ell,V\backslash U}(v)$, it connects to an unexplored vertex w_2 with probability $1-(1-p)^{m-1-\ell}$ because the edge $\{w_1, w_2\}$ should adopt a color that has not been used in this particular rainbow path from v to w_1. This probability is no less than $(m-1-\ell)p-(m-1-\ell)^2p^2$ since $0 < (m-1-\ell)p < 1$. By the induction assumption $\left|L^{[m-1]}_{\ell,V\backslash U}(v)\right| \ge \rho_\ell$ and the comment about the number of unexplored vertices at the beginning of the proof, we obtain

$$\begin{aligned}
\mathrm{E}\Big(\left|L^{[m-1]}_{\ell+1,V\backslash U}(v)\right|\Big) \ge & \mu_{\ell+1} \\
\ge & \rho_\ell \cdot pn(m-\ell-1)(1-pm) \\
& \cdot \Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big) \\
= & \frac{\rho_{\ell+1}}{1-\theta_\ell} \qquad (27)
\end{aligned}$$

by (22), where $\mu_{\ell+1} := \rho_\ell((m-1-\ell)p-(m-1-\ell)^2p^2)\cdot\Big(n-\frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}-rc^r(\ln n+(k-1)\ln\ln n)^r\Big)$.

When $\ell=1$, using (13), (23), (27) and the definitions of p and r, we have

$$\begin{aligned}
\mu_2 \ge \frac{\rho_2}{1-\theta_1} = & \alpha(\ln n+(k-1)\ln\ln n)pn(m-2)!(1-pm) \\
& \cdot\Big(1-\frac{3(k-1)\ln n}{n\ln(\ln n+(k-1)\ln\ln n)}-\frac{r}{n}c^r(\ln n+(k-1)\ln\ln n)^r\Big) \\
\ge & \alpha(\ln n+(k-1)\ln\ln n)^2\frac{c\ln(\ln n+(k-1)\ln\ln n)}{3\ln n} \\
& \cdot\Big(\frac{\ln n}{2\ln\ln n}-2\Big)(1-o(1)) \\
\ge & (\ln n)^{\frac{5}{3}} \qquad (28)
\end{aligned}$$

for all large n. When $\ell=2$, we can similarly estimate $\mu_3 \ge \frac{\rho_3}{1-\theta_2} \ge (\ln n)^{\frac{5}{3}}$. Hence, for $\ell=1,2$, it follows from Lemma 2 that

$$\begin{aligned}
\mathrm{P}\Big(\left|L^{[m-1]}_{\ell+1,V\backslash U}(v)\right|<\rho_{\ell+1}\Big) \le & \mathrm{P}\Big(\left|L^{[m-1]}_{\ell+1,V\backslash U}(v)\right|<\mu_{\ell+1}(1-\theta_\ell)\Big) \\
\le & e^{-\frac{1}{2}\theta_\ell^2\mu_{\ell+1}} \le e^{-\frac{1}{2}(\ln n)^{7/6}} = o((rn)^{-1}). \qquad (29)
\end{aligned}$$

When $\ell \ge 3$, we follow the line above and obtain $\mu_{\ell+1} \ge \frac{\rho_{\ell+1}}{1-\theta_\ell} \ge (\ln n)^{\frac{8}{3}}$. By Lemma 2 we again have

$$\begin{aligned}
\mathrm{P}\Big(\left|L^{[m-1]}_{\ell+1,V\backslash U}(v)\right|<\rho_{\ell+1}\Big) \le & \mathrm{P}\Big(\left|L^{[m-1]}_{\ell+1,V\backslash U}(v)\right|<\mu_{\ell+1}(1-\theta_\ell)\Big) \\
\le & e^{-\frac{1}{2}(\ln n)^{7/6}} = o((rn)^{-1}). \qquad (30)
\end{aligned}$$

This indicates that with probability $1-o(n^{-1})$, (26) holds for all $\ell \le r$. Hence, so far we have proved that for a given $v \in V\backslash U$, with probability $1-o(n^{-1})$ there exists $\hat{\ell} \le r$ such that (21) holds. Taking the union bound over all v readily gives the desired conclusion. □

Without loss of generality, we will assume the statement in Lemma 5 holds in the following. The next result is the final step towards the proof of Proposition 2, which establishes a possibility of finding a rainbow path in $G(n,m,p)$ with fewer vertices and one less color layer.

Lemma 6. *Let $U \subseteq V$ with $|U| \leq \frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$ and two vertices $v_1, v_2 \in V\backslash U$. Let P be a rainbow path in the induced subgraph $G(n,m,p)$ over $V\backslash U$ with colors in $[m-1]$, which starts from v_1 and has length no more than r. Then, with probability $1 - o(n^{-4})$, there exists some $\ell \leq m-1-r := s$ such that either*

(i) $L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2) \cap V(P) \neq \emptyset$;
or

(ii) $\left|L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)\right| \geq \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$

is true. Here, $C(P)$ means the set of colors used in P and $V(P) \subseteq V$ is the vertex set of P.

Proof. Given the set U, fix two vertices $v_1, v_2 \in V\backslash U$. We consider a breadth-first search starting from the vertex v_2 in the induced subgraph of $G(n,m,p)$ over $V\backslash U$. We assume $L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2) \cap V(P) = \emptyset$ for all $0 \leq \ell \leq s$ and aim to show (ii). To this end, we assume that for all $0 \leq \ell \leq s$, $\left|L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)\right| < \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$. If we can arrive at a contradiction with probability $1-o(n^{-4})$, then Lemma 6 is proved.

Define a sequence $\{\rho_\ell\}_{\ell\geq 0}$ with $\rho_0 = 1$, $\rho_1 = \alpha(\ln n + (k-1)\ln\ln n)$, and for $1 \leq \ell \leq s-1$,

$$\rho_{\ell+1} = \rho_\ell \cdot \frac{c(\ln n + (k-1)\ln\ln n)}{m}\Big(1 - \frac{1}{\ln n}\Big)^2 \beta_\ell (s-\ell)(1-\theta_\ell), \tag{31}$$

where $\beta_\ell = 1 - p(s-\ell)$, $\theta_1 = \theta_2 = (\ln n)^{-1/4}$ and $\theta_\ell = (\ln n)^{-3/4}$ for $3 \leq \ell \leq s-1$. Therefore, we calculate

$$\begin{aligned}\rho_\ell = &\alpha(\ln n + (k-1)\ln\ln n)\Big(\frac{c(\ln n + (k-1)\ln\ln n)}{m}\Big)^{\ell-1} \\ &\cdot \frac{(s-1)!}{(s-\ell)!}\Big(1 - \frac{1}{\ln n}\Big)^{2\ell-2}\prod_{i=1}^{\ell-1}\beta_i(1-\theta_i)\end{aligned} \tag{32}$$

for $1 \leq \ell \leq s$.

If there is some $\hat{s} \leq s$ such that $\left|\cup_{\ell=0}^{\hat{s}} L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)\right| \geq \frac{n}{\ln n} - \frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)}$, then there must be some $1 \leq \ell \leq s$ satisfying $\left|L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)\right| \geq \frac{1}{s}\Big(\frac{n}{\ln n} - \frac{3(k-1)\ln n}{\ln(\ln n+(k-1)\ln\ln n)} - 1\Big) \geq \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$ recalling (13) and the definition of r, which implies (ii). Hence, in the following we assume there are at least $n\Big(1 - \frac{1}{\ln n}\Big)$ vertices in $V\backslash U$ that are not explored yet in the breadth-first search until level s.

Claim: With probability $1 - o(n^{-4})$, $\left|L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)\right| \geq \rho_\ell$ for any $0 \leq \ell \leq s$.

Proof of the Claim. The claim can be seen by using Chernoff's bounds and induction on ℓ similarly as in Lemma 5. In fact, note that $\left|L^{[m-1]\backslash C(P)}_{0,V\backslash U}(v_2)\right| = \rho_0 = 1$ and $\left|L^{[m-1]\backslash C(P)}_{1,V\backslash U}(v_2)\right| \geq \rho_1 = \alpha(\ln n + (k-1)\ln\ln n)$ by Lemma 4. Suppose the inequality in the claim holds for levels up to ℓ, and we estimate for the level $\ell+1$, where $1 \leq \ell \leq s-1$. In the breadth-first search process, for each vertex $w_1 \in L^{[m-1]\backslash C(P)}_{\ell,V\backslash U}(v_2)$, it connects to an unexplored vertex $w_2 \in V\backslash(U \cup \cup_{i=0}^{\ell} L^{[m-1]\backslash C(P)}_{i,V\backslash U}(v_2))$ with probability at least $1-(1-p)^{s-\ell}$ because the edge $\{w_1, w_2\}$ should adopt a color that has not been used in this particular rainbow path from v_2 to w_1. This probability is no less than $(s-\ell)p - (s-\ell)^2p^2 = (s-\ell)\beta_\ell p$ since $0 < (s-\ell)p < 1$. Furthermore, the probability that w_2 is connected to at least one vertex in $L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)$ is no less than

$$\begin{aligned}1-(1-(s-\ell)\beta_\ell p)^{\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|} \geq&(s-\ell)\beta_\ell p\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|-(s-\ell)^2\beta_\ell^2p^2\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|^2\\ \geq&(s-\ell)\beta_\ell p\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|\Big(1-\frac{mnp}{(\ln n+(k-1)\ln\ln n)^{r-1}}\Big)\\ \geq&(s-\ell)\beta_\ell p\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|\Big(1-\frac{1}{\ln n}\Big),\end{aligned} \tag{33}$$

where $0<(s-\ell)\beta_\ell p\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|<1$, $\beta_\ell<1$, $s-\ell<m$, the assumption $\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|<\frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$, and the choice of p. By the induction assumption $\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|\geq\rho_\ell$, (31) and (33), we obtain

$$\begin{aligned}\mathrm{E}\left(\left|L^{[m-1]\setminus C(P)}_{\ell+1,V\setminus U}(v_2)\right|\right)\geq&n\Big(1-\frac{1}{\ln n}\Big)\Big(1-(1-(s-\ell)\beta_\ell p)^{\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|}\Big)\\ \geq&n\Big(1-\frac{1}{\ln n}\Big)^2p\beta_\ell(s-\ell)\left|L^{[m-1]\setminus C(P)}_{\ell,V\setminus U}(v_2)\right|\\ \geq&\rho_\ell\Big(1-\frac{1}{\ln n}\Big)^2\beta_\ell(s-\ell)\frac{c(\ln n+(k-1)\ln\ln n)}{m}\\ =&\frac{\rho_{\ell+1}}{1-\theta_\ell}:=\mu_{\ell+1}.\end{aligned} \tag{34}$$

In the case of $\ell=1$, applying (32), (34), and $\beta_1=1-o(1)$, we have for large n,

$$\begin{aligned}\mu_2=&\alpha(\ln n+(k-1)\ln\ln n)\Big(1-\frac{1}{\ln n}\Big)^2\beta_1(s-1)\frac{c(\ln n+(k-1)\ln\ln n)}{m}\\ \geq&(\ln n)^{\frac{5}{3}}.\end{aligned} \tag{35}$$

Similarly, when $\ell=2$, we have $\mu_3\geq(\ln n)^{\frac{5}{3}}$. It follows from Lemma 2 that

$$\begin{aligned}\mathrm{P}\left(\left|L^{[m-1]\setminus C(P)}_{\ell+1,V\setminus U}(v_2)\right|<\rho_{\ell+1}\right)=&\mathrm{P}\left(\left|L^{[m-1]\setminus C(P)}_{\ell+1,V\setminus U}(v_2)\right|<\mu_{\ell+1}(1-\theta_\ell)\right)\\ \leq&e^{-\frac{1}{2}\theta_\ell^2\mu_\ell}\leq e^{-\frac{1}{2}(\ln n)^{7/6}}=o(s^{-1}n^{-4}).\end{aligned} \tag{36}$$

In the case of $\ell\geq3$, we similarly derive $\mu_\ell\geq(\ln n)^{\frac{8}{3}}$ and will essentially have the same expression as (36). Taking the union bound over ℓ, we arrive at the claim. □

Apply the claim with $\ell=s$ and (32), we have the bound

$$\begin{aligned}\left|L^{[m-1]\setminus C(P)}_{s,V\setminus U}(v_2)\right|\geq\rho_s=&\alpha(\ln n+(k-1)\ln\ln n)\Big(\frac{c(\ln n+(k-1)\ln\ln n)}{m}\Big)^{s-1}\\ &\cdot(s-1)!\Big(1-\frac{1}{\ln n}\Big)^{2s-2}\prod_{i=1}^{s-1}\beta_i(1-\theta_i)\\ \geq&\alpha(\ln n+(k-1)\ln\ln n)(c(\ln n+(k-1)\ln\ln n))^{s-1}\\ &\cdot\Big(\frac{s}{m}\Big)^{s-1}e^{-s}\sqrt{2\pi s}\Big(1-\frac{1}{\ln n}\Big)^{2s-2}\prod_{i=1}^{s-1}\beta_i(1-\theta_i),\end{aligned} \tag{37}$$

where we have invoked the Stirling's formula. It is straightforward to estimate that $\sum_{i=1}^{s-1}\ln(\beta_i(1-\theta_i)) = O(1)$, $|(s-1)\ln(1-(\ln n)^{-1})| \le 2(s-1)(\ln n)^{-1} = O(1)$, and $|(s-1)\ln(s/m)| = O(r)$. It follows from (37) that

$$\begin{aligned}\left|L_{s,V\setminus U}^{[m-1]\setminus C(P)}(v_2)\right| \ge & e^{s\ln(\ln n+(k-1)\ln\ln n)+(s-1)\ln c-s+\frac{1}{2}\ln s+O(r)}\\ \ge & e^{(m-r-\frac{1}{2})\ln(\ln n+(k-1)\ln\ln n)+m(\ln c-1)+O(r)}.\end{aligned} \tag{38}$$

Consequently,

$$\begin{aligned}&\frac{(\ln n+(k-1)\ln\ln n)^{r-1}}{n}\left|L_{s,V\setminus U}^{[m-1]\setminus C(P)}(v_2)\right|\\ &\quad\ge e^{m(\ln(\ln n+(k-1)\ln\ln n)+\ln c-1)-\frac{3}{2}\ln(\ln n+(k-1)\ln\ln n)-\ln n+O(r)}.\end{aligned} \tag{39}$$

Combining (9) and (39), we obtain

$$\begin{aligned}&\frac{(\ln n+(k-1)\ln\ln n)^{r-1}}{n}\left|L_{s,V\setminus U}^{[m-1]\setminus C(P)}(v_2)\right|\\ &\quad\ge e^{\frac{3}{2}(\ln c-1)+\frac{\sqrt{\ln\ln\ln n}}{\ln\ln n}(\ln(\ln n+(k-1)\ln\ln n)+\ln c-1)+O(r)}\\ &\quad\ge e^{O(r)+\sqrt{\ln\ln\ln n}} \ge 1.\end{aligned} \tag{40}$$

Note that (40) is derived based on the claim, which means we have arrived at a contradiction with the initial assumption with probability $1-o(n^{-4})$. □

Putting these pieces together, we are now in a position to show Proposition 2.

Proof of Proposition 2. Our aim is to show that there exists a set of k internally vertex-disjoint rainbow paths between any two distinct vertices v_1 and v_2 in $G(n,m,p)$ w.h.p. Fix any such two vertices v_1 and v_2, we will find these rainbow paths one by one. According to Lemma 6, we know that for any rainbow path P starting from v_1 with length no more than r, it holds with probability $1-o(n^{-4})$ that there is $\ell(P) \le s$ satisfying $L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)\cap V(P) \ne \emptyset$ or $\left|L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)\right| \ge \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$. Here, $V(P)$ is the vertex set of P, and $\ell(P)$ is an integer determined by Lemma 6. By Lemma 3, the total number of rainbow paths starting from v_1 with length no more than r is at most

$$\Big(\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big)+\Big(\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big)^2+\cdots+\Big(\frac{2c\ln n}{\ln\ln n}+\frac{\ln n}{(\ln\ln n)^2}\Big)^r \le n. \tag{41}$$

It then follows from Lemma 6 that with probability $1-o(n^{-3})$, for any rainbow path P starting from v_1 with length no more than r, there is $\ell(P) \le s$ satisfying $L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)\cap V(P) \ne \emptyset$ or $\left|L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)\right| \ge \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$. If the former holds for one P, we have found a rainbow path linking v_1 and v_2. In the following, we only need to consider the scenario where for any such P we have $\left|L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)\right| \ge \frac{n}{(\ln n+(k-1)\ln\ln n)^{r-1}}$.

We define a set $E(v_1,v_2)$ containing the following ordered vertex pairs: If a rainbow path P starting from v_1 with length no more than r ends at a vertex w_1, and $w_2 \in L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)$, then (w_1,w_2) is put into the set $E(v_1,v_2)$. By Lemma 5, we have no less than $\frac{2\alpha c^{r-1}}{3}(\ln n+(k-1)\ln\ln n)^r$ rainbow paths starting from v_1 with length no more than r and having the other distinct end points $w_1 \in L_{\hat{\ell},V}^{[m-1]}(v_1)$ for some $\hat{\ell} \le r$. Here, $\hat{\ell}$ of course can be different for each rainbow path. By our assumption above, for each such rainbow path P, the vertices in the explored set $L_{\ell(P),V}^{[m-1]\setminus C(P)}(v_2)$

are different from its end point w_1. As commented above, we will put the pairs (w_1, w_2) into $E(v_1, v_2)$, for every $w_2 \in L^{[m-1]\backslash C(P)}_{\ell(P),V}(v_2)$. Hence, $E(v_1, v_2)$ contains at least

$$\frac{2\alpha c^{r-1}}{3}(\ln n + (k-1)\ln\ln n)^r \cdot \frac{n}{(\ln n + (k-1)\ln\ln n)^{r-1}} \geq \frac{6}{p}\ln n \tag{42}$$

distinct pairs of ordered vertices, where we have applied the bound in (13) and the choice of p. Clearly, $E(v_1, v_2)$ has at least $(3/p)\ln n$ unordered vertex pairs. Since $1 - p \leq e^{-p}$, with probability at least $1 - (1-p)^{\frac{3}{p}\ln n} \geq 1 - o(n^{-2})$, the color layer G_m has an edge linking a pair in $E(v_1, v_2)$. This gives rise to a rainbow path connecting v_1 and v_2.

So far we have shown that with probability $1 - o(n^{-2})$ we have found a rainbow path between v_1 and v_2. Denote the set of internal vertices on this path by P_1. Clearly, $|P_1| \leq m = s + r + 1 \leq \frac{3\ln n}{\ln(\ln n + (k-1)\ln\ln n)}$ by (13). Considering the induced subgraph of $G(n, m, p)$ over $V\backslash P_1$, we can repeat the above argument and find a second rainbow path P_2. Continuing this way, with a probability of at least $(1 - o(n^{-2}))^k = 1 - o(n^{-2})$, we can obtain the k rainbow paths $P_1, P_2, \cdots, P_k$ from $G(n, m, p)$. These rainbow paths are internally vertex-disjoint. Since there are no more than n^2 ways to choose $v_1, v_2 \in V$, we know that with probability $1 - o(1)$, $G(n, m, p)$ is rainbow k-connected. □

4. Numerical experiments and conclusions

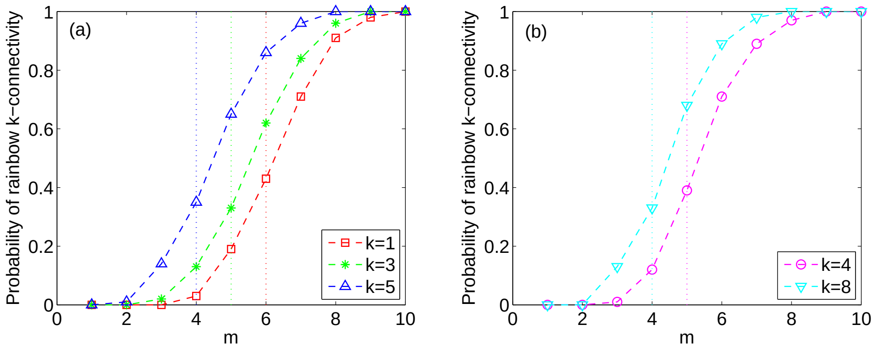

Figure 1: Probability of rainbow k-connectivity as a function of m for $G(n, m, p)$ with $p = 1.5(\ln n + (k-1)\ln\ln n)/mn$ and (a) $n = 600$ and (b) $n = 3000$.

To illustrate our theoretical results, we generate 300 multiplex random graphs $G(n, m, p)$ with two different network sizes: $n = 600$ and $n = 3000$ vertices. The edge connection probability is taken as $p = \frac{c(\ln n + (k-1)\ln\ln n)}{mn}$ with $c = 1.5$. For given $m, k \in \mathbb{N}$, we count the number of graphs that are rainbow k-connected and use the frequency to approximate the probability of rainbow k-connectivity. In Figure 1, we show the result for some different k and $m \in [10]$, where the vertical lines indicate the corresponding concentrated middle value $\hat{m} + 1$ calculated from Theorem 1. We observe the phase transition in terms of the number of layers in $G(n, m, p)$ even for relatively small number of vertices. The concentration results are also quite accurate and in line with the theoretical analysis.

The network layers in our multiplex random graphs are assumed to be binomial random graphs at the supercritical regime $p = c(\ln n + (k-1)\ln\ln n)/mn$ for some $c > 1$. It would be interesting to examine $rc_k(G(n, m, p))$ for sparse random graphs at the regime $p = (\ln n + (k-1)\ln\ln n + \omega(n))/mn$ for some $\omega(n) \to \infty$. Network layers taking other related random graph models such as the uniform random graphs and the random regular graphs [37] are also worth investigation.

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