Resilient consensus in continuous-time networks with ℓ-hop communication and time delay

Yilun Shang[1]

[1]*Department of Computer and Information Sciences, Northumbria University, Newcastle NE1 8ST, UK*
Email: `yilun.shang@northumbria.ac.uk`

Abstract

This paper studies the problem of resilient consensus in continuous-time multiagent systems against potential malicious agents. Network topology conditions have been developed to ensure asymptotic consensus of cooperative agents in the network under multi-hop communication and path-dependent heterogeneous delays. We develop two new distributed protocols based on the idea of weighted mean subsequence reduced algorithm to address scalar resilient consensus and vector resilient consensus problems in path-weighted directed networks leveraging on the capability of multi-hop communication. Our frameworks are flexible in that both linear and general nonlinear network coupling functions are featured. Resilient consensus problem for dynamical agents with higher-order integrators is solved as a byproduct. We illustrate the error introduced by the resilience mechanisms when the network is free of malicious agents through numerical simulations.

Keywords:
Resilient consensus, multiagent system, multi-hop communication, communication delay, higher-order dynamics.
2000 MSC: 93E03, 93D05

1. Introduction

To coordinate a network of dynamical agents, the fundamental challenge is generally to steer all agents to a common state cooperatively based on their local neighborhood information. Consensus problems [15, 14], in this context, deal with the design of distributed control algorithms that guarantee state agreement as time grows unbounded. The states of agents can be widely interpreted for instance as physical measurements such as position, velocity, temperature, opinions in social network, or generic data and information in multiagent applications. A variety of consensus protocols have been reported in the literature, e.g. [29, 3, 11], under different communication conditions for discrete-time and continuous-time network dynamics over the past few decades.

In the study of consensus problems, the security issue relevant to the presence of adversaries or misbehaving agents in the networks has attracted substantial interest in recent years. Malicious and Byzantine agents can deviate from the set control rules and disseminate arbitrary bad messages to their neighbors to hinder the collective consensus behavior [11]. A pioneering work on resilient consensus is often referred back to [12], where a discrete-time Weighted-Mean-Subsequence-Reduced (W-MSR) algorithm is designed to ensure the consensus of all cooperative agents in the network when the number of misbehaving agents is bounded by r locally (r-local model) or globally (r-total model). When the underlying network coupling topology is sufficiently robust, the W-MSR algorithm is effective in reaching consensus in that it disregards certain extremal (potentially corrupted) values received by a cooperative agent at each iteration. It has been extended to tackle resilient consensus under various constraints on information flows including asynchrony [20], channel noise [19], state saturation [21], and delayed information [7].

Resilience in consensus reaching against misbehaving agents depends on a critical notion of network robustness [12], which specifies a minimum required level of connectivity in the coupling topology. As dense networks are often costly and difficult to maintain, some control strategies have been explored to reduce the topological conditions. An assumption of trusted agents, which are insusceptible to attacks, is introduced in [1] to facilitate consensus. The work [33] considers a variant of trusted region formed by trusted agents. The framework of multiplex networks is adopted in [22] to reduce robustness requirement when the agents have asymmetric interaction with only neighbors having larger or smaller states. Recently, a multi-hop communication strategy, where message can be relayed over a discrete-time network, is introduced in [32]. The multi-hop communication, with an origin in computer science [26, 25], facilitates consensus over sparser networks by introducing the multi-hop network robustness [32] generalizing the original robustness notion. In [31], two-hop information is utilized to detect misbehaving agents in the networks. To tackle higher-order dynamics, an attack isolation consensus algorithm is proposed in [34] to isolate neighbors who are victims of attack by using two-hop communication.

Motivated by the above line of research, especially the multi-hop network robustness concept, we in this paper present two new W-MSR based frameworks for achieving resilient consensus in continuous-time networks. In the first framework, we consider scalar-valued states of agents over a network with ℓ-hop communication capability and time delay. The main novelties are as follows. Firstly, we focus on the r-local scenario with malicious agents, which are more deleterious than the r-total model and/or malicious agents previously considered in [32, 31, 34]. The r-local assumption is more practical especially in large networks, where estimation over the entire network is often costly or unavailable. As categorized in [12], malicious agents are less adversarial because they have to broadcast the same value to all their neighbors. Secondly, the multi-hop resilient consensus strategy proposed here favorably accommodates both linear and nonlinear protocols and makes use of the weights of edges along the communication paths as opposed to the arithmetic average scheme used in [32].

On the basis of our scalar-valued framework, we further introduce a vector-valued resilient consensus protocol, which converts the messages received from ℓ-hop neighbors to scalars and a type of W-MSR mechanism can then be applied. Compared to scalar-valued states, resilient vector consensus has been much less studied mainly due to the difficulty of implementing the filtering of W-MSR in multiple dimensions. The scalar results of resilient consensus cannot be generalized to vector cases by using conventional techniques such as the Kronecker product in canonical multiagent systems. Some remarkable works in this direction include [2, 4, 21, 23, 28], which nevertheless only address one-hop interaction and delay-free systems. Moreover, to our knowledge only average-based linear protocols have been investigated in this direction. To accommodate multi-hop communication and general weighted protocols in these works, we need to formulate an appropriate network topology structure providing more flexible communicability (c.f. Definitions 1-3) and design a new agent control protocol with multiple system transformations (c.f. Remark 1) that are amenable to W-MSR algorithms. Our second framework presents a general distributed protocol to solve the vector-valued resilient consensus under ℓ-hop communication and time delay filling these gaps. As a byproduct, higher-order integrators can also be viewed as a special case in this framework.

It is worth noting that in practical networked systems time delay is a typical threat to the performance of consensus. This is highly relevant to the multi-hop communication scenario as information transmission along different paths may have different time delays. Our frameworks cope with the challenge of coexistence of time-varying path-dependent heterogeneous delays and multi-hop communication. Time delay has also been factored in [32] for discrete-time agents.

The rest of the paper is organized as follows. Section 2 sets out the problem formulation. Section 3 presents the multi-hop resilient consensus protocols. Sections 4 and 5 are devoted to scalar and vector consensus analyses, respectively. Simulations are given in Section 6 and conclusions are drawn in Section 7.

2. Problem statement

2.1. Graph theory

Denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of reals and non-negative integers. At time $t \geq 0$, the graph representation of the coupling topology of agents is given by a directed graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$, where the node set $\mathcal{V} = \{1, 2, \cdots, n\}$ represents the group of interacting agents, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of directed edges, and the adjacency matrix $\mathcal{A}(t) = (a_{ij}(t)) \in \mathbb{R}^{N\times N}$ is defined as $a_{ij}(t) > 0$ if $(j, i) \in \mathcal{E}(t)$, i.e., node j can send a message directly to node i, and $a_{ij}(t) = 0$ otherwise. Given nodes i_0 and i_l in $\mathcal{V}$, a directed path from i_0 to i_l, namely a sequence of adjacent nodes $(i_0, i_1, \cdots, i_l)$, is called an l-hop path, where i_0 is the head, i_l is the tail, and $i_1, i_2, \cdots, i_{l-1}$ (if any) are intermediate nodes. We write $d_{ij}(t)$ for the length of the shortest directed path from j to i. Clearly $d_{ij}(t) = 1$ if and only if $(j, i) \in \mathcal{E}(t)$. Let $\mathcal{P}_{ij}(t)$ be the collection of all l-hop paths from j to i if $d_{ij}(t) = l$.

To investigate multi-hop communication, we need some appropriate definitions for long-range neighbors. Specifically, the l-hop out-neighborhood and in-neighborhood of node i are given by $\mathcal{N}_i^{l+}(t) = \{j \in \mathcal{V} : d_{ij}(t) = l\}$ and $\mathcal{N}_i^{l-}(t) = \{j \in \mathcal{V} : d_{ji}(t) = l\}$, respectively, where $l \in \mathbb{N}$. Apparently, we have $\mathcal{N}_i^{0+}(t) = \mathcal{N}_i^{0-}(t) = \{i\}$. Given $\ell \in \mathbb{N}$, the $(\leq \ell)$-hop out-neighborhood and $(\leq \ell)$-hop in-neighborhood are given by $\mathcal{N}_i^{\leq\ell+}(t) = \cup_{l=0}^{\ell}\mathcal{N}_i^{l+}(t)$ and $\mathcal{N}_i^{\leq\ell-}(t) = \cup_{l=0}^{\ell}\mathcal{N}_i^{l-}(t)$, respectively. In classic graph theory [14], $\mathcal{N}_i^{1+}(t)$ and $\mathcal{N}_i^{1-}(t)$ (i.e., with $l = 1$) are referred to as the open out-/in-neighborhoods and $\mathcal{N}_i^{\leq 1+}(t)$ and $\mathcal{N}_i^{\leq 1-}(t)$ (i.e., with $l = 1$) are the closed out-/in-neighborhoods.

The set of nodes $\mathcal{V}$ is composed of two types of nodes, namely $\mathcal{V} = \mathcal{C} \cup \mathcal{B}$, where cooperative nodes are in $\mathcal{C}$ and malicious nodes are in $\mathcal{B}$. These two types of nodes follow different rules, which will be specified in the following sections. Inspired by [12, 32], we introduce the following definitions for multi-hop reachability and robustness; see Fig. 1(a) for an illustration.

Definition 1. For $r, \ell \in \mathbb{N}$, a set of nodes $\mathcal{S} \subseteq \mathcal{V}$ is said to be r-reachable under ℓ-hop communication relative to $\mathcal{B}$ if there exits $i \in \mathcal{S}$ such that i is the tail for at least r independent $(\leq \ell)$-hop paths with heads in $\mathcal{V}\backslash\mathcal{S}$ and all intermediate nodes of them are in $\mathcal{V}\backslash\mathcal{B}$. Here, two paths are independent if they do not share any nodes except the tail.

Definition 2. For $r, \ell \in \mathbb{N}$, a graph $\mathcal{G}$ is said to be r-robust under ℓ-hop communication relative to $\mathcal{B}$ if at least one of any two mutually exclusive nonempty sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$ is r-reachable under ℓ-hop communication relative to $\mathcal{B}$.

When $\ell = 1$, the 1-hop reachability and robustness are in line with the standard notions of reachability and robustness. Similarly, the locality of the set of malicious nodes in $\mathcal{G}(t)$ can also be extended to the multi-hop setting as follows.

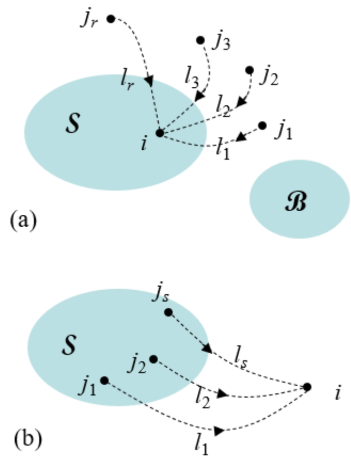

Figure 1: A schematic for Definitions 1-3. (a) The set $\mathcal{S}$ is r-reachable under ℓ-hop communication relative to $\mathcal{B}$ if the path lengths $l_1, l_2, \cdots, l_r \leq \ell$. (b) The set $\mathcal{S}$ is r-local under ℓ-hop communication if the path lengths $l_1, l_2, \cdots, l_s \leq \ell$ and $s \leq r$.

Definition 3. For $r, \ell, t \in \mathbb{N}$, a set of nodes $\mathcal{S} \subseteq \mathcal{V}$ is said to be r-local under ℓ-hop communication if $|\mathcal{S} \cap \mathcal{N}_i^{\leq \ell -}(t)| \leq r$ for every $i \in \mathcal{V} \backslash \mathcal{S}$.

The above locality is illustrated in Fig. 1(b). In the r-total model, a typical assumption is that the total number of malicious nodes is upper bounded by r, namely $|\mathcal{B}| \leq r$ [12, 20, 32, 34]. This condition requires a global knowledge of r, which is difficult in large networks, and it is more restrictive compared to requiring $\mathcal{B}$ to be r-local under multi-hop communication as there can be way more than r malicious nodes in the entire network. In practical applications such as sensor networks and autonomous vehicle systems, the parameter r in the r-local condition can often be estimated by using a distributed max-consensus process; see e.g. [8, 21].

2.2. System formulation for scalar consensus

As mentioned above, the set of nodes $\mathcal{V}$ consists of malicious nodes in $\mathcal{B}$ and cooperative nodes in $\mathcal{C}$. The number and identity of malicious nodes are hidden from the cooperative nodes and they on the other hand gain complete information of the entire network. Moreover, malicious nodes can collude with each other and send arbitrary bad messages to their neighbors. Malicious adversary models have been used to simulate the worst possible attack scenarios in computer science [11, 26]. The objective of solving resilient consensus problems is to design distributed control protocols for cooperative nodes so that they can achieve agreement regardless of the action of any potential misbehaving nodes.

Let $x_i(t) \in \mathbb{R}^m$ be the state of node $i \in \mathcal{V}$ at time t. In this subsection we consider the system formulation for the scalar consensus problem, i.e., the case of $m = 1$. Given a maximum communication capacity represented by the parameter $\ell \in \mathbb{N}$, node i in the multi-hop communication setting is able to transmit a message to each ($\leq \ell$)-hop out-neighbor j in $\mathcal{N}_i^{\leq \ell +}(t)$ along every path in $\mathcal{P}_{ji}(t)$. Along these paths, each cooperative intermediate node relays the message faithfully but malicious intermediate nodes can forge the message in an arbitrary manner. We write $x_i^p(t) \in \mathbb{R}^m$ for the received message at node j sent from head i along a path $p \in \mathcal{P}_{ji}(t)$. The time delay occurring in the transmission along the path p is denoted by $\tau_{ji}^p(t)$ satisfying $\tau_{ji}^p(t) \le \tau$ for some $\tau > 0$ (see Assumption 1 below), whose mechanism will be further specified in Section 3. In addition to the message information $x_i^p(t)$, node j also receives the path information p. We assume that an intermediate malicious node can only tamper the message information but not the path information as in [32, 26].

At any time $t \ge 0$, a cooperative node $i \in C$ performs the following three steps. 1) *Forward* step, in which node i forwards the message $x_i(t)$ to each node $j \in \mathcal{N}_i^{\le \ell +}(t)$ along every path in $\mathcal{P}_{ji}(t)$; 2) *Collate* step, in which node i collates message $x_j^p(t - \tau_{ij}^p(t))$ for all $j \in \mathcal{N}_i^{\le \ell -}(t)$ and $p \in \mathcal{P}_{ij}(t)$; 3) *Update* step, in which the state value follows

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

with $u_i(t) \in \mathbb{R}$ being the control input of i to be designed in Section 3. For a malicious node $i \in \mathcal{B}$, it can update the state in an arbitrary manner as commented above.

If $\varphi : (t_1, t_2) \to \mathbb{R}$ is continuous on the interval $t \in (t_1, t_2)$, its Dini upper right derivative is defined as

$$\mathsf{D}^+\varphi(t) = \limsup_{s \to 0^+} \frac{\varphi(t+s) - \varphi(t)}{s}. \tag{2}$$

The function φ is non-increasing if and only if $\mathsf{D}^+\varphi(t) \le 0$ for $t \in (t_1, t_2)$. The Dini derivative has the following property.

Lemma 1. ([5]) *Let $f_i(t, x) : (t_1, t_2) \times \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function and $f(t, x) := \max_{i \in \mathcal{V}} f_i(t, x)$, where $\mathcal{V} = \{1, 2, \cdots, n\}$. Suppose $x(t) \in \mathbb{R}^m$ is absolutely continuous on the interval $t \in (t_1, t_2)$. We obtain $\mathsf{D}^+ f(t, x(t)) = \max_{i \in \mathcal{I}(t)} \dot{f}_i(t, x(t))$, where $t \in (t_1, t_2)$ and $\mathcal{I}(t) := \{i \in \mathcal{V} : f_i(t, x(t)) = f(t, x(t))\}$.*

2.3. System formulation for vector consensus

In the case of multi-dimensional or vector state space of nodes, we have $x_i(t) \in \mathbb{R}^m$ as set up in Section 2.2 but with a general $m \in \mathbb{N}$. At any time $t \ge 0$, a cooperative node $i \in C$ performs the following three steps similarly. 1) *Forward* step, in which node i forwards the message $x_i(t)$ to each node $j \in \mathcal{N}_i^{\le \ell +}(t)$ along every path in $\mathcal{P}_{ji}(t)$; 2) *Collate* step, in which node i collates message $x_j^p(t - \tau_{ij}^p(t))$ for all $j \in \mathcal{N}_i^{\le \ell -}(t)$ and $p \in \mathcal{P}_{ij}(t)$; 3) *Update* step, in which the state vector follows

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{3}$$

with $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times q}$ being the system matrices and $u_i(t) \in \mathbb{R}^q$ being the control input of i to be designed in Section 3. Any malicious node $i \in \mathcal{B}$ will update the state in an arbitrary manner as commented above.

Taking $u_i(t) = Cx_i(t) + bw_i(t)$ for some $C \in \mathbb{R}^{q \times m}$ and $b \in \mathbb{R}^q$ in (3), we obtain the following single input system

$$\dot{x}_i(t) = (A + BC)x_i(t) + Bbw_i(t), \tag{4}$$

where $w_i(t) \in \mathbb{R}$ is the input value. Let $0_m \in \mathbb{R}^m$ be the vector of all zeros. If (A, B) is controllable, it is proved that [13] the matrix C can be taken as follows such that $(A + BC, Bb)$ is controllable for any $b \in \mathbb{R}^q$ satisfying $Bb \neq 0_m$:

$$C = QR^{-1}, \tag{5}$$

where $R = (B_1, AB_1, \cdots, A^{r_1-1}B_1, B_2, AB_2, \cdots, A^{r_2-1}B_2, \cdots, B_k, AB_k, \cdots, A^{r_k-1}B_k) \in \mathbb{R}^{m\times m}$ with $B_1, B_2, \cdots, B_k$ $(k \leq q)$ being the distinct columns in B, $\sum_{i=1}^{k} r_i = m$, $r_i \geq 1$ for $1 \leq i \leq k$, and $Q = (Q_1, Q_2, \cdots, Q_m) \in \mathbb{R}^{q\times m}$ with $Q_{a_j} = e_{j+1}$ for $1 \leq j \leq k-1$, $a_j = \sum_{i=1}^{j} r_i$, and $Q_a = 0_q$ for $a \in \{1, 2, \cdots, m\}\backslash\{a_1, a_2, \cdots, a_{k-1}\}$. Here, $e_{j+1} \in \mathbb{R}^q$ is the $(j+1)$-th unit vector for $1 \leq j \leq k-1$.

Since $(A+BC, Bb)$ is controllable, the matrix $F^{-1} := (Bb, (A+BC)Bb, (A+BC)^2Bb, \cdots, (A+BC)^{m-1}Bb) \in \mathbb{R}^{m\times m}$ is invertible. Let $F_m^{\mathsf{T}} \in \mathbb{R}^{1\times m}$ be the last row of the matrix F, where T is the matrix transpose. Let the characteristic polynomial of $A+BC$ be $\det(\gamma I_m - A - BC) = \sum_{k=1}^{m+1} \lambda_k \gamma^{k-1}$ with $\lambda_{m+1} = 1$ and $I_m \in \mathbb{R}^{m\times m}$ being the identity matrix. By defining the invertible matrix $S := (F_m, (A+BC)^{\mathsf{T}} F_m, (A+BC)^{2\mathsf{T}} F_m, \cdots, (A+BC)^{(m-1)\mathsf{T}} F_m)^{\mathsf{T}} \in \mathbb{R}^{m\times m}$, the system (4) can be further converted to the canonical form [10]

$$\dot{y}_i(t) = \overline{A} y_i(t) + \overline{B} w_i(t) \tag{6}$$

by applying the similarity transform $y_i(t) = S x_i(t) \in \mathbb{R}^m$, where $\overline{A} := S(A+BC)S^{-1} = \begin{pmatrix} 0 & \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{m+1} \\ -\lambda_1 & -\lambda_2 & -\lambda_3 & \cdots & -\lambda_m \end{pmatrix} \in \mathbb{R}^{m\times m}$ and $\overline{B} := SBb = (0, \cdots, 0, 0, 1)^{\mathsf{T}} \in \mathbb{R}^m$.

Remark 1. The system (3) can be viewed as the multiple-input multiple-output (MIMO) system for the networked agents under consideration. To approach the vector-valued resilient consensus problem, in the above we first introduce the transformation $u_i(t) = Cx_i(t) + bw_i(t)$ to convert (3) to the single-input multiple-output (SIMO) system (4). This is a key step in our framework to facilitate the state space analysis of the control system. The system (4) is then converted to the canonical form (6), which again is a SIMO system. It will be further reduced to a single-input single-output (SISO) framework in Section 3. This approach allows us to decouple the agent dynamics and apply the design idea of W-MSR algorithms later. As we have noted, when the original system (3) is controllable, the transformed system (4) is also controllable, which facilitates our reduction to the canonical form (6).

Remark 2. Taking into account of the time delay involved in the continuous-time multiagent systems (1) and (3), we can formally set up the framework by invoking the functional differential equation theory [9]. Write $\mathsf{C} = \mathsf{C}([-\tau, 0]; \mathbb{R}^{nm})$ for the Banach space consisting of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^{nm}$ under the norm $\|\varphi\|_{\mathsf{C}} = \sup_{-\tau\leq t\leq 0} \|\varphi(t)\|$ for $\varphi \in \mathsf{C}$. Given $x \in \mathsf{C}$ and $t \geq 0$, write $x_t(s) := x(t+s)$, $s \in [-\tau, 0]$. Given an initial function $\varphi \in \mathsf{C}$, the multiagent systems (1) and (3) can be recast as

$$\dot{x}(t) = f(x_t), \qquad t \geq 0, \tag{7}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathsf{T}} \in \mathbb{R}^{nm}$ and $f : \mathsf{C} \to \mathbb{R}^{nm}$ is a continuous functional. Here, the index i indicates the i-th component $x_i(t)$, which should not be confused with the time subscript t defined above. A solution of (7) with the initial condition φ, denoted by $x[\varphi](t)$, is defined on the interval $[-\tau, \infty)$, which satisfies the equation (7) for $t \geq 0$ and $x[\varphi](0) = \varphi$. The uniqueness of solution is established in [9].

3. ℓ-hop resilient consensus algorithms

In this section we present the ℓ-hop resilient consensus strategies in the scalar and vector node state scenarios. To thwart the potential influence of malicious nodes in multi-hop communication networks, the concept of minimum message cover [26] turns out to be useful. For a set of paths $\mathcal{P}$ with the same tail $i \in \mathcal{V}$, a message cover, denoted by $\mathcal{M}$, is a set of nodes satisfying: (i) $i \notin \mathcal{M} \subseteq \mathcal{V}$ and (ii) removing all nodes in $\mathcal{M}$ will disconnect each path $p \in \mathcal{P}$. It is clear that $\mathcal{M}$ contains at least one intermediate node or the head of each path $p \in \mathcal{P}$. A message cover with minimum cardinality is called a minimum message cover. Minimum message covers may not be unique but all minimum message covers have the same cardinality.

3.1. Scalar consensus case

We first consider the scalar consensus scenario described in Section 2.2. Given time $t \in \mathbb{N}$, multi-hop communication capacity $\ell \in \mathbb{N}$ and a parameter $r \in \mathbb{N}$, the *Update* step is implemented as follows. The cooperative node $i \in C$ sorts the values $\{x_j^p(t-\tau_{ij}^p(t))\}_{j\in\mathcal{N}_i^{\le\ell-}(t),\ p\in\mathcal{P}_{ij}(t)}$ and arranges them from the highest to the lowest in a list $L_i(t)$. We first examine the values which are higher than $x_i(t)$. Let $\mathcal{M}_i^>(t)$ be the minimum message cover for the group of paths corresponding to these values. If $|\mathcal{M}_i^>(t)| \le r$, all values larger than $x_i(t)$ are deleted from $L_i(t)$. If $|\mathcal{M}_i^>(t)| > r$, we delete the values starting from the highest in $L_i(t)$ downwards such that the corresponding minimum message cover has cardinality r but the deletion of one more value would increment the cardinality to $r+1$. We next examine the values which are lower than $x_i(t)$ likewise. Let $\mathcal{M}_i^<(t)$ be the minimum message cover for the group of paths corresponding to these values. If $|\mathcal{M}_i^<(t)| \le r$, all values lower than $x_i(t)$ are deleted from $L_i(t)$. If $|\mathcal{M}_i^<(t)| > r$, we delete the values starting from the lowest in $L_i(t)$ upwards such that the corresponding minimum message cover has cardinality r but the deletion of one more value would increment the cardinality to $r+1$. Write $\mathcal{R}_i(t) \subseteq \cup_{j\in\mathcal{N}_i^{\le\ell-}}\mathcal{P}_{ij}(t)$ for the set of paths corresponding to those deleted values from $L_i(t)$. For $i, j \in \mathcal{V}$, let $\psi_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and locally Lipschitz function. The control input in (1) is given by

$$u_i(t) = \sum_{p\in\left(\cup_{j\in\mathcal{N}_i^{\le\ell-}(t)}\mathcal{P}_{ij}(t)\right)\setminus\mathcal{R}_i(t)} a_{ij}^p(t) \cdot \psi_{ij}(x_j^p(t-\tau_{ij}^p(t)), x_i(t)), \tag{8}$$

where $a_{ij}^p(t) = a_{ii_1}(t)a_{i_1i_2}(t)\cdots a_{i_{l-1}j}(t)$ and $p = (j, i_{l-1}, i_{l-2}, \cdots, i_2, i_1, i) \in \mathcal{P}_{ij}(t)$ is an l-hop path for some $l \le \ell$. The coupling coefficient $a_{ij}^p(t)$ is path-weighted, which is of physical relevance when the edge weights are no more than 1 — Longer communication path may exert lower influence.

Assumption 1. We assume that the function ψ_{ij} in (8) satisfies (i) $\psi_{ij}(b_1, b_2) = 0$ if and only if $b_1 = b_2$ and (ii) $\psi_{ij}(b_1, b_2)(b_1 - b_2) > 0$ if $b_1 \ne b_2$. Moreover, the path-dependent time delay satisfies $\tau_{ij}^p(t) \le \tau$ for some $\tau > 0$.

3.2. Vector consensus case

We next consider the vector consensus scenario described in Section 2.3. Note that $x_i(t) \in \mathbb{R}^m$ with potentially $m > 1$ and the above algorithm cannot be directly applied here. The trick up our sleeve is a conversion as follows. Define $M := (\mu_1, \mu_2, \cdots, \mu_{m-1}, \mu_m) \in \mathbb{R}^{1\times m}$ such that the polynomial $\sum_{k=1}^m \mu_k\gamma^{k-1}$ is Hurwitz stable and $\mu_m = 1$. Recall that the above polynomial is Hurwitz stable if all coefficients are positive and the roots possess negative real parts. For $i \in \mathcal{V}$, define a scalar value $z_i(t) = MSx_i(t) \in \mathbb{R}$, where S is given in Section 2.3. Similarly, set $z_i^p(t - \tau_{ij}^p(t)) = MSx_i^p(t - \tau_{ij}^p(t)) \in \mathbb{R}$ for $i, j \in \mathcal{V}$ and $p \in \mathcal{P}_{ij}(t)$.

With this preparation, given time $t \in \mathbb{N}$, multi-hop communication capacity $\ell \in \mathbb{N}$ and a parameter $r \in \mathbb{N}$, the *Update* step can be implemented as follows. The cooperative node $i \in C$ sorts the values $\{z_j^p(t - \tau_{ij}^p(t))\}_{j \in \mathcal{N}_i^{\leq \ell -}(t),\ p \in \mathcal{P}_{ij}(t)}$ and arranges them from the highest to the lowest in a list $L_i(t)$. We first examine the values which are higher than $z_i(t)$. Let $\mathcal{M}_i^{>}(t)$ be the minimum message cover for the group of paths corresponding to these values. If $|\mathcal{M}_i^{>}(t)| \leq r$, all values larger than $z_i(t)$ are deleted from $L_i(t)$. If $|\mathcal{M}_i^{>}(t)| > r$, we delete the values starting from the highest in $L_i(t)$ downwards such that the corresponding minimum message cover has cardinality r but the deletion of one more value would increment the cardinality to $r + 1$. We next examine the values which are lower than $z_i(t)$ likewise. Let $\mathcal{M}_i^{<}(t)$ be the minimum message cover for the group of paths corresponding to these values. If $|\mathcal{M}_i^{<}(t)| \leq r$, all values lower than $z_i(t)$ are deleted from $L_i(t)$. If $|\mathcal{M}_i^{<}(t)| > r$, we delete the values starting from the lowest in $L_i(t)$ upwards such that the corresponding minimum message cover has cardinality r but the deletion of one more value would increment the cardinality to $r + 1$. Similarly, write $\mathcal{R}_i(t) \subseteq \cup_{j \in \mathcal{N}_i^{\leq \ell -}} \mathcal{P}_{ij}(t)$ for the set of paths corresponding to those deleted values from $L_i(t)$. The control input $u_i(t) \in \mathbb{R}^q$ in (3) is given by

$$\begin{aligned} u_i(t) =& (C + b\Lambda S)x_i(t) \\ &+ b \sum_{p \in \left(\cup_{j \in \mathcal{N}_i^{\leq \ell -}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} a_{ij}^p(t) \\ &\cdot \psi_{ij}(z_j^p(t - \tau_{ij}^p(t)), z_i(t)) \end{aligned} \tag{9}$$

or equivalently, the input value $w_i(t) \in \mathbb{R}$ in (6) is given by

$$\begin{aligned} w_i(t) =& \Lambda S x_i(t) + \sum_{p \in \left(\cup_{j \in \mathcal{N}_i^{\leq \ell -}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} a_{ij}^p(t) \\ &\cdot \psi_{ij}(z_j^p(t - \tau_{ij}^p(t)), z_i(t)), \end{aligned} \tag{10}$$

where $\Lambda = (\lambda_1, \lambda_2 - \mu_1, \lambda_3 - \mu_2, \cdots, \lambda_m - \mu_{m-1}) \in \mathbb{R}^{1 \times m}$, $a_{ij}^p(t) = a_{ii_1}(t) a_{i_1 i_2}(t) \cdots a_{i_{l-1} j}(t)$ and $p = (j, i_{l-1}, i_{l-2}, \cdots, i_2, i_1, i) \in \mathcal{P}_{ij}(t)$ is an l-hop path for some $l \leq \ell$.

Remark 3. The coupling function $\psi_{ij}(b_1, b_2)$ follows Assumption 1, which covers both typical linear and nonlinear protocols like $\psi_{ij}(b_1, b_2) = (b_1 - b_2)^{\gamma_1}$ for odd γ_1 and $\psi_{ij}(b_1, b_2) = \mathrm{sgn}(b_1 - b_2)|b_1 - b_2|^{\gamma_2}$ for some $\gamma_2 > 0$ [15, 14], where sgn is the signum function.

Remark 4. In our multi-hop communication framework, every cooperative node $i \in C$ has the knowledge of its ($\leq \ell$)-hop out-neighborhood $\mathcal{N}_i^{\leq \ell +}(t)$ and ($\leq \ell$)-hop in-neighborhood $\mathcal{N}_i^{\leq \ell -}(t)$. When $\ell = 1$, this reduces to the classical network consensus. As ℓ is a finite and often small number, the minimum message covers can be conveniently determined [26]. In the previous W-MSR protocols [11, 12, 21], a cooperative agent typically updates its state in each iteration by discarding the largest r and the smallest r states among its neighbors. Hence, our proposed protocols are of low complexity comparable to these W-MSR protocols, and the main computational load is again around sorting the neighboring states.

Remark 5. The control laws designed above are based on the idea of W-MSR algorithms [1, 11, 12, 21], which typically remove a set of largest and smallest values in the neighborhood of each cooperative agent. Our designed protocols in (8) and (9) still belong to this class of W-MSR algorithms although we discard extreme values based on the concept of minimum message cover and ℓ-hop neighborhoods. It is worth noting that there have been some other competing design principles tackling resilient consensus problems. One of them is the adaptive control methods [6, 30], which involve the design of distributed emulator-based adaptive controllers. These methods are generally capable of withstanding misbehaving agents influenced by exogenous disturbances and controlling the performance of the system under malicious behaviors against that of the undisturbed system. Another strategy is using reputation metrics to classify the neighbors and remove the malicious agents [17, 18]. This line of research generally takes a consensus algorithm as input, and allows cooperative agents to identify any malicious neighbors such that the consensus value can be corrected to that of the subnetwork containing only cooperative agents. The differences of these principles are summarized on a high level in Fig. 2, which is not intended to be categorical as the boundaries between these approaches presumably tend to blur.

	Number of malicious agents	Malicious behavior	Agent state space	Control objective
W-MSR	Bounded	General	Mostly scalar	Consensus of cooperative agents
Adaptive control	Unbounded	Disturbance	Vector	To ensure consensus value of the whole system sits within bounded distance from the undisturbed system
Reputation metric	Bounded	General	Vector	To identify malicious agents and correct the consensus value of cooperative agents

Figure 2: A comparison of some design principles for approaching resilient consensus problems.

In order to develop effective resilient consensus results, we in the following consider a fixed network of agents $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. As the above algorithms alter the communication edges as time goes, we essentially work with a switching network $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ as formulated above. A mild condition regarding the switching frequency is the following.

Assumption 2. We assume that the network $\mathcal{G}(t)$ switches at the following time points $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$. There is $\theta \in \mathbb{R}$ satisfying $t_{k+1} - t_k \geq \theta > 0$ for $k \in \mathbb{N}$.

The objective of resilient consensus is to show that all cooperative nodes reach an agreement and are typically within a bounded safe region. Specifically, we aim to show (i) $x_i(t)$ remains bounded for any $i \in C$ and $t \geq -\tau$ and (ii) $\lim_{t\to\infty} x_i(t) - x_j(t) = 0_m$ for any $i, j \in C$ and initial configuration $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t)) \in \mathsf{C}$, i.e., $x_i[\varphi](0) = \varphi_i$ for $i \in \mathcal{V}$.

4. Resilient scalar consensus analysis

The proposed multi-hop resilient algorithm with a scalar state space is analyzed in this section. Our main result is the following.

Theorem 1. *Consider the weighted directed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $\mathcal{V} = \mathcal{B} \cup C$, where the cooperative nodes follow the proposed ℓ-hop resilient scalar consensus strategy with (1) and (8) and the set $\mathcal{B}$ of malicious nodes is r-local under ℓ-hop communication. Suppose that*

Assumptions 1 and 2 hold. If $\mathcal{G}$ is $(2r+1)$-robust under ℓ-hop communication relative to $\mathcal{B}$, then resilient consensus is achieved.

Proof. (Boundedness) We will first show that the trajectories of the nodes are within an interval determined by the initial configuration of the cooperative nodes.

Define two quantities $\varphi_{\max} := \max_{i\in\mathcal{C},-\tau\le t\le 0}\varphi_i(t)$ and $\varphi_{\min} := \min_{i\in\mathcal{C},-\tau\le t\le 0}\varphi_i(t)$. By the system setup, $\varphi_{\min} \le x_i(t) \le \varphi_{\max}$ for $-\tau \le t \le 0$ and $i \in \mathcal{C}$. If a cooperative would ever exceed the upper bound $\varphi_{\max}$ at some time $\bar{t} > 0$, without loss of generality we can assume a node $\bar{i} \in \mathcal{C}$ satisfying $x_{\bar{i}}(\bar{t}) = \varphi_{\max}$, $\dot{x}_{\bar{i}}(\bar{t}) > 0$, and $x_i(t) \le \varphi_{\max}$ for all $i \in \mathcal{C}$ and $t \le \bar{t}$. In view of (1) and (8) we obtain

$$\begin{aligned} 0 <\dot{x}_{\bar{i}}(\bar{t}) = &\sum_{p\in\left(\cup_{j\in\mathcal{N}_{\bar{i}}^{\le\ell-}(\bar{t})}\mathcal{P}_{\bar{i}j}(\bar{t})\right)\setminus\mathcal{R}_{\bar{i}}(\bar{t})} a^p_{\bar{i}j}(\bar{t}) \\ &\cdot \psi_{\bar{i}j}\big(x_j^p(\bar{t}-\tau^p_{\bar{i}j}(\bar{t})), x_{\bar{i}}(\bar{t})\big). \end{aligned} \tag{11}$$

We take a closer look at the right-hand side of (11). Fix a node $j \in \mathcal{N}_{\bar{i}}^{\le\ell-}(\bar{t})$ and a path $p \in \mathcal{P}_{\bar{i}j}(\bar{t})\setminus\mathcal{R}_{\bar{i}}(\bar{t})$. If node $j \in \mathcal{C}$ and all intermediate nodes in p are cooperative nodes, then $x_j^p(\bar{t} - \tau^p_{\bar{i}j}(\bar{t})) = x_j(\bar{t} - \tau^p_{\bar{i}j}(\bar{t})) \le \varphi_{\max} = x_{\bar{i}}(\bar{t})$ by our assumption. If $j \in \mathcal{B}$ or there is a malicious intermediate node in p, then our proposed algorithm leads to $x_j^p(\bar{t} - \tau^p_{\bar{i}j}(\bar{t})) \le x_{\bar{i}}(\bar{t}) = \varphi_{\max}$ since $\mathcal{B}$ is r-local under ℓ-hop communication. Using Assumption 1, $\psi_{\bar{i}j}(x_j^p(\bar{t} - \tau^p_{\bar{i}j}(\bar{t})), x_{\bar{i}}(\bar{t}))$ on the right-hand side of (11) is non-positive. Since the coupling weight $a^p_{\bar{i}j}(\bar{t}) \ge 0$, we conclude the right-hand side of (11) is non-positive, which is of course a contradiction. Hence, we have proved $x_i(t) \le \varphi_{\max}$ for any $t \ge -\tau$ and $i \in \mathcal{C}$. In an analogous manner, we can derive $x_i(t) \ge \varphi_{\min}$ for any $t \ge -\tau$ and $i \in \mathcal{C}$, which concludes the boundedness of the solution.

(Agreement) To show the asymptotic agreement of cooperative nodes, we define $h_{\max}(x_t) := \max_{i\in\mathcal{C},-\tau\le s\le 0} x_i(t+s)$ and $h_{\min}(x_t) := \min_{i\in\mathcal{C},-\tau\le s\le 0} x_i(t+s)$ for $t \ge 0$ along the solution of the system (1) and (8). The continuous Lyapunov-Krasovskii functional can be taken as $g(x_t) := h_{\max}(x_t) - h_{\min}(x_t) \ge 0$.

Fix $t \ge 0$. In the light of Lemma 1, we have

$$\begin{aligned} &\mathsf{D}^+ h_{\max}(x_t) \\ &= \max_{i\in\mathcal{I}_1(t)} \dot{x}_i(t+s_1) := \dot{x}_{i_1}(t+s_1) \\ &= \sum_{p\in\left(\cup_{j\in\mathcal{N}_{i_1}^{\le\ell-}(t+s_1)}\mathcal{P}_{i_1 j}(t+s_1)\right)\setminus\mathcal{R}_{i_1}(t+s_1)} a^p_{i_1 j}(t+s_1) \\ &\quad\cdot \psi_{i_1 j}\big(x_j^p(t+s_1-\tau^p_{i_1 j}(t+s_1)), x_{i_1}(t+s_1)\big) \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\mathsf{D}^+ h_{\min}(x_t) \\ &= \min_{i\in\mathcal{I}_2(t)} \dot{x}_i(t+s_2) := \dot{x}_{i_2}(t+s_2) \\ &= \sum_{p\in\left(\cup_{j\in\mathcal{N}_{i_2}^{\le\ell-}(t+s_2)}\mathcal{P}_{i_2 j}(t+s_2)\right)\setminus\mathcal{R}_{i_2}(t+s_2)} a^p_{i_2 j}(t+s_2) \\ &\quad\cdot \psi_{i_2 j}\big(x_j^p(t+s_2-\tau^p_{i_2 j}(t+s_2)), x_{i_2}(t+s_2)\big), \end{aligned} \tag{13}$$

where $s_1, s_2 \in [-\tau, 0]$, $\mathcal{I}_1(t) = \{j \in C : x_j(t) = \max_{i \in C} x_i(t)\}$ and $\mathcal{I}_2(t) = \{j \in C : x_j(t) = \min_{i \in C} x_i(t)\}$ are two non-empty index sets. We take a closer look at the sign of $\mathsf{D}^+ h_{\max}(x_t)$ in (12). Note that the choice of $s_1 \in [-\tau, 0]$ may be not unique and let us consider the largest such s_1. Equivalently, denote by $\bar{t}$ the largest time instant in the range $[t-\tau, t]$ satisfying $x_{i_1}(\bar{t}) = \max_{i \in C, -\tau \le s \le 0} x_i(t+s)$. For the location of $\bar{t}$, we consider three cases.

(i) If $\bar{t} = t - \tau$, this means $s_1 = -\tau$ and $t - \tau$ is the unique time instant in the interval $[t-\tau, t]$ attaining the maximum $\max_{i \in C, -\tau \le s \le 0} x_i(t+s)$. Hence, $\mathsf{D}^+ h_{\max}(x_t) < 0$.

(ii) If $t - \tau < \bar{t} < \tau$, this means $-\tau < s_1 < 0$ and the maximum is attained in the middle of the interval. Hence, $\mathsf{D}^+ h_{\max}(x_t) = 0$.

(iii) If $\bar{t} = \tau$, this means $s_1 = 0$. Fix a node $j \in \mathcal{N}_{i_1}^{\le \ell -}(t+s_1)$ and a path $p \in \mathcal{P}_{i_1 j}(t+s_1) \backslash \mathcal{R}_{i_1}(t+s_1)$. If node $j \in C$ and all intermediate nodes in p are cooperative nodes, then by definition we have $x_{i_1}(t+s_1) = x_{i_1}(\bar{t}) \ge x_j^p(t + s_1 - \tau_{i_1 j}^p(t+s_1))$. If $j \in \mathcal{B}$ or there is a malicious intermediate node in p, then our algorithm yields $x_{i_1}(t+s_1) \ge x_j^p(t+s_1-\tau_{i_1 j}^p(t+s_1))$ since $\mathcal{B}$ is r-local under ℓ-hop communication. As per Assumption 1, we have $\psi_{i_1 j}(x_j^p(t+s_1-\tau_{i_1 j}^p(t+s_1)), x_{i_1}(t+s_1)) \le 0$. Since the coupling weight $a_{i_1 j}^p(t+s_1) \ge 0$, we conclude $\mathsf{D}^+ h_{\max}(x_t) \le 0$ in (12).

Combining the above comments, we derive $\mathsf{D}^+ h_{\max}(x_t) \le 0$ for all $t \ge 0$. Utilizing an analogous argument for (13), we arrive at $\mathsf{D}^+ h_{\min}(x_t) \ge 0$ for all $t \ge 0$. Hence, $\mathsf{D}^+ g(x_t) = \mathsf{D}^+ h_{\max}(x_t) - \mathsf{D}^+ h_{\min}(x_t) \le 0$ for $t \ge 0$.

From the above discussion we know that $g(x_t)$ is non-increasing. We can further claim that $\lim_{t\to\infty} \mathsf{D}^+ g(x_t) = 0$. In fact, if this is not true, we have $\liminf_{t\to\infty} \mathsf{D}^+ g(x_t) < 0$. There must exist some $\alpha < 0$ satisfying the following property: For each time instant $\bar{t} > 0$ there exits $t \ge \bar{t}$ such that $\mathsf{D}^+ g(x_t) < 2\alpha$. Take a sequence of time points $\bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_{k'} < \bar{t}_{k'+1} < \cdots$ satisfying $\mathsf{D}^+ g(x_{\bar{t}_{k'}}) < 2\alpha$ and $\bar{t}_{k'+1} - \bar{t}_{k'} > \theta_1 > 0$ for any $k' \ge 1$, where θ_1 is a constant. Recall $\{t_k\}_{k\ge1}$ is the collection of switching points specified in Assumption 2. For any interval $\mathcal{J}$ of time satisfying $\mathcal{J} \cap \{t_k\}_{k\ge1} = \emptyset$, by Assumption 1 the system solution $x(t)$ is bounded and $\mathsf{D}^+ g(x_t)$ is uniformly continuous on $\mathcal{J}$. Consequently, there is $\theta_2 > 0$ satisfying $|\mathsf{D}^+ g(x_{t'}) - \mathsf{D}^+ g(x_{t''})| \le -\alpha$ for all $t', t'' \in \mathcal{J}$ and $|t' - t''| \le \theta_2$. We can choose the above θ_2 small enough such that $[\bar{t}_{k'} - \theta_2, \bar{t}_{k'} + \theta_2] \cap \{t_k\}_{k\ge1} = \emptyset$ for all $k' \ge 1$. For any $t \in [\bar{t}_{k'} - \theta_2, \bar{t}_{k'} + \theta_2]$ with $k' \ge 1$, we obtain

$$
\begin{aligned}
\mathsf{D}^+ g(x_t) &= -\,|\mathsf{D}^+ g(x_{\bar{t}_{k'}}) - \mathsf{D}^+ g(x_{\bar{t}_{k'}}) + \mathsf{D}^+ g(x_t)| \\
&\le -\,|\mathsf{D}^+ g(x_{\bar{t}_{k'}})| + |\mathsf{D}^+ g(x_{\bar{t}_{k'}}) - \mathsf{D}^+ g(x_t)| \\
&\le 2\alpha - \alpha = \alpha,
\end{aligned} \tag{14}
$$

where we recall that $\mathsf{D}^+ g(x_{\bar{t}_{k'}}) < 2\alpha$ and $\alpha < 0$ by definition, and therefore $-|\mathsf{D}^+ g(x_{\bar{t}_{k'}})| < 2\alpha$. Invoking Assumption 2, we can choose $0 < \theta_3 < \theta$ satisfying the following two conditions: (i) the collection $\{[\bar{t}_{k'} - \theta_3, \bar{t}_{k'} + \theta_3]\}_{k'\ge1}$ forms a set of mutually exclusive intervals, and (ii) the upper bound $\mathsf{D}^+ g(x_t) \le \alpha$ in (14) is true for all $t \in [\bar{t}_{k'} - \theta_3, \bar{t}_{k'} + \theta_3]$ with $k' \ge 1$. Taking integration of the derivative $\mathsf{D}^+ g(x_t)$ over the real half-axis yields

$$
\begin{aligned}
\int_0^\infty \mathsf{D}^+ g(x_t) dt &\le \lim_{K\to\infty} \sum_{k'=1}^{K} \int_{\bar{t}_{k'}-\theta_3}^{\bar{t}_{k'}+\theta_3} \mathsf{D}^+ g(x_t) dt \\
&\le \lim_{K\to\infty} \sum_{k'=1}^{K} 2\alpha\theta_3 \\
&= \lim_{K\to\infty} 2K\alpha\theta_3 = -\infty.
\end{aligned} \tag{15}
$$

Recall that $\mathsf{D}^+ g(x_t) \le 0$ for $t \ge 0$. The infinity in (15) implies that $g(x_t)$ continue decreasing without a lower bound. This is at odds with the fact $g(x_t) = h_{\max}(x_t) - h_{\min}(x_t) \ge 0$ for $t \ge 0$. Hence, we have proved that $\mathsf{D}^+ g(x_t)$ is vanishing as t tends to infinity.

By the definition of $g(x_t)$ and the above comments, we know that $\lim_{t\to\infty} \mathsf{D}^+ h_{\max}(x_t) = \lim_{t\to\infty} \mathsf{D}^+ h_{\min}(x_t) = 0$. In view of (12) and (13), there exist two constants $\beta_{\min} \le \beta_{\max}$ such that $\lim_{t\to\infty} x_{i_1}(t) = \beta_{\max}$ and $\lim_{t\to\infty} x_{i_2}(t) = \beta_{\min}$. Note that the two nodes i_1 and i_2 may depend on t. If we can prove $\beta_{\min} = \beta_{\max}$, then the agreement is achieved for all cooperative nodes asymptotically. To this end, we note that the malicious set $\mathcal{B}$ is r-local under ℓ-hop communication and the network $\mathcal{G}$ is $(2r+1)$-robust under ℓ-hop communication relative to $\mathcal{B}$. Using our proposed algorithm, the resulting network $\mathcal{G}(t)$ becomes 1-robust under ℓ-hop communication relative to $\mathcal{B}$. For any $t \in \mathbb{N}$, we claim that $\mathcal{G}(t)$ has a directed spanning tree. In fact, if this is not true, $\mathcal{G}(t)$ contains two strongly connected components $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{N}_i^{1-}(t)\backslash\mathcal{S}_1 = \emptyset$ and $\mathcal{N}_j^{1-}(t)\backslash\mathcal{S}_2 = \emptyset$ for all $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$. This would be at odds with our assumption that $\mathcal{G}(t)$ is 1-robust under ℓ-hop communication relative to $\mathcal{B}$ for any given $\ell \ge 1$.

Using $\lim_{t\to\infty} \mathsf{D}^+ h_{\max}(x_t) = 0$ and (12), we know that $\psi_{i_1 j}(x_j^p(t+s_1-\tau_{i_1 j}^p(t+s_1)), x_{i_1}(t+s_1)) \to 0$ for any path $p \in (\cup_{j\in\mathcal{N}_{i_1}^{\le\ell-}(t+s_1)} \mathcal{P}_{i_1 j}(t+s_1))\backslash\mathcal{R}_{i_1}(t+s_1)$ as $t \to \infty$. By Assumption 1, $x_j^p(t+s_1-\tau_{i_1 j}^p(t+s_1)) - x_{i_1}(t+s_1) \to 0$ as t tends to infinity. We can repeat this argument by checking the 1-hop in-neighbors of nodes j following the dynamics (8) and so on. This means the root node of the spanning tree in $\mathcal{G}(t)$ asymptotically holds the value $\beta_{\max}$ as t tends to infinity. An analogous argument can be applied to i_2 and (13), and we conclude that the root node of the spanning tree in $\mathcal{G}(t)$ asymptotically holds the value $\beta_{\min}$ as t tends to infinity. Although the two nodes i_1, i_2 and the root node depend on $\mathcal{G}(t)$ and hence t, the above argument holds for any given t. Therefore, we have $\beta_{\max} = \beta_{\min}$ and the proof is complete. □

Remark 6. As mentioned in Section 2, the identities of malicious nodes are generally not available to the cooperative nodes. In the worst scenario, to verify the robustness condition in Theorem 1, we will have no choice but to check every possible set of $\mathcal{B}$. For instance, if for every set $\mathcal{S} \subseteq \mathcal{V}$ with $|\mathcal{S}| = r$, the network $\mathcal{G}$ is $(2r+1)$-robust under ℓ-hop communication relative to $\mathcal{S}$, then the robustness condition in Theorem 1 holds. In the event that some malicious nodes are spotted, the number of testing cases can be reduced accordingly.

5. Resilient vector consensus analysis

When the state space of nodes is multi-dimensional, we have $x_i(t) \in \mathbb{R}^m$ for $i \in \mathcal{V}$. We have the following consensus result on the basis of the proposed multi-dimensional consensus protocol in Section 3.

Theorem 2. *Consider the weighted directed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $\mathcal{V} = \mathcal{B} \cup \mathcal{C}$, where the cooperative nodes follow the proposed ℓ-hop resilient vector consensus strategy with (3) and (9) and the set $\mathcal{B}$ of malicious nodes is r-local under ℓ-hop communication. Suppose that Assumptions 1 and 2 hold, and (A, B) is controllable. If $\mathcal{G}$ is $(2r+1)$-robust under ℓ-hop communication relative to $\mathcal{B}$, then resilient consensus is achieved.*

Proof. Recall that $z_i(t) = MSx_i(t) = My_i(t) \in \mathbb{R}$, where $M = (\mu_1, \mu_2, \cdots, \mu_m) \in \mathbb{R}^{1\times m}$ and

$S \in \mathbb{R}^{m\times m}$ are defined in Section 2. By using (6) and (10), we obtain for $i \in C$,

$$\begin{aligned}
\dot{z}_i(t) =& MS\dot{x}_i(t) = M\dot{y}_i(t) = M\overline{A}y_i(t) + M\overline{B}w_i(t) \\
=& M\overline{A}y_i(t) + M\overline{B}\Bigg(\Lambda S x_i(t) \\
&+ \sum_{p\in\left(\cup_{j\in N_i^{\leq \ell^-(t)}}\mathcal{P}_{ij}(t)\right)\setminus\mathcal{R}_i(t)} a_{ij}^p(t) \\
&\cdot \psi_{ij}(z_j^p(t-\tau_{ij}^p(t)), z_i(t))\Bigg) \\
=& \sum_{p\in\left(\cup_{j\in N_i^{\leq \ell^-(t)}}\mathcal{P}_{ij}(t)\right)\setminus\mathcal{R}_i(t)} a_{ij}^p(t) \\
&\cdot \psi_{ij}(z_j^p(t-\tau_{ij}^p(t)), z_i(t)),
\end{aligned} \tag{16}$$

where we have applied the relationship $M\overline{A} = -M\overline{B}\Lambda \in \mathbb{R}^{1\times m}$ and $M\overline{B} = 1$. We observe that (16) is equivalent to the scalar consensus system (1) with (8). Therefore, by Theorem 1 we have: (i) There exists some $z \in \mathbb{R}$ such that $\lim_{t\to\infty} z_i(t) = z$ for all $i \in C$, and (ii) $MS\varphi_{\min} \leq z_i(t) \leq MS\varphi_{\max}$ for all $i \in C$ and $t \geq -\tau$.

Write $y_i(t) = (y_{i,1}(t), y_{i,2}(t), \cdots, y_{i,m})^\top \in \mathbb{R}^m$ for $i \in C$. Denote by $\eta_i(t) = (y_{i,1}(t), y_{i,2}(t), \cdots, y_{i,m-1}(t))^\top \in \mathbb{R}^{m-1}$. By (6) and $z_i(t) = My_i(t)$, we have $\dot{y}_{i,m-1}(t) = y_{i,m}(t) = \mu_m y_{i,m}(t) = z_i(t) - \sum_{k=1}^{m-1}\mu_k y_{i,k}(t)$ and

$$\dot{\eta}_i(t) = \hat{A}\eta_i(t) + \hat{B}z_i(t), \tag{17}$$

where $\hat{A} = \begin{pmatrix} 0 & \mu_m & 0 & \cdots & 0 \\ 0 & 0 & \mu_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_m \\ -\mu_1 & -\mu_2 & -\mu_3 & \cdots & -\mu_{m-1} \end{pmatrix}$
$\in \mathbb{R}^{(m-1)\times(m-1)}$ and $\hat{B} = (0, 0, \cdots, 0, 1)^\top \in \mathbb{R}^{m-1}$. By our assumption in Section 3 that $\det(\gamma I_{m-1} - \hat{A}) = \sum_{k=1}^{m}\mu_k\gamma^{k-1}$ is Hurwitz stable. Hence, $\hat{A}$ is a Hurwitz stable matrix.

Since $\mu_1 > 0$, we define the difference vector $\varepsilon_i(t) := \eta_i(t) - (\mu_1^{-1}, 0_{m-2}^\top)^\top z \in \mathbb{R}^{m-1}$ for every $i \in C$. Using (17) we have the following dynamics

$$\begin{aligned}
\dot{\varepsilon}_i(t) =& \hat{A}\eta_i(t) + \hat{B}z_i(t) \\
=& \hat{A}\varepsilon_i(t) + \hat{A}(\mu_1^{-1}, 0_{m-2}^\top)^\top z + \hat{B}z_i(t) \\
=& \hat{A}\varepsilon_i(t) + \hat{B}(z_i(t) - z),
\end{aligned} \tag{18}$$

where we have applied the expression $\hat{A}(\mu_1^{-1}, 0_{m-2}^\top)^\top + \hat{B} = 0_{m-1}$. Since $\hat{A}$ is Hurwitz stable and z is the limit of $z_i(t)$, it follows from (18) that $\lim_{t\to\infty}\varepsilon_i(t) = 0$ for $i \in C$. Therefore, $\lim_{t\to\infty}\eta_i(t) = (\mu_1^{-1}, 0_{m-2}^\top)^\top z$. Noting that $\lim_{t\to\infty} y_{i,m}(t) = \lim_{t\to\infty} z_i(t) - \lim_{t\to\infty}\sum_{k=1}^{m-1}\mu_k y_{i,k}(t) = z - z = 0$, we obtain $\lim_{t\to\infty} y_i(t) = (\mu_1^{-1}, 0_{m-1}^\top)^\top z$ for $i \in C$. Therefore,

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} S^{-1}y_i(t) = S^{-1}(\mu_1^{-1}, 0_{m-1}^\top)^\top z \tag{19}$$

for $i \in C$. Recall that $S := (F_m, (A+BC)^{\mathsf{T}} F_m, (A+BC)^{2\mathsf{T}} F_m, \cdots, (A+BC)^{(m-1)\mathsf{T}} F_m)^{-\mathsf{T}} \in \mathbb{R}^{m\times m}$. Let the first column of S^{-1} be denoted by S_1. Thus, we have $\lim_{t\to\infty} x_i(t) = \mu_1^{-1} z S_1$ for $i \in C$. The asymptotic agreement is proved.

Furthermore, noting that $z_i(t) = MS\,x_i(t)$ and $MS\varphi_{\min} \le z_i(t) \le MS\varphi_{\max}$ for all $i \in C$ and $t \ge -\tau$, it is clear that $x_i(t)$ remains in a bounded region determined by the initial configuration of the multiagent system (3) with (9). The proof is complete. □

It is clear that Remark 6 is also applicable here. Moreover, we mention that our resilient vector consensus framework can be used to solve resilient consensus for agents with higher-order dynamics. Consider the multiagent dynamics:

$$\begin{aligned} &\dot{\xi}_i(t) = \xi_i^{(1)}(t), \quad \dot{\xi}_i^{(1)}(t) = \xi_i^{(2)}(t), \cdots, \\ &\dot{\xi}_i^{(m-2)}(t) = \xi_i^{(m-1)}(t), \quad \dot{\xi}_i^{(m-1)}(t) = w_i(t), \end{aligned} \tag{20}$$

where $\xi_i(t) \in \mathbb{R}$ is the state value of node $i \in C$, $\xi_i^{(k)}(t) \in \mathbb{R}$ is the k-th derivative for $1 \le k \le m-1$, and $w_i(t) \in \mathbb{R}$ is the control input.

Set $y_i(t) := (\xi_i(t), \xi_i^{(1)}(t), \cdots, \xi_i^{(m-1)}(t))^{\mathsf{T}} \in \mathbb{R}^m$. The system (20) is equivalent to the system (6) by choosing $\lambda_k = 0$ for $1 \le k \le m$, namely

$$\dot{y}_i(t) = \overline{A} y_i(t) + \overline{B} w_i(t), \tag{21}$$

where $\overline{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m\times m}$ and $\overline{B} = (0, \cdots, 0, 0, 1)^{\mathsf{T}} \in \mathbb{R}^m$. In view of (10), the control input can be designed as follows

$$\begin{aligned} w_i(t) =& (0, -\mu_1, -\mu_2, \cdots, -\mu_{m-1}) \cdot y_i(t) \\ &+ \sum_{p \in \left(\cup_{j\in N_i^{\le \ell-}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} a_{ij}^p(t) \\ &\qquad \cdot \psi_{ij}(z_j^p(t-\tau_{ij}^p(t)), z_i(t)) \\ =& -\sum_{k=1}^{m-1} \mu_k \xi_i^{(k)}(t) \\ &+ \sum_{p \in \left(\cup_{j\in N_i^{\le \ell-}(t)} \mathcal{P}_{ij}(t)\right) \setminus \mathcal{R}_i(t)} a_{ij}^p(t) \\ &\qquad \cdot \psi_{ij}(z_j^p(t-\tau_{ij}^p(t)), z_i(t)), \end{aligned} \tag{22}$$

where $z_i(t) = My_i(t) = \sum_{k=0}^{m-1} \mu_{k+1} \xi_i^{(k)}(t)$, $z_j^p(t-\tau_{ij}^p(t)) = My_j^p(t-\tau_{ij}^p(t)) = \sum_{k=0}^{m-1} \mu_{k+1} \xi_j^{p(k)}(t-\tau_{ij}^p(t))$ for $i, j \in \mathcal{V}$ and $p \in \mathcal{P}_{ij}(t)$. Here, $\xi_i^p(t) \in \mathbb{R}$ is defined as the received message at tail j sent from head i along a path $p \in \mathcal{P}_{ji}(t)$. The following consensus result is a direct corollary of Theorem 2.

Corollary 1. *Consider the weighted directed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $\mathcal{V} = \mathcal{B} \cup C$, where the cooperative nodes follow the proposed ℓ-hop resilient vector consensus strategy with (21) and (22) and the set $\mathcal{B}$ of malicious nodes is r-local under ℓ-hop communication. Suppose that Assumptions 1 and 2 hold. If $\mathcal{G}$ is $(2r+1)$-robust under ℓ-hop communication relative to $\mathcal{B}$, then there exists $z \in \mathbb{R}$ satisfying $\lim_{t\to\infty} y_i(t) = (\mu_1^{-1}, 0_{m-1}^T)^T z$ for all $i \in C$.*

It follows from (20) that $\lim_{t\to\infty} \xi_i(t) = \mu_1^{-1}z$ and $\lim_{t\to\infty} \xi_i^{(k)}(t) = 0$ $(1 \le k \le m-1)$ for all $i \in C$. When $m = 2$, the second-order multiagent system (20) corresponds to the classical mechanics. The consensus result indicates that the cooperative nodes will stop at a common location eventually.

Remark 7. We have seen from Corollary 1 that the resilient vector consensus protocol works for higher-order agent dynamics and some nonlinear coupling functions ψ_{ij} satisfying Assumption 1 are allowed (see Remark 3). However, our protocols only work in the leaderless scenario as no effort has been made in the strategies to differentiate a leader from a malicious agent. As a result, any leader captured by some minimum message cover of a cooperative agent will be removed and thus tracking consensus may not be achieved. In the literature, several strategies have been proposed for resilient tracking consensus such as those recruiting sufficiently large number of leaders [27] and those admitting strong robustness conditions with trusted leaders [24]. However, these algorithms are tailored for discrete-time systems with nearest neighbor interactions. To realize resilient tracking consensus in our setting, new developments will be needed to either reduce the number of leaders or relax the network robustness because the multi-hop interaction would require a larger number of leaders and a more connected topology, both of which tend to be undesirable.

6. Simulations

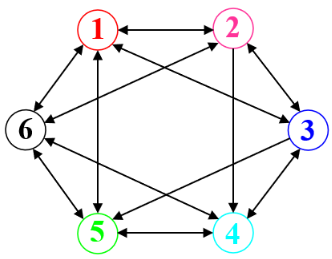

Figure 3: A network network $\mathcal{G}$ over $n = 6$ agents with $\mathcal{B} = \{1\}$ and $C = \{2, 3, 4, 5, 6\}$.

Example 1. In this example, we present simulation results on a network topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $n = 6$, $C = \{2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{1\}$; see Fig. 3. The edge weights $\{a_{ij}\}_{i,j\in\mathcal{V}}$ are taken as 0.8 if present and 0 otherwise. Note that $\mathcal{G}$ is not 3-robust under 1-hop communication. To check this we can take $\mathcal{S}_1 = \{2, 3\}$ and $\mathcal{S}_2 = \mathcal{V}\backslash\mathcal{S}_1$. Nevertheless, it is 3-robust under 2-hop communication relative to any set $\mathcal{S}$ with $|\mathcal{S}| = 1$. This network topology shows the usefulness of our ℓ-hop communication framework as it may allow resilient consensus on a sparser network. We set $\ell = 2$ and $r = 1$ accordingly.

We first consider the scalar consensus framework for the dynamics (1) and (8). Set the time delay $\tau_{ij}^p(t) \equiv \tau = 0.3$ and the coupling function $\psi_{ij}(b_1, b_2) = b_1 - b_2$ for all $i, j \in \mathcal{V}$. The initial configuration is described by $x_i(0) \in [-2, 2]$, $\varphi_i(t) = 0$ for $t \in [-\tau, 0]$ and $i \in \mathcal{V}$. The malicious node follows $\dot{x}_1(t) = -0.5\sin(3t)$ and the cooperative nodes follow (8). The consensus result is shown in Fig. 4. The cooperative nodes reach an agreement regardless of the malicious behavior of node 1 as one would expect from Theorem 1.

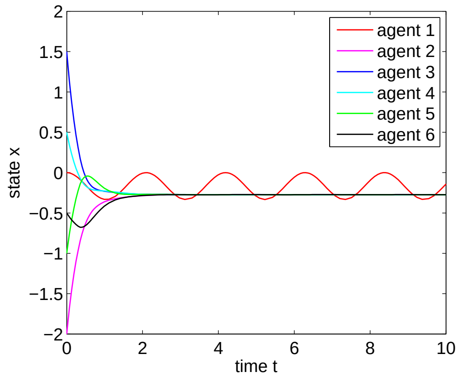

Figure 4: State trajectories for resilient scalar consensus in Example 1 following (1) and (8).

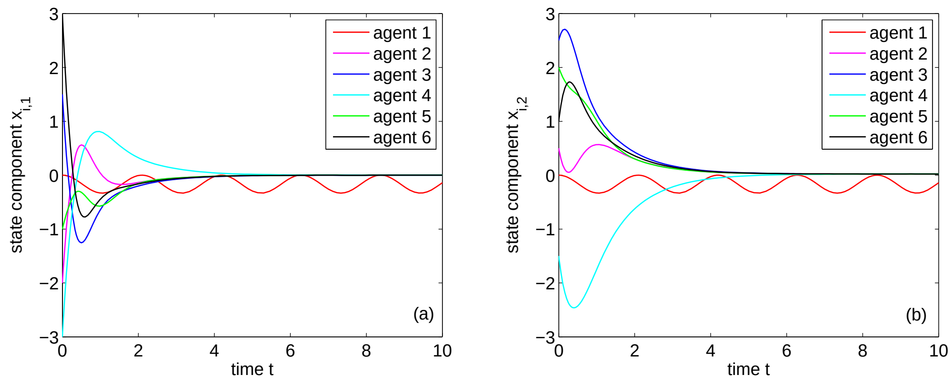

Figure 5: State trajectories for resilient vector consensus in Example 1 following (3) and (9).

Next, we examine the vector consensus framework for the dynamics (3) and (9). The time delay and coupling function are set as above. Let $m = q = 2$ and we consider the system (3) as

$$\dot{x}_i(t) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} x_i(t) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u_i(t). \quad (23)$$

It is direct to check the controllability of (A, B). To verify the matrix calculations performed in Section 2.3, we derive the following: $R = B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $C = QR^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$. Take $b = (1, 0)^\top$. We have $A + BC = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 1 & -1 \\ 0 & 0.5 \end{pmatrix}$ and $F_2 = (0, 0.5)^\top$. Moreover, $S = \begin{pmatrix} 0 & 0.5 \\ 1 & 0 \end{pmatrix}$, $\overline{A} = S(A + BC)S^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$ and $\overline{B} = SBb = (0, 1)^\top$. By calculating the characteristic polynomial, we verify $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 1$.

Write $x_i(t) = (x_{i,1}(t), x_{i,2}(t))^\top$ for $i \in \mathcal{V}$. Choose $M = (1, 1)$ and hence $\Lambda = (-2, -3)$. The initial configuration is described by $x_i(0) \in [-3, 3]^2$, $\varphi_i(t) = 0$ for $t \in [-\tau, 0]$ and $i \in \mathcal{V}$.

The malicious node follows $\dot{x}_1(t) = -0.5\sin(3t)1_2$ and the cooperative nodes follow (9), where $1_2 = (1,1)^\mathsf{T}$. We show the two components of the state vectors of nodes in Fig. 5. Resilient consensus has been achieved as predicted by Theorem 2.

Example 2. The W-MSR resilience mechanisms considered here come with a toll in terms of an added error when in the absence of malicious agents [16]. In this example, we compare our protocols for scalar consensus and vector consensus with the corresponding strategies without removing any neighbors. Namely, we compare our strategies with those taking $\mathcal{R}_i(t) = \emptyset$ in (8) for scalar consensus and in (9) for vector consensus, respectively. Specifically, we consider the same network $\mathcal{G}$ as Example 1, where $C = \{1, 2, \cdots, 6\}$. With the same system dynamics as in Example 1, we show in Fig. 6 the scalar consensus reaching processes for strategies with and without the resilience mechanism for $x_i(0) \in [-2, 2]$ ($i \in \mathcal{V}$). When the resilience mechanism is in place, the final consensus value is around −0.122 whereas it is around −0.028 without having the resilience mechanism. It is worth noting that the impact of the resilience mechanism is not only in the final consensus value but also in the consensus speed — the resilience mechanism slightly slows down the consensus seeking. This is presumably due to the removal of neighbors inherent in the W-MSR strategy.

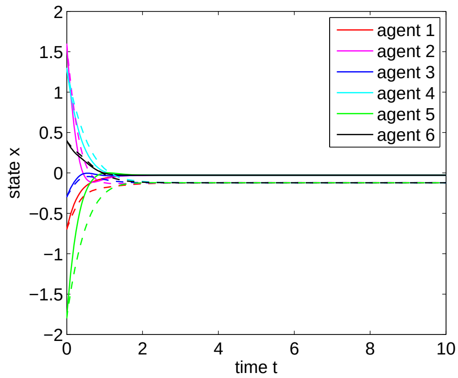

Figure 6: State trajectories for resilient scalar consensus in Example 2 with resilience mechanism (dashed curves) and without resilience mechanism (solid curves).

Similarly, in Fig. 7 we show the vector consensus reaching processes for strategies with and without the resilience mechanism for $x_i(0) \in [-3, 3]^2$ ($i \in \mathcal{V}$). When the resilience mechanism is in place, the final consensus value is around (−0.237, 0.006) whereas it is around (−0.069, 0.003) without having the resilience mechanism. We again observe a visible difference caused by the W-MSR removal mechanism.

7. Conclusion

In this paper we have addressed the resilient consensus problems in continuous-time multiagent networks under both multi-hop communication and path-dependent communication delay. Resilient consensus conditions have been developed to make use of the multi-hop communication topology. We have presented two general frameworks tailored for the scalar state space and the vector state space of agents, respectively. The proposed protocols are flexible accommodating malicious adversaries and path-dependent communication weights, and both linear and

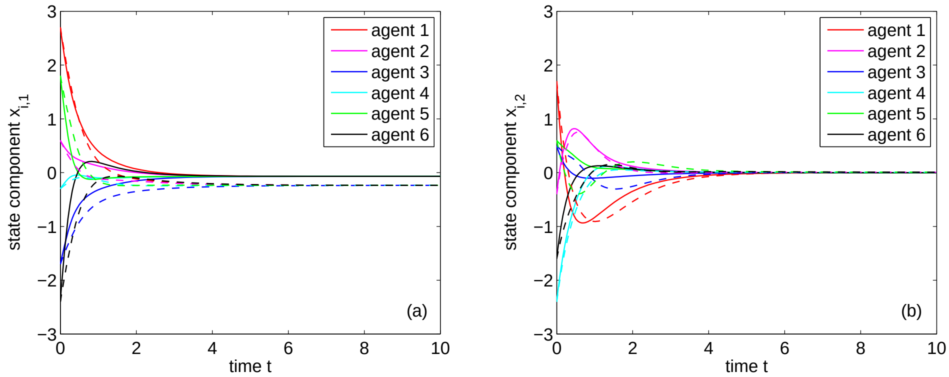

Figure 7: State trajectories for resilient vector consensus in Example 2 with resilience mechanism (dashed curves) and without resilience mechanism (solid curves).

nonlinear coupling dynamics are factored in. Higher-order agent dynamics can also be solved as a special case. In the future works, we plan to explore other effective techniques such as reputation metric-based strategies, event-triggered control and impulsive control under multi-hop communication.

CRediT authorship contribution statement

Yilun Shang: Investigation, Writing-original draft, Writing-review & editing.

Declaration of competing interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The author is grateful to the three anonymous reviewers and the editor for their constructive and insightful comments that helped improve the paper significantly.

References

[1] W. Abbas, A. Laszka, X. Koutsoukos, Improving network connectivity and robustness using trusted nodes with application to resilient consensus. *IEEE Trans. Contr. Netw. Syst.*, 5(4)(2018) 2036–2048

[2] W. Abbas, M. Shabbir, J. Li, X. Koutsoukos, Resilient distributed vector consensus using centerpoint. *Automatica*, 136(2022) 110046

[3] A. Amirkhani, A. H. Barshooi, Consensus in multi-agent systems: a review. *Artif. Intell. Rev.*, 55 (2022) 3897–3935

[4] Y. Bai, J. Wang, Resilient consensus of continuous-time linear networked systems. *IEEE Trans. Circuits Syst. Express Briefs*, 69(8)(2022) 3500–3504

[5] J. M. Danskin, The theory of max-min, with applications. *SIAM J. Appl. Math.*, 14(4)(1966) 641–664

[6] G. De La Torre, T. Yucelen, Adaptive architectures for resilient control of networked multiagent systems in the presence of misbehaving agents. *Int. J. Contr.*, 91(3)(2018) 495–507

[7] S. M. Dibaji, H. Ishii, Resilient consensus of second-order agent networks: asynchronous update rules with delays. *Automatica*, 81(2017) 123–132

[8] S. Giannini, D. D. Paloa, A. Petitti, A. Rizzo, On the convergence of the max-consensus protocol with asynchronous updates. *52nd IEEE Conf. Decision and Control*, Florence, Italy, 2013, pp. 2605–2610

[9] J. R. Haddock, T. Krisztin, J. Terjéki, J. H. Wu, An invariance principle of Lyapunov-Razumikhin type for neutral functional differential equations. *J. Differ. Equat.*, 107(1994) 395–417

[10] M. Heymann, W. Wonham, Comments on pole assignment in multi-input controllable linear systems. *IEEE Trans. Autom. Contr.*, 13(6)(1968) 748–749

[11] H. Ishii, Y. Wang, S. Feng, An overview on multi-agent consensus under adversarial attacks. *Ann. Rev. Contr.*, 53(2022) 252–272

[12] H. J. LeBlanc, H. Zhang, X. Koutsoukos, S. Sundaram, Resilient asymptotic consensus in robust networks. *IEEE J. Select. Areas Commun.*, 31(4)(2013) 766–781

[13] D. G. Luenberger, Canonical forms for linear multivariable systems. *IEEE Trans. Autom. Contr.*, 12(3)(1967) 290–293

[14] M. Mesbahi, M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*, Princeton University Press, New Jersey, 2010

[15] R. Olfati-Saber, J. A. Fax, R. M. Murray, Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(2007) 215–233

[16] G. Ramos, D. Silvestre, A. P. Aguiar, A resilient continuous-time consensus method using a switching topology. *Syst. Contr. Lett.*, 169(2022) 105381

[17] G. Ramos, D. Silvestre, C. Silvestre, A general discrete-time method to achieve resilience in consensus algorithms. *59th IEEE Conf. Decision and Control*, Jeju Island, Korea, 2020, pp. 2702–2707

[18] G. Ramos, D. Silvestre, C. Silvestre, General resilient consensus algorithms. *Int. J. Contr.*, 95(6)(2022) 1482–1496

[19] H. Rezaee, T. Parisini, M. M. Polycarpou, Almost sure resilient consensus under stochastic interation: links failure and noisy channels. *IEEE Trans. Autom. Contr.*, 66(12)(2021) 5727–5741

[20] D. M. Senejohnny, S. Sundaram, C. D. Persis, P. Tesi, Resilience against misbehaving nodes in asynchronous networks. *Automatica*, 104(2019) 26–33

[21] Y. Shang, Resilient consensus in multi-agent systems with state constraints. *Automatica*, 122(2020) 109288

[22] Y. Shang, Resilient consensus for robust multiplex networks with asymmetric confidence intervals. *IEEE Trans. Netw. Sci. Eng.*, 8(1)(2021) 65–74

[23] Y. Shang, Median-based resilient consensus over time-varying random networks. *IEEE Trans. Circuits Syst. Express Briefs*, 69(3)(2022) 1203–1207

[24] Y. Shang, Resilient tracking consensus over dynamic random graphs: a linear system approach. *Euro. J. Appl. Math.*, (2022) *doi.org/10.1017/S0956792522000225*

[25] Y. Shang, W. Luo, S. Xu, *L*-hop percolation on networks with arbitrary degree distributions and its applications. *Phys. Rev. E*, 84(2011) 031113

[26] L. Su, N. H. Vaidya, Reaching approximate Byzantine consensus with multi-hop communication. *Inf. Comput.*, 255(2017) 352–368

[27] J. Usevitch, D. Panagou, Resilient leader-follower consensus to arbitrary reference values in time-varying graphs. *IEEE Trans. Autom. Contr.*, 65(4)(2020)1755–1762

[28] J. Yan, Y. Mo, X. Li, C. Wen, A 'safe kernel' approach for resilient multi-dimensional consensus. *IFAC PapersOnLine*, 53-2(2020) 2507–2512

[29] H. Yang, Q.-L. Han, X. Ge, L. Ding, Y. Xu, B. Jiang, D. Zhou, Fault-tolerant cooperative control of multiagent systems: a survey of trends and methodologies. *IEEE Trans. Ind. Inf.*, 16(1)(2020) 4–17

[30] E. Yildirim, T. Yucelen, Control of multiagent networks with misbehaving nodes. *Int. J. Syst. Sci.*, 52(13)(2021) 2858–2874

[31] L. Yuan, H. Ishii, Secure consensus with distributed detection via two-hop communication. *Automatica*, 131(2021) 109775

[32] L. Yuan, H. Ishii, Resilient consensus with multi-hop communication. (2022) *arXiv:2201.03214*

[33] Y. Zhai, Z.-W. Liu, M.-F. Ge, G. Wen, X. Yu, Y. Qin, Trusted-region subsequence reduction for designing resilient consensus algorithms. *IEEE Trans. Netw. Sci. Eng.*, 8(1)(2021) 259–268

[34] D. Zhao, Y. Lv, X. Yu, G. Wen, G. Chen, Resilient consensus of higher order multiagent networks: an attack isolation-based approach. *IEEE Trans. Autom. Contr.*, 67(2)(2022) 1001–1007